Meromorphic continuation of zeta functions associated to elliptic operators

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Abstract

We give a new proof of the meromorphic continuation property of zeta functions associated to elliptic operators. The argument adapts easily to the hypoelliptic operators considered by Connes and Moscovici in their recent work on transverse index theory in noncommutative geometry.

Introduction

Let Δ be a Laplace-type operator on a smooth, closed manifold M, and let D be a linear differential operator on M. It is well known that the "zeta-function" $\text{Trace}(D\Delta^{-z})$, which is well-defined in some right half-plane within $\mathbb{C}$, admits a meromorphic continuation to all of $\mathbb{C}$. This phenomenon of meromorphic continuation was first observed by Minakshisundaram and Pleijel [6], and in approximately the generality we shall consider here it is due to Seeley [8].

Recent work in noncommutative geometry has renewed interest in these zeta functions: Alain Connes and Henri Moscovici have proved a "local" index theorem in noncommutative geometry [2] which describes the cyclic cohomology class associated to an elliptic operator (its Chern character) in terms of the residues of zeta functions associated to the operator. At a more basic level, Connes defines a notion of dimension for his noncommutative geometric spaces using the locations of poles of zeta functions associated to the space.

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The purpose of this note is to investigate the meromorphic continuation property in a way which is appropriate to Connes' theory of noncommutative geometry. This means that we shall emphasize Hilbert space operator theory, commutator identities, and so on. We shall also work directly with the complex powers Δ^{-z} (defined by spectral theory) rather than say the resolvents $(\lambda - \Delta)^{-1}$ or the heat operators $e^{-t\Delta}$. Our method combines an approach due to Guillemin [4] with some simple formulas that go back at least to M. Riesz [7]. It applies not just to the classical situation of Laplace-type operators on manifolds, but also to the more elaborate operators encountered by Connes and Moscovici in their study of transverse index theory for foliations.

Although we shall use operator theory language which is appropriate to Connes noncommutative-geometric notion of spectral triple, the examples which we shall consider will never be too far removed from "commutative" manifold theory. It would be interesting to see whether or not the basic method can be adapted to other more complicated situations, such as for example the $SU_q(2)$ spectral triple analyzed by Connes in [1], or (less ambitiously) the noncommutative torus.

1 Partial Differential Operators

Let M be a smooth, closed manifold which is equipped with a smooth measure, using which we can form the Hilbert space $L^2(M)$. Denote by $\mathcal{D}(M)$ the algebra of all linear partial differential operators on M, viewed as an algebra of operators on the vector space $C^\infty(M)$ and filtered by order. Let Δ be a positive, second order, elliptic partial differential operator on M. As is well known, the operator Δ is essentially self-adjoint on the domain $C^\infty(M)$. We shall use this and the following two additional properties of Δ:

(i) If D is any operator in $\mathcal{D}$ then

$$\operatorname{order}\big([\Delta, D]\big) \leq \operatorname{order}(D) + 1.$$

(ii) If $D \in \mathcal{D}$ and if $\operatorname{order}(D) \leq q$ then there is a constant $\varepsilon > 0$ such that

$$\|\Delta^{\frac{q}{2}}\phi\|_{L^2(M)} + \|\phi\|_{L^2(M)} \geq \varepsilon \|D\phi\|_{L^2(M)}, \quad \forall \phi \in C^\infty(M).$$

The second property is easily recognizable as the basic elliptic estimate for Δ. To use it, it is convenient to introduce the usual Sobolev spaces $W_s(M)$. These are, for $s \geq 0$, the completions of $C^\infty(M)$ in the norms

$$\|\phi\|^2_{W_s(M)} = \|\Delta^{\frac{s}{2}}\phi\|^2_{L^2(M)} + \|\phi\|^2_{L^2(M)}.$$

We shall also use the following terminology:

Definition 1.1. Let $m \in \mathbb{R}$. We shall say that an operator $T\colon C^\infty(M) \to C^\infty(M)$ has *analytic order* m *or less* if, for every $s \geq 0$ such that $s + m \geq 0$, the operator T extends to a continuous linear operator from $W_{m+s}(M)$ to $W_s(M)$.

Condition (ii) above is then equivalent to the assertion that every linear partial differential operator of order q or less has analytic order q or less.

The following simple algebraic fact will be at the heart of our meromorphic continuation argument.

Lemma 1.2. *There exist order zero operators* $A_1, \dots, A_N$ *in* $\mathcal{D}$ *and order one operators* $B_1, \dots, B_N$ *in* $\mathcal{D}$ *such that:*

(i) $\sum_{i=1}^N [B_i, A_i] = \dim(M) \cdot I$.

