A NOTE ON TOEPLITZ OPERATORS

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Abstract. We study Toeplitz operators on Bergman spaces using techniques from the analysis of Dirac-type operators on complete Riemannian manifolds, and prove an index theorem of Boutet de Monvel from this point of view.

Let B be a strongly pseudoconvex domain in $\mathbb{C}^n$. The *Bergman space* $\mathcal{H}^2(B)$ is the subspace of $\mathcal{L}^2(B)$ consisting of the Lebesgue square-integrable holomorphic functions on B.

Let f be a smooth function on $\overline{B}$. The *Toeplitz operator* T_f is the compression to $\mathcal{H}^2(B)$ of the operator of pointwise multiplication by f. If $F = [f_{ij}]$ is a smooth $N \times N$ matrix-valued function (= matrix of smooth functions) then denote by T_F the operator matrix with entries $T_{f_{ij}}$. View it as an operator on a direct sum of N copies of $\mathcal{H}^2(B)$.

In this note we are concerned with the following result, the first part of which is due to Venugopalkrishna [10], and the second to Boutet de Monvel [3].

Theorem. *Suppose that the restriction of F to the boundary ∂B is an invertible matrix-valued function. Then the operator T_F is Fredholm and*

$$\operatorname{Index}(T_F) = \frac{-(n-1)!}{(2n-1)!(2\pi i)^n} \int_{\partial B} \operatorname{trace}((F^{-1}dF)^{2n-1}).$$

We shall equip B with a complete Hermitian metric, a poor man's version of the Bergman metric, and borrow an estimate from a paper of Donnelly and Fefferman [5] to exhibit a gap in the spectrum of the Dolbeaut operator on B. This is the main novelty in our paper: it reduces the theorem to a result about the Dolbeaut operator, to which standard calculations apply. The index formula itself follows from the Atiyah-Singer theorem.

1. A Complete Hermitian Metric

Let B be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Let r be a smooth, real-valued function on $\mathbb{C}^n$ such that

$$B = \{p \in \mathbb{C}^n \mid r(p) > 0\}$$

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and such that grad(r) is nowhere vanishing on ∂B. Recall that B is *strongly pseudoconvex* if the following condition (which depends only on B, not the choice of r) is satisfied at every point $p \in \partial B$:

$$\text{(1.1)} \qquad \textit{If } a \in \mathbb{C}^n \textit{ is non-zero and } \sum_i a_i \frac{\partial r}{\partial z_i} = 0 \textit{ then } \sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j < 0.$$

Replace r by $r - Cr^2$ in a neighbourhood of ∂B, where C is a sufficiently large positive constant. Then the following stronger condition holds at every $p \in \partial B$:

$$\text{(1.2)} \qquad \textit{If } a \in \mathbb{C}^n \textit{ is any non-zero vector then } \sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j < 0.$$

By continuity the inequality (1.2) holds in a neighbourhood U of ∂B in $\overline{B}$. It is a simple matter to modify r so that (1.2) holds throughout $\overline{B}$; see Proposition 10.4 in [6]. *From now on we shall assume that* (1.2) *holds at every point in* $\overline{B}$.

Lemma 1. *The form*

$$\text{(1.3)} \qquad \sum_{ij} h_{ij} dz_i \otimes d\bar{z}_j = -\sum_{ij} \frac{\partial^2 \log(r)}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$$

defines a Hermitian metric on B.

Proof. Calculating the derivatives we find that

$$\text{(1.4)} \qquad h_{ij} = \frac{1}{r^2} \frac{\partial r}{\partial z_i} \frac{\partial r}{\partial \bar{z}_j} - \frac{1}{r} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}.$$

Since r is real-valued the first term is positive semidefinite. The second (including the minus sign) is positive definite by (1.2). □

Lemma 2.

(1) *The real part of* h_{ij} *is a complete Riemannian metric on* B.

(2) *If* f *is any smooth function on* $\overline{B}$ *then with respect to this Riemannian metric the gradient of* f *on* B *vanishes at infinity.*

Proof. Let $c(t)$ $(0 \le t \le 1)$ be a curve in B. Write $c(t) = (c_1(t), \ldots, c_n(t))$, where the $c_j(t)$ are complex valued functions. Then

$$\text{length}\,(c) = \int_0^1 \sqrt{\sum h_{ij} \frac{dc_i}{dt} \overline{\frac{dc_j}{dt}}}\, dt.$$

But (1.4) shows that

$$\sum h_{ij} \frac{dc_i}{dt} \overline{\frac{dc_j}{dt}} \ge \sum \frac{1}{r^2} \frac{\partial r}{\partial z_i} \frac{\partial r}{\partial \bar{z}_j} \frac{dc_i}{dt} \overline{\frac{dc_j}{dt}} = \left| \frac{1}{r} \frac{dr}{dt} \right|^2$$

(we compose r with c to get a function of t). Therefore

$$\text{length}\,(c) \geq \int_0^1 \left| \frac{1}{r}\frac{dr}{dt} \right| \, dt \geq \left| \int_0^1 \frac{1}{r}\frac{dr}{dt}\, dt \right| = |\log(r_1) - \log(r_0)|,$$

where $r_1 = r(c(1))$ and $r_0 = r(c(0))$. Since $|\log(r)| \to \infty$ at ∂B this estimate shows that bounded sets in B lie within compact subsets $\{p : |\log(r(p))| \leq C\}$ of B, which proves completeness.

