JOURNAL OF FUNCTIONAL ANALYSIS 73, 107–112 (1987)

On a Technical Theorem of Kasparov

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Communicated by A. Connes

Received July 27, 1985

A short proof of the technical theorem underlying G. G. Kasparov's *KK*-theory is presented. The theorem is concerned with the separation of two orthogonal subalgebras of an "outer multiplier algebra" $M(D)/D$ by an element in a relative commutant. The proof given here is a simple application of the notion of quasicentral approximate unit.

INTRODUCTION

The purpose of this note is to give what is, we hope, a short, simple proof of the main result underlying the construction of the product in Kasparov's *KK*-theory, namely [4, Sect. 3, Theorem 4]. It has been noted by Skandalis [6] that a simpler version of this result suffices, and since the proof of it is a little more straightforward we have considered it separately.

PRELIMINARIES

If D is a C^*-algebra then denote by $M(D)$ its multiplier algebra. Recall that the *strict topology* on $M(D)$ is the vector space topology generated by the seminorms $x \mapsto \|x d\| + \|dx\|$ $(d \in D)$. With respect to it, multiplication is continuous on bounded sets and also, the set of positive elements is closed. Furthermore, $M(D)$ is complete in this topology (see [2]). By an approximation argument this implies the following convergence criterion for *bounded, self-adjoint sequences* $\{x_n\}_{n=1}^{\infty}$ in $M(D)$: if d is a strictly positive element for D (i.e., $d \geqslant 0$ and $d \cdot D$ is dense in D), and if $\{x_n d\}_{n=1}^{\infty}$ converges in the norm topology then $\{x_n\}_{n=1}^{\infty}$ converges in the strict topology.

Suppose that A is a C^*-subalgebra of a C^*-algebra B and that $\mathscr{B}$ is a compact subset of B such that $\mathscr{B} \cdot A$ and $A \cdot \mathscr{B}$ are subsets of A. Suppose further that A has a strictly positive element; then there exists a sequential,

0022-1236/87 $3.00

positive, increasing approximate unit $\{u_n\}_{n=1}^{\infty}$ for A such that $\lim_{n\to\infty} \|u_n b - b u_n\| = 0$ for all b in $\mathscr{B}$ (see [1] or [5]; such an approximate unit is said to be *quasicentral*).

We will be considering $Z/2$-graded C^*-algebras; for the definition of these see [3]. In the above paragraph, if B is a $Z/2$-graded C^*-algebra and A is a graded subalgebra then there exist quasicentral approximate units whose terms u_n are of degree zero. Indeed, it is easy to see that an approximate unit for the degree zero subalgebra $A^{(0)}$ of A is an approximate unit for A whilst by [1] there exists a quasicentral approximate unit within the convex hull of any approximate unit. Also, A has a strictly positive element of degree zero if it has one at all.

Any $Z/2$-grading on D extends uniquely to one on $M(D)$; the sets of degree zero and degree one elements are closed in the strict topology.

The brackets [,] will denote the graded commutator of elements, subspaces, etc. Thus: $[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx$ if x and y are homogeneous. Of course this is just the ordinary commutator if one of x or y is degree zero.

LEMMA [1, Lemma to Theorem 2]. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if x and y are elements of a C^*-algebra with $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, $x \geqslant 0$ (and* $\deg(x) = 0$ *if the algebra is graded) and if $\|[x, y]\| < \delta$ then $\|[x^{1/2}, y]\| < \varepsilon$.*

THE TECHNICAL THEOREMS

In the following proofs, all approximate units are, without further mention, assumed to be positive, increasing, and consisting of degree zero elements. We begin with the formulation of Kasparov's theorem used by Skandalis.

