BIVARIANT K-THEORY AND THE NOVIKOV CONJECTURE

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Abstract. Kasparov's bivariant K-theory is used to prove two theorems concerning the Novikov higher signature conjecture. The first generalizes a result of J. Roe and the author on amenable group actions. The second is a C^*-algebraic counterpart of a theorem of G. Carlsson and E. Pedersen.

1. Introduction

In a preliminary section of a recent paper [23], Guoliang Yu introduced a property of discrete metric spaces which guarantees the existence of a uniform embedding into Hilbert space. John Roe and the author observed in [14] that, applied to the metric space underlying a finitely generated discrete group G, Yu's property is equivalent to the topological amenability of the translation action of G on its Stone-Čech compactification. Using the main theorem of Yu's paper we were then able to conclude that if G is any group for which BG is a finite complex, and if G acts amenably on some compact Hausdorff space, then the Novikov higher signature conjecture is true for G. The main purpose of this note is to provide a second proof of this result, and in fact to strengthen the theorem, mainly by removing the hypothesis that BG be finite:

1.1. Theorem. *Let G be a countable discrete group. If there exists a topologically amenable action of G on some compact Hausdorff space then, for every separable G-C^*-algebra A, the Baum-Connes assembly map*

$$\mu : KK^G_*(\mathcal{E}G, A) \to KK(\mathbb{C}, C^*_r(G, A))$$

is split injective.

The construction of the Baum-Connes assembly map is reviewed in Section 2 below. It is known that injectivity of the Baum-Connes assembly map implies the Novikov higher signature conjecture. See [3].

Theorem 1.1 applies to amenable groups, since the trivial action of an amenable group on a point is topologically amenable. The Novikov conjecture for (countable) amenable groups was proved by Gennadi Kasparov and the author in [12], using the infinite-dimensional Bott periodicity argument of [11], and the fact that every amenable group admits a proper, affine-isometric action on a Hilbert space. Here we shall prove Theorem 1.1 by appealing to a result of Jean-Louis Tu [21], who

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has extended the theorem proved in [12] from groups to *groupoids.* Actually, our approach to Theorem 1.1 is not unrelated to the one followed by Roe and the author in [14], which relied on Yu's paper [23]. Indeed Yu's paper, like Tu's, depends upon infinite-dimensional Bott periodicity, and the property of Yu which guarantees a uniform embedding into Hilbert space is very suggestive of the Følner set definition of amenability.

Theorem 1.1 bears at least a superficial resemblance to an elegant theorem of Gunnar Carlsson and Erik Pedersen [4], in which injectivity of the assembly map in L-theory and algebraic K-theory is deduced from the existence of a suitable action of G on a compact space, although now the hypotheses on the action are geometric, not harmonic analytic, in nature. The second purpose of this note is to sketch a proof of the Carlsson-Pedersen theorem, as adapted to the Baum-Connes assembly map:

1.2. Theorem. *Let G be a countable group and suppose that the classifying space for proper actions $\mathcal{E}G$ is G-compact. Suppose that $\mathcal{E}G$ admits a metrizable compactification $\overline{\mathcal{E}G}$ which is a G-space and which is H-equivariantly contractible, for every finite subgroup H of G. If $\overline{\mathcal{E}G}$ has the property that $\lim_{g\to\infty} \operatorname{diam}_{\overline{\mathcal{E}G}}(gK) = 0$, for every compact subset K of $\mathcal{E}G$, then the Baum-Connes assembly map*

$$\mu\colon KK_*^G(\mathcal{E}G, A) \to KK(\mathbb{C}, C_r^*(G, A))$$

is injective.

Thanks to the work of Roe [17] and others, the relationship between the Carlsson-Pedersen theory and C^*-algebras is already quite well understood. It particular it is known how to directly translate the original arguments of Carlsson and Pedersen into the language of C^*-algebras and operator K-theory, using a 'coarse geometric' formulation of the Baum-Connes assembly map [13]. Because of this we will give only a very concise account of our approach, in which Kasparov's bivariant equivariant K-theory serves as a substitute for the original constructions of Carlsson and Pedersen involving spectra. But we hope that the novelty of the argument justifies the duplication of effort as regards the final result.

In Section 2 we review, as briefly as possible, the definition of the Baum-Connes assembly map. Section 3 proves Theorem 1.1. Section 4 provides further necessary information on bivariant C^*-algebra K-theory, and in the final section we sketch our approach to the Carlsson-Pedersen theorem.

2. Bivariant K-theory and assembly

Consider the category of separable C^*-algebras and $*$-homomorphisms. Kasparov's KK-theory [15] is a functor from this category into an additive category whose objects are again the separable C^*-algebras (the functor is the identity on objects) and whose morphism groups are denoted $KK(A, B)$. Now, fix a countable discrete[1] group G and consider the category of separable G-C^*-algebras and equivariant $*$-homomorphisms. Kasparov's equivariant KK-theory is a functor from

[1]The general theory encompasses also second countable, locally compact groups, but the case of discrete groups is adequately general for the purposes of this paper.

this category into an additive category whose objects are the separable G-C^*-algebras (the functor is again the identity on objects) and whose morphism groups are denoted $KK^G(A,B)$.