Let X be a tangent vector field on $\overline{B}$. From (1.4) we get

$$|X| \geq \mathit{constant} \cdot r^{-1/2} |X|_{Eucl},$$

where on the left hand side we have the norm induced from the metric (1.3), and on the right we have the ordinary Euclidean norm. So for a cotangent vector field ω on $\overline{B}$ we have $|\omega| \leq \mathit{constant} \cdot r^{1/2} |\omega|_{Eucl},$. Apply this to $\omega = df$ to prove part (2) of the lemma. $\square$

An Estimate for the Dolbeaut Operator on B

The material in this section is adapted from a paper of H. Donnelly and C. Fefferman [5].

Let TB be the (real) tangent bundle of B, and $\bigwedge^* T^*_{\mathbb{C}} B$ the complexified exterior algebra bundle. Both receive inner products from the Hermitian metric (1.3).[1] Denote by $A^{p,q}$ the space of smooth compactly supported forms on B of type (p, q). Having specified a Hermitian metric on B the space $A^{p,q}$ has a natural inner product. Denote by $\mathcal{A}^{p,q}_h$ the Hilbert space completion.

From the Hermitian metric (1.3) we obtain a Hodge operator

$$\bar{*} \colon A^{p,q} \to A^{n-p,n-q}$$

(see [11]). It is a conjugate linear, isometric isomorphism. We recall that

$$\bar{\partial}^* = -\bar{*}\bar{\partial}\bar{*}, \tag{2.1}$$

where $\bar{\partial}^*$ denotes the formal adjoint of $\bar{\partial}$ with respect to the given inner products on $A^{p,q}$.

Our analysis of Toeplitz operators relies on the following estimate.

Proposition. *If* $\omega \in A^{n,q}$ *then*

$$\|\bar{\partial}\omega\|^2 + \|\bar{\partial}^*\omega\|^2 \geq \frac{q}{2}\|\omega\|^2. \tag{2.2}$$

For $\eta \in T^*_{\mathbb{C}} B$ and $\omega \in \bigwedge^*_{\mathbb{C}} T^* B$ we define *interior product* $\eta \lrcorner\, \omega$ by the adjoint relation

$$(\eta \lrcorner\, \omega, \omega') = -(\omega, \eta \wedge \omega')$$

[1] The conventions are such that in the standard Hermitian metric $h_{ij} = \delta_{ij}$ on $\mathbb{C}^n$ we have $|dz_i|^2 = 2$ (pointwise norm). This accounts for the presence of some factors of 2 later on.

(note that $\eta \lrcorner \omega$ is *conjugate* linear in η). Using this we define

$$c(\eta)\omega = \eta \wedge \omega + \eta \lrcorner \omega$$

and also

$$\tilde{c}(\eta)\omega = \eta \wedge \omega - \eta \lrcorner \omega.$$

If $\eta, \xi \in T^*B \subset T^*_{\mathbb{C}}B$ then the operators $c(\eta)$ and $\tilde{c}(\xi)$ anticommute. In addition, $\tilde{c}(\xi)$ is self-adjoint whereas $c(\eta)$ is skew-adjoint.

Let f be a C^∞ function on B. Recall that the *Hessian* of f at a point $p \in B$ is the symmetric bilinear form

$$H_f \colon TB_p \times TB_p \to \mathbb{R}$$

given by the formula

$$H_f(X, Y) = X(\tilde{Y}(f)) - (\nabla_X \tilde{Y})(f).$$

Here ∇ is the Levi-Cevita affine connection and $\tilde{Y}$ denotes an extension of Y to a vector field defined near p (the formula does not depend on the choice of extension).

Let $X_1, \ldots, X_{2n}$ be a local frame for TB and denote by $\eta_1, \ldots, \eta_{2n}$ the dual frame for T^*B. We define a self-adjoint endomorphism of the exterior algebra bundle of B by the formula

$$\boldsymbol{H}_f = \sum_{i,j} H_f(X_i, X_j) c(\eta_i) \tilde{c}(\eta_j)$$

(it does not depend on the choice of local frames).