THEOREM. *Let D be a $Z/2$-graded C^*-algebra with a strictly positive element; let E_1 and E_2 be graded subalgebras of $M(D)$ with strictly positive elements; and let $\mathscr{F}$ be a separable, graded, linear subspace of $M(D)$. If $E_1 \cdot E_2 \subset D$ and $[\mathscr{F}, E_1] \subset E_1$ then there exists a degree zero element $N \in M(D)$ such that $1 \geqslant N \geqslant 0$, $(1-N) \cdot E_1 \subset D$, $N \cdot E_2 \subset D$, and $[N, \mathscr{F}] \subset D$.*

Proof. Since $[\mathscr{F}, E_1] \subset D$ implies $[\mathscr{F}^*, E_1] \subset D$, by replacing $\mathscr{F}$ with $\mathscr{F} + \mathscr{F}^*$ we may assume that $\mathscr{F}$ is self-adjoint. Denote by $\mathrm{Alg}(\mathscr{F})$ the algebra generated by $\mathscr{F}$; then since $\mathscr{F}$ (graded) commutes with E_1, modulo E_1, it follows that $\overline{\mathrm{Alg}(\mathscr{F})\, E_1 + E_1}$ is a C^*-algebra, and we may replace E_1 with this larger algebra and still retain the properties of E_1 (note that a strictly positive element for the old E_1 is a strictly positive element for the new one). Also, not only do we have $[\mathscr{F}, E_1] \subset E_1$, but now $\mathscr{F} \cdot E_1 \subset E_1$. Having made these simplifications we may proceed: let d, e_1, and e_2 be strictly positive elements of degree zero and norm less than or equal to 1 for D, E_1 and E_2, respectively, and let F be a compact subset of the unit ball of $\mathscr{F}$ whose linear span is dense in $\mathscr{F}$. It suffices to find a degree zero element N such that $1 \geqslant N \geqslant 0$, $Ne_1 - e_1 \in D$, $Ne_2 \in D$, and $[N, F] \subset D$. Let $\{u_n\}_{n=1}^{\infty}$ be an approximate unit for E_1 such that for all n:

(i) $\| u_n e_1 - e_1 \| < 2^{-n}$; and

(ii) $\| [u_n, x] \| < 2^{-n}$ if $x \in F$.

Let $\{w_n\}_{n=1}^{\infty}$ be an approximate unit for D such that for all n:

(iii) $\| w_n x - x \| < 2^{-2n}$ if $x = d$, $u_n e_2$, or $u_{n+1} e_2$ (by hypothesis these products are all in D); and

(iv) if $x = e_1$ or e_2, or $x \in F$ then $\| [w_n, x] \|$ is so small that (making use of the lemma), $\| [d_n, x] \| < 2^{-n}$, where $d_n = (w_n - w_{n-1})^{1/2}$ (and $w_0 = 0$).

The partial sums of the infinite series $\sum_{n=1}^{\infty} d_n u_n d_n$ are positive, and since $\sum_{n=1}^{N} d_n u_n d_n \leqslant \sum_{n=1}^{N} d_n^2 = w_N$, they are bounded in norm by 1. Condition (iii) implies that $\| d_n u_n d_n d \| < 5 \times 2^{-n}$, so the term-wise product of $\sum_{n=1}^{\infty} d_n u_n d_n$ with d converges in norm. Therefore $\sum_{n=1}^{\infty} d_n u_n d_n$ converges in the strict topology to some degree zero element $N \in M(D)$, and $1 \geqslant N \geqslant 0$, since a strict limit of positive elements is positive. Since multiplication is continuous on bounded sets in the strict topology, $(1-N) \cdot e_1 = \sum_{n=1}^{\infty} (d_n^2 - d_n u_n d_n) e_1$, $Ne_2 = \sum_{n=1}^{\infty} d_n u_n d_n e_2$, and $[N, x] = \sum_{n=1}^{\infty} [d_n u_n d_n, x]$, if $x \in F$. But

$$(d_n^2 - d_n u_n d_n) e_1 = d_n [d_n, e_1] - d_n u_n [d_n, e_1] + d_n (e_1 - u_n e_1) d_n, \quad (1)$$

$$d_n u_n d_n e_2 = d_n u_n [d_n, e_2] + d_n u_n e_2 d_n, \quad (2)$$

$$[d_n u_n d_n, x] = d_n u_n [d_n, x] + [d_n, x] u_n d_n + d_n [u_n, x] d_n, \quad (3)$$

and by using (iv) and (i) on (1), (iv) and (iii) on (2), and (iv) and (ii) on (3), we see that each of the three series above converges absolutely, the nth term being bounded in norm by $2^{-n} \times$ (constant). Since all of the terms are in D, so are the limits. Q.E.D.