Equivariant and non-equivariant KK-theory are related in several ways. If G is the trivial group then G-equivariant KK-theory and KK-theory agree with one another. If H is any subgroup of G then there is a forgetful functor from G-C^*-algebras to H-C^*-algebras, and a corresponding functor from KK^G-theory to KK^H-theory, giving *resriction maps*

$$KK^G(A,B) \to KK^H(A,B).$$

The category of C^*-algebras may be included into the category of G-C^*-algebras as the full subcategory of algebras equipped with trivial G-actions. There is a corresponding inclusion of KK-theory into KK^G-theory (however it is not full), giving maps

$$KK(A,B) \to KK^G(A,B),$$

when A and B are trivial G-C^*-algebras. Most importantly, and most interestingly, there is a functor from G-C^*-algebras to C^*-algebras which associates to each G-C^*-algebra A the (reduced) crossed product C^*-algebra $C^*_r(G,A)$, and there is a compatible functor from the G-equivariant KK-category to KK-theory, providing *descent homomorphisms*

$$KK^G(A,B) \to KK(C^*_r(G,A), C^*_r(G,B)).$$

Both KK-theory and equivariant KK-theory are homotopy invariant, and in addition they have other important homological properties which allow one to regard them as bivariant K-homology/K-cohomology theories for C^*-algebras. For our immediate purposes we shall need to record only some properties of the following equivariant K-homology theory for proper G-spaces. Let Z be a proper G-space (see [3]) for which the quotient Z/G is compact and metrizable. If A is a separable G-C^*-algebra then define

$$K^G(Z,A) = KK^G(C_0(Z), A),$$

which we shall call the *equivariant K-homology of Z with coefficients in A*. If W is any proper G-space, not necessarily G-compact, then define

$$K^G(W,A) = \varinjlim_{Z \subset W} K^G(Z,A),$$

where the direct limit is over subsets Z of W with Z/G compact and metrizable. Observe that even if W is second countable and locally compact, $K^G(W,A)$ is *not* the same as $KK^G(C_0(W),A)$. However there is always a *comparison homomorphism*

$$K^G(W,A) \to KK^G(C_0(W),A).$$

The groups $K^G(W,A)$ constitute the even groups $K^G_{2i}(W,A)$ in a 2-periodic generalized homology theory $K^G_*(W,A)$ on proper G-spaces. The odd groups in the homology theory are

$$K^G_{2i+1}(W,A) = K^G(W,SA),$$

where

$$SA = C_0(0,1) \otimes A = \left\{ f\colon [0,1] \to A \ : \ \begin{array}{c} f \text{ is continuous} \\ f(0) = 0 = f(1) \end{array} \right\}.$$

Associated to any decomposition $W = W_1 \cup W_2$ of W, as a union of closed G-subsets, there is a Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to K^G_j(W_1 \cap W_2, A) \to K^G_j(W_1,A) \oplus K^G_j(W_2,A) \\ \to K^G_j(W_1 \cup W_2, A) \to K^G_{j-1}(W_1 \cap W_2, A) \to \cdots . \end{aligned}$$

Any equivariant $*$-homomorphism $A_1 \to A_2$ (or indeed any morphism in the group $KK^G(A_1,A_2)$) gives a natural transformation from $K^G_*(W,A_1)$ to $K^G_*(W,A_2)$ and in particular a commuting diagram relating the Mayer-Vietoris sequences in the two theories.

In order to calculate equivariant K-homology, the following simple and well-known result is often useful.

2.1. Lemma. *Let H be a finite subgroup of a countable group G and let V be a compact metrizable space equipped with an action of H. Form the induced proper G-space $G \times_H V$ by dividing $G \times V$ by the diagonal action of H, where H acts on G by left-translations and observe that V is included in $G \times_H V$ as an open, H-invariant subset. If A is any separable G-C^*-algebra then the composition*

$$K^G(G\times_H V, A) \to K^H(G\times_H V, A) \to KK^H(C_0(G\times_H V), A) \to KK^H(C(V), A),$$

in which the first map is the forgetful functor from G-C^*-algebras to H-C^*-algebras, the second is the comparison homomorphism, and the last is induced from the inclusion of $C(V)$ as an ideal in $C_0(G \times_H V)$, is an isomorphism of abelian groups. □

This is proved in [11] (see Lemma 13.11) for E-theory, a variant of KK-theory that we will discuss in Section 4. The proof for KK-theory is essentially the same.

With these preliminaries about KK-theory and equivariant K-homology in hand we can now turn to the formulation of the Baum-Connes conjecture [3]. If Z is a proper G-space with Z/G compact and metrizable, and if φ is a non-negative, compactly supported function on Z such that $\sum_{g\in G} g(\varphi^2) = 1$, then the formula

$$p = \sum_{g\in G} \varphi g(\varphi)[g]$$

defines a projection in the crossed product $C^*_r(G,Z) = C^*_r(G, C_0(Z))$, and hence a $*$-homomorphism from $\mathbb{C}$ to $C^*_r(G,Z)$.

2.2. Definition. The *Baum-Connes assembly map* for the space Z and the auxilliary G-C^*-algebra A is the composition

$$K^G(Z,A) \xrightarrow{\text{descent}} KK(C^*_r(G,Z), C^*_r(G,A)) \xrightarrow{p} KK(\mathbb{C}, C^*_r(G,A)),$$

where the second map is composition with the $*$-homomorphism determined by the above projection p (at the level of KK-theory, the choice of function φ used to define p is immaterial).

If W is any proper G-space, not necessarily G-compact, then the assembly maps for each of its subsets Z, with Z/G compact and metrizable, combine to form an assembly map for W,

$$\varinjlim_{Z\subset W} K^G(Z,A) \xrightarrow{\text{descent}} \varinjlim_{Z\subset W} KK(C^*_r(G,Z), C^*_r(G,A)) \xrightarrow{p} KK(\mathbb{C}, C^*_r(G,A)).$$

2.3. Baum-Connes Conjecture. *Let G be a countable discrete group and denote by $\mathcal{E}G$ a universal proper G-space* [3]*. If A is any separable G-C^*-algebras then the assembly map*

$$K^G(\mathcal{E}G, A) \to KK(\mathbb{C}, C^*_r(G,A)),$$

is an isomorphism of abelian groups

To be accurate, the above is the Baum-Connes conjecture for discrete groups, 'with coefficients' in any separable G-C^*-algebra A.

Theorems 1.1 and 1.2 assert that under suitable hypotheses on G this Baum-Connes assembly map is *injective.* It is known that for any given group G, the injectivity of the Baum-Connes map implies the Novikov higher signature conjecture for the group G.