(ii) If $D \in \mathcal{D}$ *and if* $\operatorname{order}(D) \leq q$ *then*

$$qD = \sum_{i=1}^N [D, A_i] B_i + R,$$

where $R \in \mathcal{D}$ *and* $\operatorname{order}(R) \leq q - 1$.

Proof. If there existed global coordinates $x_1, \dots, x_d$ on M then we could simply take

$$A_i = x_i \quad \text{and} \quad B_i = \frac{\partial}{\partial x_i}.$$

As it is, we can achieve the same thing using a partition of unity $\{\phi_\alpha\}$ subordinate to a cover $\{U_\alpha\}$ of M by coordinate charts. Let ψ_α be a smooth function, supported in U_α, for which $\phi_\alpha \psi_\alpha = \phi_\alpha$, and then form the operators

$$A_{i,\alpha} = \psi_\alpha \cdot x_i \quad \text{and} \quad B_{i,\alpha} = \phi_\alpha \cdot \frac{\partial}{\partial x_i},$$

using local coordinates in U_α. □

2 Statement of the Main Theorem

We want to apply the functional calculus to the operator Δ; in particular we want to form its complex powers. In order to do so it is convenient to modify Δ a little in order to guarantee invertibility.

Let $K\colon C^\infty(M) \to C^\infty(M)$ be a positive operator of analytic order $-\infty$ (thus K extends to a continuous map from $L^2(M)$ into $C^\infty(M)$), chosen so that the operator

$$\Delta_1 = \Delta + K$$

is bounded below. Consider the Cauchy integral

$$\Delta_1^{-z} = \frac{1}{2\pi i}\int \lambda^{-z}(\lambda - \Delta_1)^{-1}\, d\lambda,$$

in which the contour of integration is a downwards pointing vertical line in $\mathbb{C}$ separating 0 from the spectrum of Δ_1. If $\mathrm{Re}(z) > 0$, then the integral is absolutely convergent in the operator norm topology of $L^2(M)$, and indeed in the operator norm topology of each $W_s(M)$. It follows from the Sobolev theory that $\cap_{s\geq 0} W_s(M) = C^\infty(M)$, and we therefore obtain a well defined operator Δ_1^{-z} on the vector space $C^\infty(M)$. It has analytic order $-2\,\mathrm{Re}(z)$.

If $D \in \mathcal{D}$, then it follows from the Rellich Lemma in Sobolev theory that the operator $D\Delta_1^{-z}$ extends to a trace-class operator on $L^2(M)$, whenever $\mathrm{Re}(z)$ is sufficiently large. (In fact if D has order q then $D\Delta_1^{-z}$ is trace-class whenever $\mathrm{Re}(z) > \frac{n+q}{2}$.) The function $\mathrm{Trace}(D\Delta_1^{-z})$ is therefore defined and holomorphic in some right half-plane in $\mathbb{C}$. We shall prove the following result:

Theorem 2.1. *The function* $\mathrm{Trace}(D\Delta_1^{-z})$ *extends to a meromorphic function on* $\mathbb{C}$*, it has only simple poles, and if* $\mathrm{order}(D) = q$ *and* $\dim(M) = n$*, then these are located within the sequence of points*

$$\frac{q+n}{2}, \frac{q-1+n}{2}, \frac{q-2+n}{2}, \ldots.$$

Remark 2.2. So far we have considered the case of operators acting on scalar functions, but with trivial changes the proof we shall give below adapts to operators acting on sections of bundles. It also adapts to more complicated situations, as we shall point out later in the article.

3 Sketch of the Proof

According to Lemma 1.2, if D is a differential operator of order q or less, then

$$\boxed{qD = \sum_{i=1}^{N} [D, A_i]B_i + R}$$

where the remainder R is a differential operator of order $q-1$ or less. As a result of this and the identity $\sum_i [B_i, A_i] = nI$, it follows by simple algebra that

$$(q+n)D = \sum_{i=1}^{N} [D, A_i B_i] + \sum_{i=1}^{N} [B_i, A_i D] + R,$$

for the same remainder R. Hence $(q+n)D$ is equal to R, modulo commutators.

We are going to demonstrate that the boxed identity holds not just for differential operators but for families like $T_z = D\Delta_1^{-z}$, to which we shall assign the "order" $q-2z$ if $\text{order}(D) = q$. We will therefore be able to conclude that

$$(q+n-2z)T_z = R_z, \quad \text{modulo commutators,}$$

for some remainder family R_z of "order" $q-1-2z$. If $\text{Re}(z) \ll 0$, then we can take traces to obtain the formula

$$\text{Trace}(T_z) = \frac{1}{q+n-2z} \text{Trace}(R_z)$$

(since the trace of a commutator vanishes). Repeating this argument $\ell+1$ times we can write

$$\text{Trace}(T_z) = \frac{1}{q+n-2z} \frac{1}{q-1+n-2z} \cdots \frac{1}{q-\ell+n-2z} \text{Trace}(S_z),$$

where $\{S_z\}$ has "order" $q-\ell-2z$. But since $\text{Trace}(S_z)$ is defined and holomorphic whenever $\text{Re}(z) > \frac{q-\ell+n}{2}$, we see that by making ℓ large we can meromorphically extend $\text{Trace}(T_z)$ to as large a right half-plane as we please. This proves the theorem.