Next is the more general technical theorem. The proof is similar in most respects to the one above, so we will be quite brief.

THEOREM. *Let D be a $Z/2$-graded C^*-algebra; let E_1 and E_2 be graded C^*-subalgebras of $M(D)$, E_1 possessing a strictly positive element and E_2 separable; let E be a graded (closed, two-sided) ideal in E_1; and let $\mathscr{F}$ be a* *graded, separable linear subspace of* $M(D)$. *If* $E_1 \cdot E_2 \subset E$, $[\mathscr{F}, E_1] \subset E_1$ *and* $D \subset E_1 + E_2$ *then there exists a degree zero element* $N \in M(D)$ *such that* $1 \geqslant N \geqslant 0$, $(1-N) \cdot E_1 \subset E$, $N \cdot E_2 \subset E$, *and* $[N, \mathscr{F}] \subset E$.

Note that $(1-N) \cdot E_1 \subset E$ and $E \lhd E_1$ imply that $N \cdot E_1 \subset E_1$ (the conclusion of [3, Sect. 3, Theorem 4] not listed above). Also, $[\mathscr{F}, E_1] \subset E_1$ implies that $[\mathscr{F}, E] \subset E$ (the hypothesis of [3, Sect. 3, Theorem 4] not listed above). The theorem implies a version of itself where the hypothesis of separability of E_2 is replaced by the existence of a strictly positive element: given such an E_2, replace it with the C^*-algebra generated by a degree zero strictly positive element.

Proof. We may assume that $\mathscr{F}$ is self-adjoint, and by replacing E_2 with the smallest C^*-algebra E_2' containing it for which $[\mathscr{F}, E_2'] \subset E_2'$ we may assume that $[\mathscr{F}, E_2] \subset E_2$ (it is not hard to check that $E_1 \cdot E_2' \subset E$). Any approximate unit for E_i is an approximate unit for the C^*-algebra $\overline{\mathrm{Alg}(\mathscr{F})\, E_i + E_i}$ $(i = 1, 2)$. Therefore since, as remarked in the preliminaries, there exists a quasicentral approximate unit within the convex hull of any approximate unit, there exist approximate units for the E_i which are quasicentral with respect to elements of $\mathscr{F}$. So let e_1, e_2, and $F \subset \mathscr{F}$ be as in the proof of the previous theorem and let $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be approximate units for E_1 and E_2, respectively, such that for all n:

(i) $\|u_n e_1 - e_1\| < 2^{-n}$;

(ii) $\|[d_n, e_2]\| < 2^{-n}$ and $\|[d_n, x]\| < 2^{-n}$ $(x \in F)$, where $d_n = (u_n - u_{n-1})^{-1/2}$, $u_0 = 0$;

(iii) $\|w_n e_2 - e_2\| < 2^{-n}$; and

(iv) $\|[w_n, x]\| < 2^{-n}$ $(x \in F)$.

It follows that the termwise products of the series $\sum_{n=1}^{\infty} d_n(1-w_n)\, d_n$ with e_1 and e_2 converge in norm to elements which are, respectively, equal to e_1 and 0, modulo E. Therefore, if $y \in E_1$ or $y \in E_2$ then the termwise product of $\sum_{n=1}^{\infty} d_n(1-w_n)\, d_n$ with y converges in norm to y or 0, modulo E, respectively. Thus, the termwise product converges for every element of $E_1 + E_2$, and since $D \subset E_1 + E_2$, the series converges in the strict topology to some degree zero element $N \in M(D)$ for which $1 \geqslant N \geqslant 0$, $(1-N)\, E_1 \subset E$, and $NE_2 \subset E$.

If $x \in F$ then

$$[N, x] = \sum_{n=1}^{\infty} [d_n^2, x] - \sum_{n=1}^{\infty} [d_n w_n\, d_n, x],$$

where both series are absolutely convergent by conditions (ii) and (iv) above. The terms of the second series are in E and therefore so is the sum.