3. Amenable Actions

We begin by recalling the definition of a (topologically) amenable action on a compact space (see Definition 2.2 in [14] and Definition 2.2.7 in [2]).

3.1. Definition. An action of a countable discrete group G by homeomorphisms on a compact Hausdorff space X is *amenable* if there is a sequence of weak*-continuous maps $\mu^n\colon X \to \operatorname{prob}(G)$ such that for every $g \in G$,

$$\lim_{n\to\infty} \sup_{x\in X} \|g\mu^n_x - \mu^n_{gx}\|_1 = 0.$$

Here $\operatorname{prob}(G)$ denotes the space of Borel probability measures on G. Of course, since G is discrete these are nothing more than the non-negative functions on G which sum to 1. The space $\operatorname{prob}(G)$ is a subset of the dual of $C_0(G)$, and so has a natural weak*-topology. Since G is discrete, the dual is in fact $\ell^1(G)$, and the norm $\|\,\|_1$ is the usual ℓ^1-norm.

There are various interesting examples of amenable actions. If G is a discrete subgroup of a Lie group $\mathcal{G}$ then G acts amenably on the Furstenburg boundary of

$\mathcal{G}$ [7]. If G is a word-hyperbolic group then G acts amenably on its Gromov boundary [1]. If G is a finitely generated group with finite asymptotic dimension [8][22] then G acts amenably on its Stone-Cech compactification [14]. Finally, if G is amenable then of course it acts amenably on the one point space.

We aim to prove the following result, which is an extension of Theorem 1.2 in [14].

3.2. Theorem. *Suppose that the countable group G admits an amenable action on a compact Hausdorff space. If A is any separable G-C^*-algebra then the Baum-Connes assembly map*

$$\mu\colon K_*^G(\mathcal{E}G, A) \to KK_*(\mathbb{C}, C_r^*(G, A))$$

is split injective.

Kasparov and the author proved the following result in [12]:

3.3. Theorem. *If G is a countable amenable group, and if A is any separable G-C^*-algebra, then the Baum-Connes assembly map*

$$\mu\colon K^G(\mathcal{E}G, A) \to KK(\mathbb{C}, C_r^*(G, A))$$

is an isomorphism of abelian groups.

This settles one case of Theorem 3.2. Jean-Louis Tu has given an interesting extension of our result to groupoids [21]. A special case of it is as follows:

3.4. Theorem. *If G is a countable group and X is a compact, metrizable and amenable G-space, then for every separable G-C^*-algebra A the assembly map*

$$K^G(\mathcal{E}G, C(X) \otimes A) \longrightarrow KK(\mathbb{C}, C_r^*(G, C(X) \otimes A))$$

is an isomorphism.

This specialization of the main theorem in [21] can be proved following the original argument presented by Kasparov and the author in [12]. The hypothesis of amenability implies that there is a proper affine-isometric action of G on a bundle $X \times H$ of affine Hilbert spaces over X. The proof of Theorem 3.4 is then an X-parametrized version of the proof given in [12].[2]

To deduce Theorem 3.2 from Theorem 3.4 we shall, in two steps, replace the compact amenable space X with a more manageable compact and amenable G-space.

[2]The hypothesis that G acts amenably on some compact space implies that G is C^*-exact, as described in [10]; see also [16]. It follows that the technical difficulties which arose in [12] when analyzing the *reduced* C^*-algebra of G, as opposed to the *full* group C^*-algebra, do not arise in this case.

3.5. Lemma. *Suppose that G admits an amenable action on a compact Hausdorff space X. Then G admits an amenable action on some second countable, compact Hausdorff space Y.*

Proof. Suppose that X is an amenable G space and let $\mu^n \colon X \to \operatorname{prob}(G)$ be a sequence of continuous maps of the sort appearing in Definition 3.1. Since compact subsets of $\operatorname{prob}(G)$ are second countable in the weak*-topology, the weakest topology $\mathcal{T}$ on X for which the maps $x \mapsto \mu^n_{gx}$ are all continuous is second countable, and of course compact. Furthermore G acts on X by $\mathcal{T}$-homeomorphisms. If we define an equivalence relation on X by deeming that $x_1 \sim x_2$ whenever $f(x_1) = f(x_2)$, for every real-valued, $\mathcal{T}$-continuous function on X, and if Y is the associated quotient space, then Y is a compact, second countable, Hausdorff G-space, and the maps μ^n descend to Y, proving that Y is amenable. □

3.6. Lemma. *Suppose that G admits an amenable action on a second countable compact Hausdorff space Y. Then the induced action of G on the compact second countable space* $\operatorname{prob}(Y)$ *of Borel probability measures on Y is also amenable.*

Proof. Let $\mu^n \colon Y \to \operatorname{prob}(G)$ be a sequence of continuous maps of the sort appearing in Definition 3.1. Define $\mu^n \colon \operatorname{prob}(Y) \to \operatorname{prob}(G)$ by

$$\int_G f(g)\, d\mu^n_\nu(g) = \int_Y \left(\int_G f(g)\, d\mu^n_y(g) \right) d\nu(y),$$

for every $\nu \in \operatorname{prob}(Y)$. These maps are continuous (from the weak*-topology on $\operatorname{prob}(Y)$ to the weak*-topology on $\operatorname{prob}(G)$) and asymptotically G-equivariant, as in Definition 3.1. Hence $\operatorname{prob}(Y)$ is amenable, as claimed. □

3.7. Proposition. *If X is a compact metrizable G-space, and if the collapsing map from X to a point is an H-equivariant homotopy equivalence, for every finite subgroup H of G, then for every separable G-C^*-algebra A the homomorphism*

$$K^G(\mathcal{E}G, A) \to K^G(\mathcal{E}G, C(X) \otimes A)$$

induced from the collapsing map (or equivalently, from the inclusion of $\mathbb{C}$ into $C(X)$ as constant functions) is an isomorphism of abelian groups.