In the ensuing sections we shall make clear precisely what families T_z we shall be dealing with, and we shall verify that the boxed formula holds for them, as required. But first we shall review a "Taylor expansion" due to Connes and Moscovici [2, Appendix B] which we shall need in the argument.

4 Taylor Expansion

Definition 4.1. If $T\colon C^\infty(M) \to C^\infty(M)$ is any linear operator, then denote by $T^{(k)}$ the k-fold commutator of T with $\Delta\colon C^\infty(M) \to C^\infty(M)$. Thus $T^{(0)} = T$ and $T^{(k)} = [\Delta, T^{(k-1)}]$ for $k \geq 1$.

Observe that if $D \in \mathcal{D}$, and if $\operatorname{order}(D) \leq q$, then $\operatorname{order}(D^{(k)}) \leq q + k$, and therefore the analytic order of $D^{(k)}$ is similarly bounded.

For a nonnegative integer p we shall use Cauchy's formula

$$\binom{z}{p}\Delta_1^{z-p} = \frac{1}{2\pi i}\int \lambda^z(\lambda - \Delta_1)^{-p-1}\, d\lambda,$$

where the contour of integration is a downward-pointing vertical line in the complex plane which separates 0 from the spectrum of Δ_1. Here $\binom{z}{p}$ is the usual binomial coefficient. If $\operatorname{Re}(z) + p > 0$ then integral converges in the norm topology of bounded operators from H^{s+p} to H^s, and the restriction of the integral to $C^\infty(M)$ is $\binom{z}{p}\Delta_1^{z-p}$, as previously defined (by a different Cauchy integral).

Lemma 4.2. *Let* $D \in \mathcal{D}$ *and let* $z \in \mathbb{C}$. *For every positive integer* $k > \operatorname{Re}(z)$ *there is an identity*

$$\begin{aligned}[\Delta_1^{-z}, D] = \binom{-z}{1}D^{(1)}\Delta_1^{-z-1} + \binom{-z}{2}D^{(2)}\Delta_1^{-z-2} + \cdots \\ + \binom{-z}{k}D^{(k)}\Delta_1^{-z-k} + \frac{1}{2\pi i}\int \lambda^{-z}(\lambda - \Delta_1)^{-1}D^{(k)}(\lambda - \Delta_1)^{-k}\, d\lambda.\end{aligned}$$

The contour integral is to be computed down the same vertical line in $\mathbb{C}$ as above. If $\operatorname{order}(D) \leq q$ then $\operatorname{order}(D^{(k)}) \leq q + k$ and the integral converges in the norm-topology of operators from $W_{q+k}(M)$ to $L^2(M)$, or indeed in the norm-topology of operators from $W_{q+k+s}(M)$ to $W_s(M)$, for every s (we can therefore say that the integral converges in the topology of operators of analytic order $q+k$ or less). The integral therefore determines a well-defined operator on $C^\infty(M)$.

To understand the content of the lemma one should observe that if $D \in \mathcal{D}$ and if $\operatorname{order}(D) \leq q$, then

$$\begin{aligned}\text{analytic order}\big(D^{(j)}\Delta_1^{z-j}\big) &\leq \text{analytic order}\big(D^{(j)}\big) + \text{analytic order}\big(\Delta_1^{z-j}\big) \\ &\leq q + j + 2(\operatorname{Re}(z) - j) \\ &= q + 2\operatorname{Re}(z) - j.\end{aligned}$$

Hence the terms in the expansion are of decreasing analytic order. Moreover the "remainder term" has analytic order no more than $q + 2\operatorname{Re}(z) - k - 1$. To see this, apply Taylor's formula with $k+\ell$ in place of k to get

$$\begin{aligned}\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta_1)^{-1}D^{(k)}(\lambda-\Delta_1)^{-k}\,d\lambda \\ = \tbinom{-z}{k+1}D^{(k+1)}\Delta_1^{-z-k-1}+\cdots+\tbinom{-z}{k+\ell}D^{(k+\ell)}\Delta_1^{-z-k-\ell} \\ +\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta_1)^{-1}D^{(k+\ell)}(\lambda-\Delta_1)^{-k-\ell}\,d\lambda.\end{aligned}$$

The final integral converges in the topology of operators of analytic order $q+k-\ell$ or less, and so by making ℓ large we can make its analytic order as small as we please.