Lemma 1. *Let $D = d + d^*$ be the de Rham operator. Then*

$$(D + \tilde{c}(df))^2 = D^2 + \boldsymbol{H}_f + |df|^2,$$

where $|df|^2$ denotes the pointwise norm of df, acting as an operator on forms by pointwise multiplication.

Proof. This follows immediately from the formula

$$D = \sum_i c(\eta_i) \nabla_{X_i},$$

where ∇ is the affine connection on the exterior algebra bundle induced from the Levi-Cevita connection on B. $\square$

We now specialize to

$$f = \frac{1}{2} \log(r),$$

where r is the defining function for ∂B, as in the previous section.

Since f is real-valued we have that $|df|^2 = 2|\bar{\partial} f|^2$, and we compute:

$$|\bar{\partial} \log(r)|^2 = \sup \left\{ r^{-2} \left| \sum a_i \tfrac{\partial r}{\partial z_i} \right|^2 : r^{-2} \left| \sum a_i \tfrac{\partial r}{\partial z_i} \right|^2 - r^{-1} \sum a_i \tfrac{\partial^2 r}{\partial z_i \partial \bar{z}_j} \bar{a}_j \leq 2 \right\} \leq 2$$

(see the proof of Lemma 1.1). Consequently

$$|df|^2 \leq 1. \tag{2.3}$$

Lemma 2. *Let ω be a form of type $(0, n-q)$ on B. Then*

$$\boldsymbol{H}_f\omega = 2q\omega \;+\; \text{a form of type } (1, n-q-1).$$

Proof. Recall that a coordinate system is *normal* if the Hermitian metric on B has the form

$$h_{ij} = \delta_{ij} + O(|z^2|)$$

in that system. Recall also that in a Kähler manifold every point is the origin of a normal coordinate system. Let $z_i = x_i + \sqrt{-1}\, y_i$ be normal coordinates at $p \in B$. It is easily checked that if

$$d\bar{z}_I = d\bar{z}_{i_1} \dots d\bar{z}_{i_{n-q}} \qquad (I = \{1_1 < \cdots < i_{n-q}\})$$

then

$$c(dx_i)\tilde{c}(dx_i)d\bar{z}_I = \left\{ \begin{array}{rl} -d\bar{z}_I & \text{if } i \notin I \\ \text{a } (1, n-q-1)\text{-form} & \text{if } i \in I \end{array} \right\} \quad \text{at } p.$$

The same holds for $c(dy_i)\tilde{c}(dy_i)d\bar{z}_I$. Modulo forms of type $(1, n-q-1)$, we obtain the formula

$$\boldsymbol{H}_f d\bar{z}_I = -\sum_{i \in I} H_f(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i})d\bar{z}_I - \sum_{i \in I} H_f(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i})d\bar{z}_I \quad \text{at } p.$$

But

$$H_f(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}) + H_f(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i}) = 4\frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} \quad \text{at } p,$$

and it follows from the definition of the metric (1.3) that at the origin of any normal coordinate system we have

$$\frac{\partial^2 \log(r)}{\partial z_i \partial \bar{z}_j} = \delta_{ij}.$$

This proves the lemma. □

Proof of the Proposition. We first prove the related estimate

$$\|\bar{\partial}\omega\|^2 + \|\bar{\partial}^*\omega\|^2 \geq \frac{q}{2}\|\omega\|^2, \quad \forall \omega \in A^{0,n-q}. \tag{2.4}$$

The operator $D + \tilde{c}(df)$ is symmetric, and so

$$\langle (D + \tilde{c}(df))^2\omega, \omega\rangle \geq 0$$

(the angle brackets denote Hilbert space inner products). Applying Lemma 1 we get

$$\langle D^2\omega, \omega\rangle + \langle \boldsymbol{H}_f\omega, \omega\rangle + \langle |df|^2\omega, \omega\rangle \geq 0.$$

On a Kähler manifold such as B we have

$$\langle D^2\omega, \omega\rangle = 2\|\bar{\partial}\omega\|^2 + 2\|\bar{\partial}^*\omega\|^2,$$

and so Lemma 2, along with (2.3), gives

$$2\|\bar\partial\omega\|^2 + 2\|\bar\partial^*\omega\|^2 \geq 2q\langle\omega,\omega\rangle - \langle|df|^2\omega,\omega\rangle \geq (2q-1)\langle\omega,\omega\rangle, \qquad \forall\omega\in A^{0,n-q}. \tag{2.5}$$

When $q=0$ the estimate (2.4) has no content. When $q>0$ we have $2q-1\geq q$, and so (2.4) follows from (2.5).