The terms of the first series are in E_1, therefore so is the sum. But if $y \in E_1$ is of degree zero then

$$y \sum_{n=1}^{\infty} [d_n^2, x] = \sum_{n=1}^{\infty} ((y\, d_n^2)x - [y, x]\, d_n^2 - x(y\, d_n^2))$$

and since $[y, x] \in E_1$, it follows that the three sums on the right converge to yx, $-[y, x]$ and xy. Thus $y \sum_{n=1}^{\infty} [d_n^2, x] = 0$ for every degree zero $y \in E_1$, from which it follows that $\sum_{n=1}^{\infty} [d_n^2, x] = 0$, and so $[N, x] \in E$.

Q.E.D.

Remarks

We are indebted to Georges Skandalis for the following observation: if E is an ideal in D (and this is the case in most applications), then the second of the theorems above follows from the first. To see this, we note first that all C^*-algebras in the statement of the theorem may be assumed to be separable. Indeed, let $A = C^*(\mathscr{F}, e_1, e_2)$; let E_i' be the ideal in A generated by e_i; let $E' = E_1' \cap E_2'$; and let $D' = E_1' + E_2'$. Then $M(D')$ injects into $M(D)$, and E', E_i', and $\mathscr{F}$ are all contained in $M(D') \subset M(D)$. If $N \in M(D')$ is as in the conclusion of the second theorem for E', E_i', and $\mathscr{F}$, then N works for E, E, and $\mathscr{F}$ as well. Assume then these reductions are made. Now, if E is an essential ideal in D then $M(D)$ is the subalgebra of $M(E)$ consisting of those elements x for which $xD + Dx \subset D$. If $N \in M(E)$ satisfies $NE_2 \subset E$ and $(1-N)\, E_1 \subset E$, then in fact $N \in M(D)$ (and so the existence of N as in the second theorem follows from the first theorem). Indeed, $NE_2 \subset D$ and $(1-N)\, E_1 \subset D$, and so $ND \subset D$, since $D = E_1 + E_2$. If E is not an essential ideal then we must add to it the annihilator ideal $I = \{x \in D \mid xE = 0 = Ex\}$. Since $D = E_1 + E_2$ and $E_1 E_2 \subset E$, the ideal I splits as $(E_1 \cap I) \oplus (E \cap I)$. Choosing $N \in M(E)$ as in the first theorem, let $N = N \oplus 1 \oplus 0$ in the multiplier algebra of $E \oplus (E_1 \cap I) \oplus (E_2 \cap I)$. This N lies in $M(D)$ and satisfies the conclusions of the second theorem.

The proofs of the above two theorems can easily be made to accommodate actions of compact groups and "real" structures (as in [3]) by using invariant approximate units.

Although there do not necessarily exist approximate units which are invariant under the action on a noncompact group, there do exist approximate units $\{u_\lambda\}_{\lambda \in \Lambda}$ which are "quasi-invariant" in the sense that $\lim_{\lambda \to \infty} \| \alpha_g(u_\lambda) - u_\lambda \| = 0$ (where α is the action of the group). The proof is a simple modification of the proof of the existence of quasicentral approximate units, as in say [1]. This, together with a modification of the lemma in the preliminaries, where $\alpha_g(x) - x$ replaces $xy - yx$, allows a proof of the "equivariant" technical theorem of [4] along the lines of the proof above.

References

1. W. B. Arveson, Notes on extensions of C^*-algebras, *Duke Math. J.* **44** (1977), 329–355.
2. R. C. Busby, Double centralizers and extensions of C^*-algebras, *Trans. Amer. Math. Soc.* **132** (1968), 79–99.
3. G. G. Kasparov, The operator K-functor and extensions of C^*-algebras, *Math. USSR-Izv.* **16** (1981), 513–572. [English Translation]
4. G. G. Kasparov, K-theory, group C^*-algebras and higher signatures (conspectus), preprint.
5. G. K. Pedersen, C^*-algebras and their automorphism groups, *in* "LMS Monographs," No. 14, Academic Press, London/New York, 1979.
6. G. Skandalis, Some remarks on Kasparov theory, *J. Funct. Anal.* **56** (1984), 337–347.