Proof of Lemma 4.2. Suppose first that $\mathrm{Re}(z)>0$. Applying Cauchy's formula in the case $p=0$ we get

$$\begin{aligned}[\Delta_1^{-z},D] &= \frac{1}{2\pi i}\int \lambda^{-z}\big[(\lambda-\Delta_1)^{-1},D\big]\,d\lambda \\ &= \frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta_1)^{-1}D^{(1)}(\lambda-\Delta_1)^{-1}\,d\lambda \\ &= D^{(1)}\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta_1)^{-2}\,d\lambda \\ &\quad+\frac{1}{2\pi i}\int \lambda^{-z}\big[(\lambda-\Delta_1)^{-1},D^{(1)}\big](\lambda-\Delta_1)^{-1}\,d\lambda \\ &= \tbinom{-z}{1}D^{(1)}\Delta_1^{-z-1}+\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta_1)^{-1}D^{(2)}(\lambda-\Delta_1)^{-2}\,d\lambda.\end{aligned}$$

By carrying out a sequence of similar manipulations on the remainder integral we arrive at

$$\begin{aligned}[\Delta_1^{-z},D] = \tbinom{-z}{1}D^{(1)}\Delta_1^{-z-1}+\tbinom{-z}{2}D^{(2)}\Delta_1^{-z-2}+\cdots \\ +\tbinom{-z}{k}D^{(k)}\Delta_1^{-z-k}+\frac{1}{2\pi i}\int \lambda^{z}(\lambda-\Delta_1)^{-1}D^{(k)}(\lambda-\Delta_1)^{-k}\,d\lambda,\end{aligned}$$

as required. This proves the lemma for $\mathrm{Re}(z)>0$. The full result follows by uniqueness of analytic continuations. □

There is also an "opposite" version of the Taylor expansion, in which the powers of Δ_1 appear to the left of the operators $D^{(j)}$:

Lemma 4.3. *Let* $D \in \mathcal{D}$ *and let* $z \in \mathbb{C}$. *For every positive integer* $k > \mathrm{Re}(z)$ *there is an identity*

$$[\Delta_1^{-z}, D] = -\binom{-z}{1}\Delta_1^{-z-1}D^{(1)} + \binom{-z}{2}\Delta_1^{-z-2}D^{(2)} - \cdots \\ + (-1)^k\binom{-z}{k}\Delta_1^{-z-k}D^{(k)} + \frac{(-1)^k}{2\pi i}\int \lambda^{-z}(\lambda - \Delta_1)^{-k}D^{(k)}(\lambda - \Delta_1)^{-1}\, d\lambda.$$

□

This is proved in exactly the same way.

5 Proof of the Main Theorem

The following definitions describe the sorts of families we shall be dealing with.

Definition 5.1. Let $T_z \colon C^\infty(M) \to C^\infty(M)$ be a family of linear operators, parametrized by $z \in \mathbb{C}$ and let $k \in \mathbb{Z}$. We shall call $\{T_z\}$ an *elementary holomorphic family of type* k, if it is of the form

$$T_z = p(z)D\Delta_1^{-z-n}$$

where p is a polynomial, $D \in \mathcal{D}$ and $\mathrm{order}(D) \leq 2n + k$.

Observe that the analytic order of T_z is no more than $k - 2\,\mathrm{Re}(z)$. Obviously, if $\{T_z\}$ is an elementary holomorphic family of type k then it is also an elementary holomorphic family of type $k + 1$.

Definition 5.2. Let $T_z \colon C^\infty(M) \to C^\infty(M)$ be a family of linear operators, parametrized by $z \in \mathbb{C}$. We shall call $\{T_z\}$ a *holomorphic family of type* k if, in every complex half-plane $\mathrm{Re}(z) > a$, and for every $m \in \mathbb{R}$, we can write

$$T_z = T_z^1 + \cdots + T_z^n + R_z,$$

for some n, where each $\{T_z^j\}$ is an elementary holomorphic family of type k, and $z \mapsto R_z$ is a holomorphic map from the half-plane $\mathrm{Re}(z) > a$ to the operators of analytic order m or less.

Once again, if $\{T_z\}$ is a holomorphic family of type k, then each T_z has analytic order no more than $k - 2\,\mathrm{Re}(z)$. Moreover sums of holomorphic families of type k are again holomorphic of type k, and every holomorphic family of type k is also holomorphic of type $k + 1$.