Proof. Since $K^G(\mathcal{E}G, A)$ is a direct limit of groups $K^G(Z, A)$, where Z is a proper and G-compact G-space, and since the same holds for $K^G(\mathcal{E}G, C(X) \otimes A)$, it suffices to prove the proposition with such a space Z in place of $\mathcal{E}G$. Suppose first that Z is induced from the action of a finite subgroup H on a compact space V:

$$Z = G \underset{H}{\times} V.$$

Then, thanks to Lemma 2.1, there are natural isomorphisms

$$K^G(Z, A) \cong KK^H(C(V), A)$$

and

$$K^G(Z, C(X) \otimes A) \cong KK^H(C(V), C(X) \otimes A)$$

and so the result is proved for spaces Z of this special type. But a general G-compact proper G-space Z is a finite union of closed subspaces of the type $G \times_H V$, so the result for general Z follows from the Mayer-Vietoris long exact sequence and the five lemma. □

Proof of Theorem 3.2. Let Y be a compact, second countable Hausdorff space on which G acts amenably and let $X = \operatorname{prob}(Y)$. If H is any finite subgroup of G then there is an H-fixed point in X (simply average any point of X over H). The linear contraction to a fixed point shows that X is H-equivariantly homotopy equivalent to a point, for every H. Hence in the diagram

$$\begin{array}{ccc} K^G(\mathcal{E}G, A) & \xrightarrow{\text{assembly}} & KK(\mathbb{C}, C^*_r(G, A)) \\ {\scriptstyle\text{collapse}}\downarrow & & \downarrow{\scriptstyle\text{collapse}} \\ K^G(\mathcal{E}G, C(X) \otimes A) & \xrightarrow[\text{assembly}]{} & KK(\mathbb{C}, C^*_r(G, C(X) \otimes A)) \end{array},$$

the left-hand vertical map is an isomorphism. But according to Theorem 3.4, the bottom horizontal map is an isomorphism. It follows then that the top horizontal map is injective—in fact split injective. □

4. Further Properties of Bivariant K-Theory

There is a variant of Kasparov's KK-theory, named E-theory [5][6][10], which has some technical advantages over Kasparov's theory in certain situations (as well as some drawbacks in others). It will be convenient to use E-theory in the following section when we discuss the Carlsson-Pedersen theorem.

As with Kasparov's theory, G-equivariant E-theory is a functor from separable G-C^*-algebras to an additive category whose objects remain the separable G-C^*-algebras. The morphism groups are denoted $E^G(A, B)$ and if G is the trivial group then $E^G(A, B)$ is simply denoted $E(A, B)$. All the properties of Kasparov's theory mentioned in Section 2 carry over to E-theory, with one small exception: there is a descent functor

$$E^G(A, B) \to E(C^*(G, A), C^*(G, B))$$

involving the *full* crossed product C^*-algebra $C^*(G, A)$, but it is not known if there is, in general, a similar descent functor for the *reduced* crossed product $C^*_r(G, A)$. See [10]. For Kasparov's theory there are descent functors for both the reduced and the full crossed products; this is one respect in which KK-theory has a technical advantage over E-theory.[3]

[3] As we have already noted, this potential shortcoming of E-theory is not present for the groups G considered in the previous section, since the hypothesis that G act amenably on a compact space implies that a reduced descent functor exists for G.

Following the prescription in Section 2, we may use E-theory to define an equivariant homology theory $E^G(W, A)$ for proper G-spaces W. Specializing to $W = \mathcal{E}G$, there is an assembly map

$$E^G(\mathcal{E}G, A) \to E(\mathbb{C}, C^*(G, A)),$$

which is defined exactly as is its KK-theoretic counterpart. By composing with the quotient map $C^*(G, A) \to C^*_r(G, A)$ we obtain an E-theoretic Baum-Connes assembly map,

$$E^G(\mathcal{E}G, A) \to E(\mathbb{C}, C^*_r(G, A)),$$

for the reduced crossed product.

Equivariant E-theory and equivariant KK-theory are very close to one another. The functor from separable G-C^*-algebras to the equivariant E-theory category factors through the equivariant KK-theory category, and in many situations the associated maps

$$KK^G(A, B) \to E^G(A, B),$$

which are compatible with descent, are isomorphisms. This is so if G is finite and if A is a *nuclear* (for example, commutative) G-C^*-algebra. In particular, in the commutative diagram

$$\begin{array}{ccc} K^G(\mathcal{E}G, A) & \longrightarrow & KK(\mathbb{C}, C^*_r(G, A)) \\ \downarrow & & \downarrow \\ E^G(\mathcal{E}G, A) & \longrightarrow & E(\mathbb{C}, C^*_r(G, A)) \end{array}$$

relating the assembly maps in E-theory and KK-theory, the two vertical maps are isomorphisms. For the one on the right, this is immediate from what we have just said. For the one on the left, the assertion follows from Lemma 2.1 and its analogue in E-theory. Thus the E-theoretic and KK-theoretic formulations of the Baum-Connes conjecture are equivalent to one another.

One of the key properties of both E-theory and KK-theory is *stability,* which may be formulated as follows (we shall concentrate on E-theory but the result in KK-theory is the same). Denote by $\mathcal{K}(H)$ the C^*-algebra of compact operators on a separable Hilbert space H.

4.1. Lemma. *Let H be a separable G-Hilbert space. If $f\colon A_1 \to A_2$ is an equivariant $*$-homomorphism between separable G-C^*-algebras with the property that the tensor product $f \otimes 1\colon A_1 \otimes \mathcal{K}(H) \to A_2 \otimes \mathcal{K}(H)$ is equivariantly homotopic to a $*$-isomorphism, then f determines an isomorphism in equivariant E-theory.* □

See Proposition 7.10 in [10].