To complete the proof, use the Hodge operator $\bar*$ and the identity

$$\|\bar\partial\omega\|^2 + \|\bar\partial^*\omega\|^2 = \|\bar\partial\bar*\omega\|^2 + \|\bar\partial^*\bar*\omega\|^2,$$

which is a consequence of (2.1). □

Toeplitz Operators

In this section we shall prove that the Toeplitz operator T_F is Fredholm if the "symbol" F is invertible on ∂B.

Form the *twisted Dolbeaut operators*

$$D_+ = \bar\partial + \bar\partial^*\colon \bigoplus_{q \text{ even}} A^{n,q} \to \bigoplus_{q \text{ odd}} A^{n,q}$$

$$D_- = \bar\partial + \bar\partial^*\colon \bigoplus_{q \text{ odd}} A^{n,q} \to \bigoplus_{q \text{ even}} A^{n,q}$$

and view them as unbounded operators on the Hilbert spaces $\oplus_{q \text{ even}}\mathcal{A}_h^{n,q}$ and $\oplus_{q \text{ odd}}\mathcal{A}_h^{n,q}$. Denote by $\mathcal{D}_\pm$ the closures of these operators in the sense of unbounded operator theory [8]. We note that

$$\|\mathcal{D}_\pm\omega\|^2 = \|\bar\partial\omega\|^2 + \|\bar\partial^*\omega\|^2, \qquad \forall\omega\in\oplus_{q \text{ even/odd}}A^{n,q}.$$

So the proposition in the previous section implies that the kernel of $\mathcal{D}_+$ is concentrated in bidegree $(n,0)$. In other words the kernel of $\mathcal{D}_+$ is precisely the space of holomorphic square-integrable forms of type $(n,0)$.

The map

$$f(z) \mapsto 2^{n/2} f(z)dz_1dz_2\dots dz_n$$

gives a unitary isomorphism from $L^2(B)$ (formed using Lebesgue measure on B) to $\mathcal{A}_h^{n,0}$. The Bergman space $\mathcal{H}^2(B)$ is mapped isomorphically onto the space of holomorphic forms in $\mathcal{A}_h^{n,0}$. It follows that the Toeplitz operator T_f on Bergman space is unitarily equivalent to the compression to the kernel of $\mathcal{D}_+$ of the operator of multiplication by f on $\oplus_{q \text{ even}}\mathcal{A}_h^{n,q}$. *For the rest of the paper we shall use the notation T_f for this latter Toeplitz operator, and work exclusively with it.*

It follows from the proposition in the previous section that the operator $\mathcal{D}_-$ is bounded below (by $1/\sqrt{2}$). So we can form a "generalized inverse"

$$E\colon \bigoplus_{q \text{ even}} \mathcal{A}_h^{n,q} \to \bigoplus_{q \text{ odd}} \mathcal{A}_h^{n,q}$$

by projecting orthogonally onto the range of $\mathcal{D}_-$ and then applying $\mathcal{D}_-^{-1}$. It is a bounded Hilbert space operator whose range is the domain of $\mathcal{D}_-$.

Lemma 1. *Denote by*

$$P\colon \bigoplus_{q \text{ even}} \mathcal{A}_h^{n,q} \to \bigoplus_{q \text{ even}} \mathcal{A}_h^{n,q}$$

the orthogonal projection onto the kernel of $\mathcal{D}_+$. *Then*

$$P = I - \mathcal{D}_- E \tag{3.1}$$

Proof. The manifold B is complete in Riemannian metric given by (1.3), so by a well known result [4] the operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_+ \\ \mathcal{D}_- & 0 \end{pmatrix}$$

is self-adjoint. Consequently $\mathcal{D}_+^* = \mathcal{D}_-$, and the lemma follows from the fact that the kernel of an operator is the orthogonal complement of the range of its adjoint. □

Since $\mathcal{D}_-$ is bounded below and $\mathcal{D}_+$ is its adjoint we see that $\mathcal{D}_+\mathcal{D}_-$ is also bounded below. Hence its spectrum is bounded away from 0. Since the spectra of $\mathcal{D}_-\mathcal{D}_+$ and $\mathcal{D}_+\mathcal{D}_-$ coincide, except for 0, it follows that the operators $\mathcal{D}$ and

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_-\mathcal{D}_+ & 0 \\ 0 & \mathcal{D}_+\mathcal{D}_- \end{pmatrix}$$

are bounded below on the orthogonal complement of their common kernel. This observation will be needed in the next section.