The virtue of Definition 5.2 is that it provides a notion of holomorphic family which is closed under some simple operations. Here is a useful example of this:

Lemma 5.3. *If* $\{T_z\}$ *is a holomorphic family of type* k*, and if* $D \in \mathcal{D}$ *is of order* q *or less, then* $\{T_z D\}$ *and* $\{D T_z\}$ *are holomorphic families of type* $k+q$. □

This follows from the Taylor expansion given in Lemma 4.2. Here, then, is the crucial computation:

Lemma 5.4. *Assume that there exist operators* $A_1, \ldots, A_N$ *and* $B_1, \ldots, B_N$ *in the algebra* $\mathcal{D}$ *of generalized differential operators as in items* (i) *and* (iii) *of the statement of Theorem* 2.1. *If* $\{T_z\}$ *is a holomorphic family of type* k *then*

$$\sum_{i=1}^{N} [T_z, A_i] B_i = (k - 2z) T_z + R_z,$$

where R_z *is a holomorphic family of type* $k-1$.

Remark 5.5. In the following computation we shall need to know the orders of various differential operators, and for this we note that if $D \in \mathcal{D}$, then

$$\operatorname{order}\big([A_i, D]\big) \leq \operatorname{order}(D) - 1 \quad \text{and} \quad \operatorname{order}\big([B_i, D]\big) \leq \operatorname{order}(D),$$

since A_i is a function and B_i is a vector field.

Proof of the Lemma. It suffices to prove this in the case where T_z is an elementary holomorphic family of type k. Thus let us write

$$T_z = p(z) D \Delta_1^{-z-n},$$

where $\operatorname{order}(D) \leq 2n + k$. In fact we can go further and to drop the term $p(z)$: it clearly suffices to prove the result for the simpler elementary family

$$T_z = D \Delta_1^{-z-n}.$$

Starting with this family we get

$$\sum_{i=1}^{N} \big[D\Delta_1^{-z-n}, A_i\big] B_i = \sum_{i=1}^{N} D[\Delta_1^{-z-n}, A_i] B_i + \sum_{i=1}^{N} [D, A_i] \Delta_1^{-z-n} B_i,$$

and therefore

$$\begin{aligned} \sum_{i=1}^{N} \big[D\Delta_1^{-z-n}, A_i\big] B_i = \sum_{i=1}^{N} D\big[\Delta_1^{-z-n}, A_i\big] B_i &+ \sum_{i=1}^{N} \big[D, A_i\big] B_i \Delta_1^{-z-n} \\ &+ \sum_{i=1}^{N} \big[D, A_i\big] \big[\Delta_1^{-z-n}, B_i\big]. \end{aligned} \tag{5.1}$$

We are now going to analyze the three sums which appear on the right hand side of (5.1) separately.

If we develop the quantities $D[\Delta_1^{-z-n}, A_i]B_i$ using the 'opposite' Taylor series of Lemma 4.3, and apply Lemma 5.3, then we obtain the formula

$$\begin{aligned}\sum_{i=1}^{N} D[\Delta_1^{-z-n}, A_i]B_i &= \sum_{i=1}^{N} \binom{-z-n}{1} D\Delta_1^{-z-n-1}[\Delta_1, A_i]B_i + \text{remainder} \\ &= 2(-z-n)D\Delta_1^{-z-n} + \text{remainder},\end{aligned}$$

where the remainder is in each case a holomorphic family of type $k-1$. Next, according to our hypotheses, since $\text{order}(D) \leq 2n+k$, the middle sum in (5.1) has the form

$$\sum_{i=1}^{N} [D, A_i]B_i\Delta_1^{-z-n} = (2n+k)D\Delta_1^{-z-n} + \text{remainder}$$

where again the remainder is a holomorphic family of type $k-1$ (in fact an elementary holomorphic family). Finally by developing each of the terms $[\Delta_1^{-z-n}, B_i]$ as a Taylor series, we see that the last sum in (5.1) is a holomorphic family of type $k-2$.

Putting everything together, we see that

$$\begin{aligned}\sum_{i=1}^{N} [D\Delta_1^{-z-n}, A_i]B_i &= 2(-z-n)D\Delta_1^{z-n} + (2n+k)D\Delta_1^{-z-n} + \text{remainder} \\ &= (k-2z)D\Delta_1^{-z-n} + \text{remainder},\end{aligned}$$

where the remainder is a holomorphic family of type $k-1$, as required. □

With this lemma in hand, the proof of Theorem 2.1 is completed as in Section 3. (There is perhaps one other point which deserves comment: the vanishing of traces of commutators, which is crucial for the argument in Section 3, is not altogether trivial here since some of the operators involved are unbounded. But one can convert these to traces of commutators of bounded operators using Sobolev spaces, and invoking the easily proved fact that the trace of an operator of sufficiently low analytic order, viewed as an operator on any $W_s(M)$, is independent of s.)

6 Generalized Differential Operators

In this section we shall develop an abstract notion of elliptic linear partial differential operator which will allow us to formulate a generalization of Theorem 2.1.