In particular, by considering the inclusions of $\mathbb{C}$ and H into the direct sum Hilbert space $\mathbb{C} \oplus H$, we see that there are canonical isomorphisms

$$E^G(A \otimes \mathcal{K}(H), B) \cong E^G(A, B) \cong E^G(A, B \otimes \mathcal{K}(H)).$$

We shall use the following additional consequence in the next section. If $g_1, g_2 \in G$ then let us introduce the notation $e_{g_1 g_2} \in \mathcal{K}(\ell^2(G))$ for the rank one operator (a 'matrix element') which maps δ_{g_2} to δ_{g_1}, and which maps all other generators $\delta_g \in \ell^2(G)$ to zero.

4.2. Corollary. *Let Z be a proper G-space with Z/G compact and metrizable, and let φ be a non-negative, compactly supported function on Z such that $\sum_{g\in G} g(\varphi^2) = 1$. The formula*

$$f \mapsto \sum_{g_1,g_2\in G} g_1(\varphi)g_2(\varphi)f \otimes e_{g_1 g_2}$$

determines an equivariant $*$-homomorphism $q\colon C_0(Z) \to C_0(Z) \otimes \mathcal{K}(\ell^2(G))$ which induces the canonical isomorphism

$$E^G(C_0(Z) \otimes \mathcal{K}(\ell^2(G)), A) \cong E^G(C_0(Z), A). \qquad \square$$

See Section 10 of [10].

Turning to other properties of bivariant K-theory, the Bott periodicity theorem may be formulated as an isomorphism

$$\mathbb{C} \cong S^2 = C_0(0,1) \otimes C_0(0,1)$$

in the equivariant E-theory or KK-theory categories. To the (maximal) C^*-algebra tensor product there corresponds a tensor product functor

$$E^G(A_1, B_1) \otimes E^G(A_2, B_2) \to E^G(A_1 \otimes A_2, B_1 \otimes B_2)$$

(there is a similar tensor product in KK-theory[4]), and combining this with Bott periodicity we obtain isomorphisms

$$E^G(SA, B) \cong E^G(A, SB),$$

where the suspensions SA and SB are as in Section 2. There are, of course, similar isomorphisms in KK-theory.

An important property of E-theory which is *not* shared by KK-theory is excision. Given a short exact sequence of separable G-C^*-algebras

$$0 \to J \to B \to B/J \to 0$$

there is a certain corresponding morphism $S\,B/J \to J$ in the equivariant E-theory category, or in other words, a class in the group $E^G(S\,B/J, J)$, with the property that associated to a commuting diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & B/J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J' & \longrightarrow & B' & \longrightarrow & B'/J' & \longrightarrow & 0 \end{array}$$

[4]The minimal tensor product also defines a tensor product in KK-theory, but this is *false* for E-theory.

there is a commuting diagram

$$\begin{array}{ccc} SB/J & \longrightarrow & J \\ \big\downarrow & & \big\downarrow \\ SB'/J' & \longrightarrow & J' \end{array}$$

in the equivariant E-theory category. See Section 6 of [10]. This is functorial with respect to tensor products. In addition, associated to a short exact sequence there is a long exact sequence of abelian groups

$$\ldots \to E^G(B,A) \to E^G(J,A) \to E^G(SB/J,A) \to E^G(SB,A) \to \ldots$$

in which the 'boundary map' $E^G(J,A) \to E^G(SB/J,A)$ is composition with the class in $E^G(SB/J,J)$ of the extension. These long exact sequences are known not to exist, in general, in KK-theory [18].

We need one final property, an interesting universality condition which in effect says that E-theory is not too far from the category of C^*-algebras and $*$-homomorphisms (there is a similar assertion for KK-theory). For various results along these lines, including the theorem below, see [9][6][10][19] and [20].

4.3. Theorem. *Let A_1 and A_2 be separable G-C^*-algebras and let*

$$\mathbf{A}\colon E^G(A_2,\,\cdot\,) \to E^G(A_1,\,\cdot\,)$$

be a natural transformation of abelian group-valued functors on the category of separable G-C^*-algebras. There is then a morphism $\alpha \in E^G(A_1,A_2)$ such that $\mathbf{A}$ is composition with α. □

In other words, two natural transformations $E^G(A_2,\,\cdot\,) \to E^G(A_1,\,\cdot\,)$ agree if they agree on the identity morphism in $E^G(A_2,A_2)$.

5. The Carlsson-Pedersen Theorem

Fix a countable group G. Throughout this section we shall assume that a fixed universal proper G-space $\mathcal{E}G$ has been chosen, for which the quotient space $\mathcal{E}G/G$ is *compact and metrizable*. Thus $\mathcal{E}G$ is a second countable, G-compact, proper G-space and the domain of the Baum-Connes assembly map is the group

$$E^G(\mathcal{E}G,A) = E^G(C_0(\mathcal{E}G),A).$$

5.1. Definition. A compactification[5] $\overline{\mathcal{E}G}$ of $\mathcal{E}G$ is *admissible* if

(1) $\overline{\mathcal{E}G}$ is a metrizable compact space;
(2) the action of G on $\mathcal{E}G$ extends to a continuous action on $\overline{\mathcal{E}G}$; and
(3) if K is any compact subset of $\mathcal{E}G$ then $\lim_{g\to\infty}\operatorname{diam}(gK) = 0$, where diameter is measured using any metric on $\overline{\mathcal{E}G}$ which generates the topology of $\overline{\mathcal{E}G}$.

[5]A *compactification* of a locally compact space Y is a compact space $\overline{Y}$ containing Y as an open dense subset.

Compare [4]. The standard example to bear in mind is that of a cocompact subgroup of $SL(2,\mathbb{R})$, for which $\mathcal{E}G$ is the Poincaré disk and $\overline{\mathcal{E}G}$ is the closure of the disk in the plane.