Lemma 2. *If* φ *is any continuous function on* B *(or more generally, any vector bundle endomorphism over* B*) which vanishes at infinity then the operators* φP *and* φE *are compact.*

Proof. By an approximation argument it suffices to prove the lemma when φ is supported in a compact set. The basic elliptic estimate

$$C(\|\mathcal{D}_- u\| + \|u\|) \geq |||u|||,$$

where the triple bars denote the norm in the Sobolev space W^1, implies that both E and P are continuous when viewed as operators from L^2 into W_1. But Rellich's lemma implies that the natural inclusion of W^1 into L^2, followed by pointwise multiplication with a compactly supported function, is a compact operator. For further details, see for example [9]. □

Proposition. *If* f *is a smooth function on* $\overline{B}$ *then the commutator* $fP - Pf$ *is a compact operator.*

Proof. It suffices to show that $Pf(I-P)$ is compact, for every f. Using the fact that $E\mathcal{D}_-$ together with the formula (3.1) for P we get

$$Pf(I-P) = [f, \mathcal{D}_-]E - D_- E[f, \mathcal{D}_-]E.$$

But

$$[f, \mathcal{D}_-]\omega = \bar\partial f \wedge \omega + \bar\partial \bar f \lrcorner\, \omega. \tag{3.2}$$

By Lemma 1.2, $\bar\partial f$ vanishes at infinity, and so by Lemma 2 the operator $[f, \mathcal{D}_-]E$ is compact. Since D_-E is a bounded operator the result follows. □

Passing to Toeplitz operators, the above proposition implies (in the notation of the introduction) that

$$T_{F_1} T_{F_2} = T_{F_1 F_2}, \quad \text{modulo compact operators,}$$

and also that

$$T_F \text{ is compact if } F \text{ vanishes on } \partial B.$$

It follows that if F_1 is invertible on ∂B then T_{F_1} is invertible modulo compact operators—an inverse modulo compacts is T_{F_2}, where F_2 is any smooth matrix valued function such that $F_1 F_2 = I$ on ∂B. Hence we recover the first part of the theorem of the introduction:

Theorem. *If F is a smooth matrix valued function on $\overline{B}$ whose restriction to ∂B is invertible then the Toeplitz operator T_F is Fredholm.* □

The Index Theorem

For the rest of the section, fix a smooth, matrix-valued function

$$F \colon \overline{B} \to M_N(\mathbb{C})$$

which is invertible on ∂B. In what follows, we shall argue as if F were a scalar function rather than a vector valued function. Thus, for instance, what are in fact operators on a direct sum of N copies of a Hilbert space $\mathcal{H}$ we shall treat as operators on a single copy of $\mathcal{H}$. This considerably streamlines the notation.

As a first step towards calculating the index of T_F we form the operator

$$\mathcal{D}_F = \begin{pmatrix} F & \mathcal{D}_- \\ \mathcal{D}_+ & -F^* \end{pmatrix}$$

acting on $\oplus_q \mathcal{A}_h^{n,q}$.

Lemma. *The operator $\mathcal{D}_F$ is Fredholm, in the sense of unbounded operator theory.*[2] *Furthermore* $\operatorname{Index}(T_F) = \operatorname{Index}(\mathcal{D}_F)$.

Proof. Decompose $\mathcal{D}_F$ as a sum

$$\mathcal{D}_F = \mathcal{D} + \mathcal{F}$$

[2] A closed, densely defined operator is *Fredholm* if its range is closed and if its kernel and cokernel are finite dimensional.

in the obvious way. Denote by P the projection[3] onto the kernel of $\mathcal{D}$ and denote by Q the complementary projection so that $Q\mathcal{D} = \mathcal{D} = \mathcal{D}Q$. Of course,

$$\mathcal{D}_F = P\mathcal{D}_F P + P\mathcal{D}_F Q + Q\mathcal{D}_F P + Q\mathcal{D}_F Q.$$

The compression $P\mathcal{D}_F P$ is equal to T_F. The operators $P\mathcal{D}_F Q$ and $Q\mathcal{D}_F P$ are compact, by the Proposition in the previous section. As for the operator $Q\mathcal{D}_F Q$, we calculate that

$$(Q\mathcal{D}_F Q)^*(Q\mathcal{D}_F Q) = Q\mathcal{D}^2 Q + Q(\mathcal{D}\mathcal{F} + \mathcal{F}^*\mathcal{D})Q + Q\mathcal{F}^* Q\mathcal{F} Q \tag{4.1}$$

and

$$\mathcal{D}\mathcal{F} + \mathcal{F}^*\mathcal{D} = \begin{pmatrix} 0 & [F^*, \mathcal{D}_-] \\ [\mathcal{D}_+, F] & 0 \end{pmatrix}.$$

The norms of $[F^*, \mathcal{D}_-]$ and $[\mathcal{D}_+, F]$ are bounded by a multiple of the sup-norm of $\operatorname{grad}(F)$ (compare equation (3.2)). So if the gradient of F is small then the middle term in (4.1) is small too. Since the first term in (4.1) is bounded below and the last one is positive semidefinite, we see that $(Q\mathcal{D}_F Q)^*(Q\mathcal{D}_F Q)$ is bounded below if the gradient of F is sufficiently small. A similar calculation applies to $(Q\mathcal{D}_F Q)(Q\mathcal{D}_F Q)^*$. Hence $Q\mathcal{D}_F Q$ is invertible if $\operatorname{grad}(F)$ is sufficiently small.