Let H be a complex Hilbert space. We shall assume as given a fixed unbounded, positive, self-adjoint operator Δ on H.

Definition 6.1. Denote by $H^\infty \subseteq H$ the space of vectors common to the domains of all the powers of Δ:

$$H^\infty = \cap_{k=1}^\infty \operatorname{dom}(\Delta^k).$$

This is a dense subspace of H and has the structure of a Frechet space, although we shall treat it simply as a complex vector space below.

We shall assume as given a fixed associative algebra $\mathcal{D}$ of linear operators on the vector space H^∞.

Definition 6.2. We shall call Δ a *generalized Laplace operator of order* r, and $\mathcal{D}$ an *algebra of generalized differential operators*, if the following conditions are satisfied:

(i) The algebra $\mathcal{D}$ is filtered,

$$\mathcal{D} = \cup_{q=0}^\infty \mathcal{D}_q \quad \text{(an increasing union).}$$

We shall write $\operatorname{order}(D) \leq q$ to denote the relation $D \in \mathcal{D}_q$.

(ii) If D is any operator in $\mathcal{D}$ then $[\Delta, D] \in \mathcal{D}$ and

$$\operatorname{order}\big([\Delta, D]\big) \leq \operatorname{order}(D) + r - 1.$$

(iii) If $D \in \mathcal{D}$ and if $\operatorname{order}(D) \leq q$ then there is a constant $\varepsilon > 0$ such that

$$\|\Delta^{\frac{q}{r}} v\| + \|v\| \geq \varepsilon \|Dv\|, \quad \forall v \in H^\infty,$$

where $r = \operatorname{order}(\Delta)$.

Example 6.3. If M is a complete Riemannian manifold, Δ is the Laplace operator, and $\mathcal{D}$ is the algebra of compactly supported partial differential operators, then all the axioms hold (here $r = 2$). Item (iii) is a formulation of the basic elliptic estimate Δ.

Definition 6.4. For $s \geq 0$ let us denote by H^s the linear space

$$H^s = \mathrm{dom}(\Delta^{\frac{s}{r}}) \subseteq H,$$

which is a Hilbert space in the norm $\|v\|_s^2 = \|v\|^2 + \|\Delta^{\frac{s}{r}} v\|^2$.

Of course, in the standard example these are the ordinary Sobolev spaces.

Lemma 6.5. *If $D \in \mathcal{D}$ and if $\mathrm{order}(D) \leq q$, then for every $s \geq 0$, the operator X extends to a continuous linear operator from H^{s+q} to H^s.* □

Assume that for $D \in \mathcal{D}$ the operator $D\Delta_1^{-z}$ is trace-class, for $\mathrm{Re}(z) \gg 0$. The arguments used in the preceding sections prove the following result:

Theorem 6.6. *Suppose that there exist operators $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$ in the algebra $\mathcal{D}$ of generalized differential operators such that*

(i) If $D \in \mathcal{D}$, then

$$\mathrm{order}\big([A_i, D]\big) \leq \mathrm{order}(D) - 1 \quad \textit{and} \quad \mathrm{order}\big([B_i, D]\big) \leq \mathrm{order}(D).$$

(ii) $\sum_{i=1}^N [B_i, A_i] = d \cdot I$, for some real number d.

(iii) If $D \in \mathcal{D}$ and if $\mathrm{order}(D) \leq q$ then

$$\sum_{i=1}^N [D, A_i] B_i = qD + R,$$

where $\mathrm{order}(R) \leq q - 1$.

Then, for every generalized differential operator D, the function $\mathrm{Trace}(D\Delta_1^{-z})$, which is defined when $\mathrm{Re}(z) \ll 0$, extends to a meromorphic function of $z \in \mathbb{C}$ with only simple poles, all located in the sequence

$$\frac{q+d}{r}, \frac{q-1+d}{r}, \frac{q-2+d}{r}, \ldots,$$

where $q = \mathrm{order}(D)$. □

7 The Example of Connes and Moscovici

Let M be a smooth manifold of dimension n. Assume that an *integrable* smooth vector sub-bundle $F \subseteq TM$ is given.

Let $\mathcal{D} = \mathrm{Diff}(M, F)$ be the algebra of compactly supported differential operators on M, as in the standard example which was discussed in the previous section. We use the bundle F to provide $\mathrm{Diff}(M, F)$ with a new filtration, as follows.

(i) If f is a C^∞-function on M then $\mathrm{order}(f) = 0$.

(ii) If D is a C^∞-vector field on M then $\mathrm{order}(D) \leq 2$.

(iii) If D is a C^∞-vector field on M which is everywhere tangent to F then $\mathrm{order}(D) \leq 1$.