Here are two equivalent formulations of item (3) which do not explicitly mention the metric on $\overline{\mathcal{E}G}$:

(3′) For every compact subset $K \subset \mathcal{E}G$, for every $y \in \partial\mathcal{E}G$, and for every neighbourhood U of y in $\overline{\mathcal{E}G}$, there is a smaller neighbourhood V of y in $\overline{\mathcal{E}G}$ such that if gK intersects V then gK is wholly contained in U.

(3″) If f is a continuous, complex-valued function on $\overline{\mathcal{E}G}$, and if K is a compact subset of $\mathcal{E}G$ then $\lim_{g\to\infty} \sup_{x,y\in K} |f(gx) - f(gy)| = 0$.

Let us restate here, but in the language of E-theory, Theorem 1.2 of the introduction:

5.2. Theorem. *Suppose that $\mathcal{E}G$ is a second countable, G-compact, proper G-space and that $\mathcal{E}G$ has an admissible compactification $\overline{\mathcal{E}G}$ which is H-equivariantly contractible, for every finite subgroup H of G. Then, for every separable G-C^*-algebra A, the assembly map*

$$\alpha\colon E^G(\mathcal{E}G, A) \to E(\mathbb{C}, C^*_r(G, A))$$

is injective.

Here is an overview of the proof, in which we denote by $\partial G = \overline{\mathcal{E}G} \setminus \mathcal{E}G$ the boundary of $\mathcal{E}G$, and in which we shall for simplicity set $A = \mathbb{C}$. The admissibility of the compactification $\overline{\mathcal{E}G}$ allows us to define a G-equivariant extension

$$0 \to \mathcal{K}(\ell^2(G)) \to D \to C(\partial G) \otimes C^*_r(G) \to 0$$

(where $C^*_r(G)$ is given the trivial G-action) as follows. Fix any point $x \in \mathcal{E}G$ and define a representation

$$\pi\colon C(\overline{\mathcal{E}G}) \to \mathcal{B}(\ell^2(G))$$

by associating to $f \in C(\overline{\mathcal{E}G})$ the operator of pointwise multiplication by the function $g \mapsto f(gx)$. This representation is covariant for the left regular representation of G on $\ell^2(G)$.

5.3. Lemma. *If ρ denotes the* right *regular representation of G on $\ell^2(G)$ then for every $f \in C(\overline{\mathcal{E}G})$ and every $a \in C^*_r(G)$ the additive commutator $\pi(f)\rho(a) - \rho(a)\pi(f)$ is a compact operator.*

Proof. With this notation, considering $a = \delta_h \in C^*_r(G)$ we have that

$$\pi(f)\rho(\delta_h) - \rho(\delta_h)\pi(f) = \sum_{g\in G} (f(gh^{-1}x) - f(gx))e_{gh^{-1}g},$$

and so property (3″) in the definition of admissibility of the compactification $\overline{\mathcal{E}G}$ shows that the commutator is compact. □

It follows from the lemma that the two representations π and ρ together define an equivariant $*$-homomorphism

$$\pi \otimes \rho \colon C(\overline{\mathcal{E}G}) \otimes C_r^*(G) \to \mathcal{B}(\ell^2(G))/\mathcal{K}(\ell^2(G))$$

(where $C_r^*(G)$ is given the trivial G-action). Since π actually maps the ideal $C_0(\mathcal{E}G)$ to the compact operators, the tensor product $\pi \otimes \rho$ may be viewed as a representation of $C(\partial G) \otimes C_r^*(G)$ in $\mathcal{B}(\ell^2(G))/\mathcal{K}(\ell^2(G))$. We can now define the required extension as a pull-back, as in the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}(\ell^2(G)) & \longrightarrow & D & \longrightarrow & C(\partial G) \otimes C_r^*(G) & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\pi\otimes\rho} & & \\
0 & \longrightarrow & \mathcal{K}(\ell^2(G)) & \longrightarrow & \mathcal{B}(\ell^2(G)) & \longrightarrow & \mathcal{B}(\ell^2(G))/\mathcal{K}(\ell^2(G)) & \longrightarrow & 0
\end{array}$$

As we explained in Section 4, the extension determines an element

$$\varphi \in E^G(SC(\partial G) \otimes C_r^*(G), \mathcal{K}(\ell^2(G))),$$

and using φ we define the following *partial splitting* of the assembly map:

$$\begin{array}{ccc}
E(\mathbb{C}, C_r^*(G)) & \xrightarrow{1_{SC(\partial G)}\otimes -} & E^G(SC(\partial G), SC(\partial G) \otimes C_r^*(G)) \\
\text{partial splitting}\Big\downarrow & & \Big\downarrow\varphi \\
E^G(SC(\partial G), \mathbb{C}) & \xrightarrow[\cong]{} & E^G(SC(\partial G), \mathcal{K}(\ell^2(G))),
\end{array}$$

where the top map is tensor product with the identity morphism for $SC(\partial G)$ and the right vertical map is composition with φ (the reason for the name will become clear in a moment).

Now, associated to the short exact sequence of G-C^*-algebras

$$0 \to C_0(\mathcal{E}G) \to C(\overline{\mathcal{E}G}) \to C(\partial G) \to 0$$

is a long exact sequence in E-theory

$$\cdots \to E^G(C(\overline{\mathcal{E}G}), \mathbb{C}) \to E^G(C_0(\mathcal{E}G), \mathbb{C}) \to E^G(SC(\partial G), \mathbb{C}) \to \cdots .$$

In particular there is a *boundary map*

$$E^G(C_0(\mathcal{E}G), \mathbb{C}) \xrightarrow{\text{boundary}} E^G(SC(\partial G), \mathbb{C}).$$

Here is the crucial calculation in the proof of Theorem 5.2:

5.4. Lemma. *The diagram*

$$\begin{array}{ccc} E^G(C_0(\mathcal{E}G),\mathbb{C}) & \xrightarrow{\text{assembly}} & E(\mathbb{C},C_r^*(G)) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \text{partial splitting}} \\ E^G(C_0(\mathcal{E}G),\mathbb{C}) & \xrightarrow[\text{boundary map}]{} & E^G(SC(\partial G),\mathbb{C}) \end{array}$$

commutes.