Thus in this case, decomposing $\oplus_q \mathcal{A}_h^{n,q}$ into the direct sum of the ranges of P and Q, we have

$$\mathcal{D}_F = T_F \oplus (\textit{invertible operator}), \quad \text{modulo compacts},$$

which proves that $\mathcal{D}_F$ is Fredholm, with the same index as T_F.

In the general case, since B is complete, and since the gradient of F vanishes at infinity, there exists for any $\varepsilon > 0$ a smooth, compactly supported function φ on B such that the function

$$F' = (1 - \varphi)F$$

has gradient everywhere less that ε. For suitable ε the argument above shows that $\mathcal{D}_{F'}$ is Fredholm with $\operatorname{Index}(\mathcal{D}_{F'}) = \operatorname{Index}(T_{F'})$. But the operator

$$\mathcal{D}_F = \mathcal{D}_{F'} + \varphi\mathcal{F}$$

is a relatively compact perturbation of $\mathcal{D}_{F'}$.[4] So by perturbation theory the operator $\mathcal{D}_F$ is Fredholm, with the same index as $\mathcal{D}_{F'}$ (see Theorem 5.26 in [8]). Since $\operatorname{Index}(T_F) = \operatorname{Index}(T_{F'})$ the lemma is proved. □

Now let

$$K = \{p \in B : F \text{ is singular at } p\},$$

[3]This projection differs in a minor way from the projection called P in the previous section: it is defined on all of $\oplus_q \mathcal{A}_h^{n,q}$, not just $\oplus_{q\ \text{even}} \mathcal{A}_h^{n,q}$.

[4]This means that if v_n is a bounded sequence in the domain of $\mathcal{D}_{F'}$ with $\mathcal{D}_{F'} v_n$ bounded, then the sequence $\varphi\mathcal{F} v_n$ has a convergent subsequence. The proof follows from the basic elliptic estimate and the Rellich Lemma—c.f. the proof of Lemma 3.2.

and let U be a neighbourhood of K in B which has compact closure in B. Let V be a neighbourhood of K with compact closure in U, and with smooth boundary. Thus:

$$K \subset V \subset\subset U \subset\subset B.$$

There exists:

(i) a compact Riemannian manifold $\hat{B}$,

(ii) an elliptic partial differential operator $\hat{\mathcal{D}}_+ \colon \hat{S}_+ \to \hat{S}_-$ on $\hat{B}$ (here $\hat{S}_\pm$ denote Hermitian vector bundles on whose sections $\hat{\mathcal{D}}_+$ acts),

(iii) an open subset $\hat{U}$ of $\hat{B}$ and an isometry $\hat{U} \to U$, lifting to the appropriate vector bundles, which identifies the operators $\mathcal{D}_+$ on U and $\hat{\mathcal{D}}_+$ on $\hat{U}$.

The details of the construction (which is easily done by "doubling" a suitable subdomain of B) do not concern us: any manifold $\hat{B}$ and operator $\hat{\mathcal{D}}_+$, etc, will do.

Define vector bundles $\hat{E}_\pm$ on $\hat{B}$ as follows:

(iv) $\hat{E}_+$ is the trivial vector bundle, which we equip with the standard inner product and affine connection.

(v) $\hat{E}_-$ is the bundle formed by clutching together trivial bundles over $\hat{V}$ and $\hat{B} \setminus \hat{V}$ using the restriction of the function F to $\partial\hat{V}$. Thus a smooth section of $\hat{E}_-$ is a pair s, s' of smooth vector valued functions, on *closure* $(\hat{V})$ and $\hat{B} \setminus \hat{V}$ respectively, such that $s = Fs'$ on $\partial\hat{V}$. We equip $\hat{E}_-$ with any metric and connection which restricts to the standard structure over $\hat{V}$, where the bundle is canonically trivialized.

Define a vector bundle homomorphism

$$\hat{F} \colon \hat{E}_+ \to \hat{E}_-, \qquad \hat{F} = \begin{cases} F & \text{on } \hat{V}, \\ I & \text{on } \hat{B} \setminus \hat{V}. \end{cases}$$

According to the definition of $\hat{E}_-$ this is a well-defined and smooth map.