The order of an operator can be described more concretely using local coordinates. For this purpose we shall work only with coordinate systems $x_1, \ldots, x_d$ on M with the property that the first p coordinate vector fields $\frac{\partial}{\partial x_1} \ldots, \frac{\partial}{\partial x_p}$ span the integrable bundle F. If $D \in \mathrm{Diff}(M, F)$ then in these local coordinates we can write D as a sum

$$D = \sum_{\alpha} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

where α is a nonnegative integer multi-index. The operator D has order q or less if in every coordinate system we have

$$D = \sum_{|\alpha|_F \leq q} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

where

$$|\alpha|_F = \alpha_1 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_n.$$

This notion of order applies perfectly well to any differential operator on M, not just one which is compactly supported, and in the following definition we shall consider non-compactly supported operators.

Definition 7.1. An differential operator D on M is *elliptic of order* r *(relative to* F*)* if it is of order r or less, and if in every coordinate system, as above, and at every point x in the domain of the coordinate system,

$$\left| \sum_{|\alpha|=r} a_\alpha(x) \xi^\alpha \right| \geq \varepsilon_x \left(|\xi_1|^4 + \cdots |\xi_p|^4 + |\xi_{p+1}|^2 + \cdots + |\xi_n|^2 \right)^{\frac{r}{4}}$$

for some $\varepsilon_x > 0$ and all $\xi \in \mathbb{R}^n$.

Example 7.2. If $F = TM$ then the above definition coincides with the usual definition of ellipticity (and at the same time our new notion of order is the usual notion of order for differential operators).

It is easy to construct examples of elliptic operators which are of order 4. For example this can be done by adding together the Laplace operator for some metric on M with the square of a leafwise Laplace operator.

It is a simple matter to develop elliptic regularity theory for this new sort of elliptic operator along the standard lines. Having fixed a smooth measure on M the kth Sobolev norm on the smooth functions which are compactly supported in a coordinate chart is defined by

$$\|\phi\|^2_{W_k(M,F)} = \sum_{|\alpha|_F \le k} \left\| \frac{\partial^\alpha \phi}{\partial x^\alpha} \right\|^2_{L^2(M)}.$$

Using partitions of unity we can now extend the definition to all smooth functions on M which are supported in some compact set K. The resulting Sobolev Hilbert spaces $W^k(K, F)$ depend only on M and F, as topological vector spaces, and

$$W_{2k}(K) \subseteq W_k(K, F) \subseteq W_k(K),$$

where the $W^k(K)$ are the usual Sobolev spaces of functions supported within K, from which it follows that

$$\cap_{k\ge 0} W_k(K, F) = \cap_{k \ge 0} W_k(K) \subseteq C^\infty(K).$$

The standard argument now easily adapts to show that if Δ is elliptic of order r then

$$\|\Delta\phi\|^2_{W_k(K,F)} + \|\phi\|^2_{L^2(K)} \ge \varepsilon \|\phi\|_{W_{r+k}(K,F)}$$

for some $\varepsilon > 0$ and every smooth function ϕ supported within K. Thus if Δ is in addition positive, as a Hilbert space operator, and essentially self-adjoint on $C^\infty_c(M)$, then we obtain an example of a generalized Laplace operator, in the sense of Definition 6.2.

Now if $X \in \mathrm{Diff}(M, F)$, if $\mathrm{order}(F) \le q$, and if X is supported in a coordinate chart, then

$$\sum_{i=1}^{p} [D, x_i] \frac{\partial}{\partial x_i} + 2 \sum_{i=p+1}^{n} [D, x_i] \frac{\partial}{\partial x_i} = qD - R,$$

where $\mathrm{order}(R) \leq q-1$. We can therefore fulfill the hypotheses of Theorem 2.1 (for the subalgebra of operators supported in our chart) by defining $\{A_i\}$ and $\{B_i\}$ to be the lists

$$\{x_1, x_2, \ldots, x_p, x_{p+1}, x_{p+1}, \ldots, x_n, x_n\}$$

and

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_{p+1}}, \frac{\partial}{\partial x_{p+1}}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_n}\right\}$$

Thus each 'longitudinal' coordinate is listed once and each 'transverse' coordinate is listed twice.

As in the standard example, by covering M with coordinate charts, and making use of a subordinate partition of unity we can now fulfill the hypotheses of Theorem 2.1 (for operators supported in a given compact subset of M) by setting the A-operators equal to $\psi_\alpha \cdot x_i$ and the corresponding B-operators equal to $\phi_\alpha \cdot \frac{\partial}{\partial x_i}$ (with the same notation as before). As above every transverse operator should be listed twice.