Proof. This is a simple direct calculation in E-theory, using the explicit form of the morphism associated to an extension (see [10]). But since we have not given any concrete description of the morphisms in E-theory, let us follow an alternate and more functorial approach. We begin by noting that if A is any G-C^*-algebra then, generalizing the construction just given, there is an extension

$$0 \to \mathcal{K}(\ell^2(G)) \otimes A \to D \to C(\partial G) \otimes C_r^*(G, A) \to 0.$$

It is defined from representations of $C(\overline{\mathcal{E}G})$ and $C_r^*(G,A)$ in the multiplier algebra $\mathcal{M}(\mathcal{K}(\ell^2(G))\otimes A)$, the first given by pointwise multiplication of $g \mapsto f(gx)$ on $\ell^2(G)$, as before, and the second by the right regular representation of G on $\ell^2(G)$ and the representation $a \mapsto \sum_G e_{gg} \otimes g(a)$ of A in $\mathcal{M}(\mathcal{K}(\ell^2(G)) \otimes A)$. As in the proof of Lemma 5.3, the two representations commute, modulo the ideal $\mathcal{K}(\ell^2(G)) \otimes A$ in $\mathcal{M}(\mathcal{K}(\ell^2(G))\otimes A)$, and so the required extension is defined as a pull-back, as before. From the extension we obtain an equivariant E-theory element

$$\varphi_A \in E^G(SC(\partial G) \otimes C_r^*(G,A), A \otimes \mathcal{K}(\ell^2(G)))$$

(the crossed product algebra is given the trivial G-action) and hence a partial splitting map from $E(\mathbb{C}, C_r^*(G,A))$ to $E^G(SC(\partial G), A)$. We can now consider the commutativity of the general diagram

$$\begin{array}{ccc} E^G(C_0(\mathcal{E}G),A) & \xrightarrow{\text{assembly}} & E(\mathbb{C},C_r^*(G,A)) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \text{partial splitting}} \\ E^G(C_0(\mathcal{E}G),A) & \xrightarrow[\text{boundary map}]{} & E^G(SC(\partial G),A). \end{array}$$

But since all the maps are functorial in A (for equivariant $*$-homomorphisms) it follows from Theorem 4.3 that to prove commutativity it suffices to set $A = C_0(\mathcal{E}G)$ and show that $1 \in E^G(C_0(\mathcal{E}G), C_0(\mathcal{E}G))$ is mapped to the same element of $E^G(SC(\partial G), C_0(\mathcal{E}G))$ by the two composite maps presented in the diagram.

The boundary map sends $1 \in E^G(C_0(\mathcal{E}G), C_0(\mathcal{E}G))$ to the class of the extension

$$0 \to C_0(\mathcal{E}G) \to C(\overline{\mathcal{E}G}) \to C(\partial G) \to 0$$

in $E^G(SC(\partial G), C_0(\mathcal{E}G))$. The assembly map sends $1 \in E^G(C_0(\mathcal{E}G), C_0(\mathcal{E}G))$ to the class in $E(\mathbb{C}, C^*(G, \mathcal{E}G))$ of the $*$-homomorphism $p\colon \mathbb{C} \to C^*(G, \mathcal{E}G)$. But there is a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\mathcal{E}G) & \longrightarrow & C(\overline{\mathcal{E}G}) & \longrightarrow & C(\partial G) & \longrightarrow & 0 \\ & & {\scriptstyle q}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1\otimes p} & & \\ 0 & \longrightarrow & \mathcal{K}(\ell^2(G)) \otimes C_0(\mathcal{E}G) & \longrightarrow & D & \longrightarrow & C(\partial G) \otimes C^*_r(G, \mathcal{E}G) & \longrightarrow & 0 \end{array}$$

relating the extensions we are analyzing, in which the left vertical map is the $*$-homomorphism defined in Corollary 4.2. It follows from Corollary 4.2, and from functoriality of the E-theory morphism associated to an extension, that the partial splitting map sends $p \in E^G(\mathbb{C}, C^*_r(G, \mathcal{E}G))$ to the class of the extension determined by the boundary of G, as required. $\square$

It is clear from the lemma that injectivity of the assembly map would follow from injectivity of the boundary map. So is the boundary map injective? From the long exact sequence, injectivity of the boundary map would follow from vanishing of the preceeding map

$$E^G(C(\overline{\mathcal{E}G}), \mathbb{C}) \to E^G(C_0(\mathcal{E}G), \mathbb{C})$$

in the long exact sequence, and this in turn would follow from vanishing of the *group* $E^G(C_0(\mathcal{E}G), C(\overline{\mathcal{E}G}))$, since the above map is simply composition with a certain element from this group (namely the inclusion morphism from $C_0(\mathcal{E}G)$ to $C(\overline{\mathcal{E}G})$). But now we have the following variant of Proposition 3.7, which is proved by exactly the same Mayer-Vietoris argument:

5.5. Lemma. *If B is a separable G-C^*-algebra which is H-equivariantly contractible, for every finite subgroup H of G, then $E^G(\mathcal{E}G, B) = 0$.* $\square$

This would seem to complete the argument, given that in our situation we want to take $B = C(\overline{\mathcal{E}G})$, and we have hypothesized that $\overline{\mathcal{E}G}$ is H-equivariantly contractible, for every finite $H \subset G$. Unfortunately there is a difference between contractibility in the sense of spaces (= homotopy equivalent to a point) and contractibility in the sense of C^*-algebras (= homotopy equivalent to the zero C^*-algebra, not to the C^*-algebra $\mathbb{C}$ corresponding to a point), and because of this it is not generally true that $E^G(C_0(\mathcal{E}G), C(\overline{\mathcal{E}G})) = 0$. Indeed, if G is the trivial group, which is not excluded by our hypotheses, then $\mathcal{E}G = \overline{\mathcal{E}G} = Pt$, and the E-theory group $E^G(C_0(\mathcal{E}G), C(\overline{\mathcal{E}G}))$ is $\mathbb{Z}$.