Lift $\hat{\mathcal{D}}_+$ to an operator

$$\hat{\mathcal{D}}_+ \otimes \hat{E}_+ : \hat{S}_+ \otimes \hat{E}_+ \longrightarrow \hat{S}_- \otimes \hat{E}_+.$$

Lift its adjoint to an operator

$$\hat{\mathcal{D}}_- \otimes \hat{E}_- : \hat{S}_- \otimes \hat{E}_- \longrightarrow \hat{S}_+ \otimes \hat{E}_-.$$

Define an operator

$$\hat{\mathcal{D}}_F : (\hat{S}_+ \otimes \hat{E}_+) \oplus (\hat{S}_- \otimes \hat{E}_-) \longrightarrow (\hat{S}_+ \otimes \hat{E}_-) \oplus (\hat{S}_- \otimes \hat{E}_+)$$

by

$$\hat{\mathcal{D}}_F = \begin{pmatrix} \hat{F} & \hat{\mathcal{D}}_- \otimes \hat{E}_- \\ \hat{\mathcal{D}}_+ \otimes \hat{E}_+ & -\hat{F}^* \end{pmatrix}.$$

By standard elliptic theory $\hat{\mathcal{D}}_F$ is a Fredholm operator.

Lemma. $\operatorname{Index}(\mathcal{D}_F) = \operatorname{Index}(\hat{\mathcal{D}}_F)$.

Proof. For $t > 0$ define operators

$$\mathcal{D}_{tF} = \begin{pmatrix} tF & \mathcal{D}_- \\ \mathcal{D}_+ & -tF^* \end{pmatrix} \quad \text{and} \quad \hat{\mathcal{D}}_{tF} = \begin{pmatrix} t\hat{F} & \hat{\mathcal{D}}_- \otimes \hat{E}_- \\ \hat{\mathcal{D}}_+ \otimes \hat{E}_+ & -t\hat{F}^* \end{pmatrix}.$$

These operators are Fredholm (by the previous lemma and discussion) and have indices independent of t (by the continuity property of the Fredholm index). Let φ be a smooth real-valued function, compactly supported in V, such that $\varphi \equiv 1$ on K. Using the fact that V and $\hat{V}$ are isometric, and the fact that the bundles $\hat{E}_\pm$ are canonically trivialized over $\hat{V}$, we can define an operator

$$G_t = \begin{pmatrix} \mathcal{D}_{tF} & -t\varphi \\ t\varphi & \hat{\mathcal{D}}^*_{tF} \end{pmatrix}.$$

This is a relatively compact perturbation of $\mathcal{D}_{tF} \oplus \hat{\mathcal{D}}^*_{tF}$, and so it suffices to show that for large enough t this operator is invertible. We calculate:

$$G_t^* G_t = \begin{pmatrix} \mathcal{D}^*_{tF}\mathcal{D}_{tF} + t^2\varphi^2 & t(\varphi\hat{\mathcal{D}}^*_{tF} - \mathcal{D}^*_{tF}\varphi) \\ t(\hat{\mathcal{D}}_{tF}\varphi - \varphi\mathcal{D}_{tF}) & \hat{\mathcal{D}}_{tF}\hat{\mathcal{D}}^*_{tF} + t^2\varphi^2 \end{pmatrix}, \tag{4.2}$$

and proceed to analyze the terms in this matrix. The isometry $U \cong \hat{U}$ identifies $\mathcal{D}_{tF}$ and $\hat{\mathcal{D}}_{tF}$ over U, so

$$\begin{aligned} \varphi\mathcal{D}_{tF} - \hat{\mathcal{D}}_{tF}\varphi &\cong \varphi\mathcal{D}^*_{tF} - \mathcal{D}^*_{tF}\varphi \\ &= \begin{pmatrix} 0 & \varphi\mathcal{D}_- - \mathcal{D}_-\varphi \\ \varphi\mathcal{D}_+ - \mathcal{D}_+\varphi & 0 \end{pmatrix}, \end{aligned}$$

which is a bounded operator, with norm independent of t. This, and a similar calculation, show that the off-diagonal terms in (4.2) are bounded operators, their norms being multiples of t. As for the diagonal terms in (4.2), we have

$$\mathcal{D}^*_{tF}\mathcal{D}_{tF} + t^2\varphi^2 = \begin{pmatrix} \mathcal{D}_-\mathcal{D}_+ + t^2(F^*F + \varphi^2) & F^*\mathcal{D}_- - \mathcal{D}_-F^* \\ \mathcal{D}_+F - F\mathcal{D}_+ & \mathcal{D}_+\mathcal{D}_- + t^2(FF^* + \varphi^2) \end{pmatrix}. \tag{4.3}$$

The off-diagonal terms in *this* 2×2-matrix are bounded uniformly in t, while the diagonal terms are bounded below by a multiple of t^2, since

$$\mathcal{D}_-\mathcal{D}_+ + t^2(F^*F + \varphi^2) \geq t^2(F^*F + \varphi^2),$$

and $F^*F + \varphi^2 > 0$, by our choice of φ. Consequently, for large t, the matrix (4.3) is bounded below by some multiple of t^2. Returning to the matrix in (4.2), the diagonal terms are bounded below by a multiple of t^2, for large t, while the off-diagonal terms are bounded above by a multiple of t. It follows that for large enough t the entire matrix in (4.2) is bounded below. A similar calculation applies to $G_t G_t^*$, which proves that G_t is invertible for large t. □

It remains to calculate the index of $\hat{\mathcal{D}}_F$. This is a simple application of the Atiyah-Singer theorem[5] and we merely sketch the method.

The diagonal terms in the matrix defining $\hat{\mathcal{D}}_F$ are order zero operators and can be dropped without altering the Fredholm index. Using this, together with the fact that the index of an operator is minus the index of its adjoint, we get

$$\mathrm{Index}(\hat{\mathcal{D}}_F) = \mathrm{Index}(\hat{\mathcal{D}}_+ \otimes \hat{E}_+) - \mathrm{Index}(\hat{\mathcal{D}}_+ \otimes \hat{E}_-).$$

So according to the index theorem,

$$\mathrm{Index}(\hat{\mathcal{D}}_F) = \int_{T^*\hat{B}} \mathrm{Todd}(T_{\mathbb{C}}\hat{B})\,\mathrm{ch}(\sigma(\hat{\mathcal{D}}_+))(\mathrm{ch}(\hat{E}_+) - \mathrm{ch}(\hat{E}_-)).$$

If we choose connections on $\hat{E}_\pm$ which are isomorphic (via $\hat{F}$) outside of $\hat{V}$, then $\mathrm{ch}(\hat{E}_+) - \mathrm{ch}(\hat{E}_-)$ (viewed as a differential form) vanishes outside of $\hat{V}$. The integral is then equal to

$$\int_{T^*B} \mathrm{Todd}(T_{\mathbb{C}}B)\,\mathrm{ch}(\sigma(\mathcal{D}_+))(\mathrm{ch}(E_+) - \mathrm{ch}(E_-)),$$

where $E_\pm$ denote trivial bundles on B equipped with connections which are isomorphic (via F) outside of a compact set. Standard calculations (compare Section 4 of [1]) reduce this last integral to

$$\int_B \mathrm{ch}(E_+) - \mathrm{ch}(E_-)$$

(when B is a domain in $\mathbb{C}^n$). From an explicit calculation of the Chern character (compare [7]) and Stoke's theorem we recover the second part of the theorem of the introduction:

Theorem. *If F is a smooth matrix valued function on $\overline{B}$ whose restriction to ∂B is invertible then the index of the Toeplitz operator T_F is given by*

$$\mathrm{Index}(T_F) = \frac{-(n-1)!}{(2n-1)!(2\pi i)^n} \int_{\partial B} \mathrm{trace}((F^{-1}dF)^{2n-1}).$$

5. Remarks

Relative K-Homology. Our calculations fit very well with the Baum-Douglas approach to relative K-homology [2]. The arguments in Section 3 show that the partial isometric part in the polar decomposition of $\mathcal{D}_+$ is a cycle for the relative K-homology group $K_0(B, \partial B)$. The Baum-Douglas theory then reduces the calculation of Toeplitz indices to the Atiyah-Singer theorem, much as do our arguments in Section 4.

[5] In our situation, where B is a domain in $\mathbb{C}^n$, the calculation can easily be reduced to the Bott Periodicity theorem: the full Atiyah-Singer theorem is not really needed.

Generalizations. Our methods adapt to more general Toeplitz index problems, in which for example the domain B is replaced by a strongly pseudoconvex domain in a general complex manifold. A Hermitian metric analogous to (1.3) can be constructed in a neighborhood W of ∂B. Since $B \setminus W$ is compact, a relative compactness argument (using the basic elliptic estimate and Rellich's lemma, once more) shows that 0 is an isolated point in the *essential* spectrum of the Dolbeaut operator $\mathcal{D}$. This is enough for the arguments in Sections 3 and 4 and we obtain an index formula for Toeplitz operators on the space of square integrable holomorphic sections of the canonical line bundle on B (the actual formula involves the Todd genus of B, and so is more complicated than the one in the introduction).

To obtain an index formula for holomorphic *functions* one generalizes the entire discussion by introducing, at the beginning, an auxiliary Hermitian holomorphic bundle V on a neighbourhood of $\overline{B}$. Since V is asymptotically flat in the metric (1.3), the estimate in Section 2 carries over (using say the constant $q/4$ in place of $q/2$) for forms with coefficients in V which are supported near ∂B. Once again, 0 is an isolated point in the essential spectrum of the Dolbeaut operator, this time twisted by V, and the remainder of our argument carries through to produce an index formula.

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