To remedy the problem we must work with *reduced* homology, which in the present context means replacing the exact sequence

$$0 \to C_0(\mathcal{E}G) \to C(\overline{\mathcal{E}G}) \to C(\partial G) \to 0$$

with the exact sequence

$$0 \to C_0(\mathcal{E}G)^{(1)} \to C(\overline{\mathcal{E}G})^{(1)} \to C(\partial G)^{(1)} \to 0,$$

where

$$C_0(\mathcal{E}G)^{(1)} = \{\, f\colon [0,1] \to C_0(\mathcal{E}G) \;:\; f(0) = f(1) = 0 \,\}$$
$$C(\overline{\mathcal{E}G})^{(1)} = \{\, f\colon [0,1] \to C(\overline{\mathcal{E}G}) \;:\; f(0) = 0, f(1) \in \mathbb{C} \,\}$$
$$C(\partial G)^{(1)} = \{\, f\colon [0,1] \to C(\partial G) \;:\; f(0) = 0, f(1) \in \mathbb{C} \,\}$$

(all the functions f here are presumed to be continuous). Here $\mathbb{C}$ is included in $C(\overline{\mathcal{E}G})$ and $C(\partial G)$ as constant functions. Of course, $C_0(\mathcal{E}G)^{(1)}$ is just $SC_0(\mathcal{E}G)$. Now it *does* follow from Lemma 5.5 that under the contractibility hypotheses of Theorem 5.2 the boundary map

$$E^G(C_0(\mathcal{E}G)^{(1)}, A) \xrightarrow{\text{boundary map}} E^G(SC(\partial G)^{(1)}, A)$$

is injective. Let us consider now the diagram

$$(\square)\qquad \begin{array}{ccc} E^G(SC_0(\mathcal{E}G), SA) & \xrightarrow{\text{assembly}} & E(S, SC^*_r(G,A)) \\ \cong\big\downarrow & & \big\downarrow\text{partial splitting} \\ E^G(C_0(\mathcal{E}G)^{(1)}, SA) & \xrightarrow[\text{boundary map}]{} & E^G(SC(\partial G)^{(1)}, SA), \end{array}$$

whose constituent maps are defined as follows. The assembly map is obtained from the commuting diagram

$$\begin{array}{ccc} E^G(C_0(\mathcal{E}G), A) & \xrightarrow{\text{assembly}} & E(\mathbb{C}, C^*_r(G,A)) \\ 1_S\otimes-\big\downarrow\cong & & \cong\big\downarrow 1_S\otimes- \\ E^G(SC_0(\mathcal{E}G), SA) & \xrightarrow[\text{assembly}]{} & E(S, SC^*_r(G,A)), \end{array}$$

in which top map is the assembly map defined in Section 2 and the vertical maps are tensor product with the identity on S. By Bott periodicity, the vertical maps are isomorphisms. The partial splitting is defined from an extension

$$0 \to SA \otimes \mathcal{K}(\ell^2(G)) \to D \to C(\partial G)^{(1)} \otimes C^*_r(G,A) \to 0$$

and the diagram

$$\begin{array}{ccc} E(S, SC^*_r(G,A)) & \xrightarrow{1_{C(\partial G)^{(1)}}\otimes-} & E^G(SC(\partial G)^{(1)}, SC(\partial G)^{(1)} \otimes C^*_r(G,A)) \\ \text{partial splitting}\big\downarrow & & \big\downarrow\varphi \\ E^G(SC(\partial G)^{(1)}, SA) & \xrightarrow[\cong]{} & E^G(SC(\partial G)^{(1)}, SA \otimes \mathcal{K}(\ell^2(G))), \end{array}$$

in which the right hand map is composition with the E-theory class of the extension. The extension is, in turn, constructed from the pair of $*$-homomorphisms

$$\pi^{(1)}\colon C(\overline{\mathcal{E}G}) \to C([0,1], \mathcal{M}(A \otimes \mathcal{K}(\ell^2(G))))$$
$$\rho^{(1)}\colon C^*_r(G,A) \to C([0,1], \mathcal{M}(A \otimes \mathcal{K}(\ell^2(G))))$$

defined by

$$\pi^{(1)}(f)(t) = \pi(f(t))$$
$$\rho^{(1)}(a)(t) = \rho(a),$$

where $t \in [0,1]$ and π and ρ are the $*$-homomorphisms defined in the proof of Lemma 5.4. The $*$-homomorphisms $\pi^{(1)}$ and $\rho^{(1)}$ commute, modulo the ideal $SA \otimes \mathcal{K}(\ell^2(G))$, and furthermore $\pi^{(1)}$ maps $C_0(\mathcal{E}G)^{(1)}$ into $SA \otimes \mathcal{K}(\ell^2(G))$. Hence we obtain an extension as required.

The boundary map in (□) is induced from the short exact sequence

$$0 \to C_0(\mathcal{E}G)^{(1)} \to C(\overline{\mathcal{E}G})^{(1)} \to C(\partial G)^{(1)} \to 0.$$

We have already noted that, under the hypotheses of Theorem 5.2, this boundary map is injective. The proof of commutativity of the diagram (□) is done just as in the proof of Lemma 5.4, and therefore the assembly map in (□) is injective. But by periodicity this implies injectivity of the Baum-Connes assembly map defined in Section 2. The proof of Theorem 5.2 is now complete.

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