It follows that all the functions $\mathrm{Trace}(D\Delta_1^{-z})$ are meromorphic, with only simple poles, located within the sequence

$$\frac{q+d}{r}, \frac{q-1+d}{r}, \frac{q-2+d}{r}, \ldots .$$

Here q is the order of D and r is the order of Δ. In contrast to the classical case, the integer d is not the dimension n of M but $p + 2(n-p)$.

8 Spectral Triples

In this section we shall briefly remark how Connes' spectral triples (his noncommutative geometric spaces) provide examples of algebras of differential operators of the sort considered in Section 6. For background information on spectral triples see the text [3].

Definition 8.1. Let (A, H, D) be a spectral triple with the property that every $a \in A$ maps H^∞ into itself. The *algebra of differential operators* associated to (A, H, D) is the smallest algebra $\mathcal{D}$ of operators on H^∞ containing A and $[D, A]$ and closed under the operation $T \mapsto [D^2, T]$.

Remark 8.2. The above is in some sense the minimal reasonable definition of an algebra of differential operators. Note however that the operator D is not necessarily included in $\mathcal{D}$.

The algebra of differential operators is filtered, as follows. We require that elements of A and $[D, A]$ have order zero and that the operation of commutator with $\Delta = D^2$ raises order by at most one. Thus the spaces $\mathcal{D}_k$ of operators of order k or less are defined inductively as follows:

(i) $\mathcal{D}_0 =$ the algebra generated by $A + [D, A]$.

(ii) $\mathcal{D}_1 = [\Delta, \mathcal{D}_0] + \mathcal{D}_0[\Delta, \mathcal{D}_0]$.

(iii) $\mathcal{D}_k = \sum_{j=1}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{k-j} + [\Delta, \mathcal{D}_{k-1}] + \mathcal{D}_0[\Delta, \mathcal{D}_{k-1}]$.

Definition 8.3. A spectral triple (A, H, D) with the property that every $a \in A$ maps H^∞ into itself satisfies the *basic estimate* if for every differential operator of order q there is an $\epsilon > 0$ such that

$$\|D^q v\| + \|v\| \geq \varepsilon \|Xv\|,$$

for all $v \in H^\infty$.

The condition that $A \cdot H^\infty \subseteq H^\infty$ is implied by the stronger condition of *regularity*, for which we refer the reader to [3]. Here we just wish to note the following fact:

Theorem 8.4. *Let (A, H, D) be a spectral triple with the property that every $a \in A$ maps H^∞ into itself. It is regular if and only if it satisfies the basic estimate.*

This is basically due to Connes and Moscovici in [2], but for a complete proof see [5].

9 A Small Refinement of the Main Theorem

As noted in Theorem 2.1, if D has order q then the poles of $\mathrm{Trace}(D\Delta_1^{-z})$ are all simple and are located at the points

$$\frac{q+n}{2}, \frac{q-1+n}{2}, \frac{q-2+n}{2}, \ldots.$$

A slightly more careful analysis of proof shows that many of these singularities are in fact removable. The reason for this is first that the formula

$$\sum_{i=1}^{N} [D, A_i] B_i = qD + \text{remainder},$$

for order q differential operators is exact when $q = 0$ (since there are no order -1 differential operators), and second that the Taylor expansion introduces scalar factors $\binom{-z-n}{m}$ which are of course zero at $z = -n, -n-1, \ldots$, and which cancel the potential poles produced in the course of the proof of Theorem 2.1.

Here are the details, which involve retracing the steps taken in the proof of Theorem 2.1. Consider first the family $D\Delta_1^{-z}$, where D is an order 0 operator. Let us show that $\operatorname{Trace}(D\Delta_1^{-z})$ has no pole at $z = 0$. If we look at the key equation (5.1) in the proof of Theorem 2.1 then we see that the second and third sums are identically zero, thanks to the fact that $[A_i, D] = 0$. As for the first sum, we note that the remainder integral in Taylor's formula vanishes at $z = 0$ (since all the other terms in Taylor's formula vanish there). So if the remainder integral, considered as a function of z, is holomorphic as a map from a given half-plane in $\mathbb{C}$ to operators of a given analytic order, then we can in fact write it as z times a function with the same analytic property. As a result, when we employ Taylor's formula to analyze the first sum in equation (5.1), we can conclude that

$$\sum_{i=1}^{N} [D\Delta_1^{-z}, A_i]B_i = \sum_{i=1}^{N} D[A_i, \Delta_1^{z}]B_i = -2zD\Delta_1^{-z} + zR_z,$$

where $\{R_z\}$ is a holomorphic family of type -1. Continuing with proof of Theorem 2.1 with this extra factor of z in hand we get

$$\operatorname{Trace}(D\Delta^{-z}) = z \cdot \frac{1}{2z}\frac{1}{2z-1}\cdots\frac{1}{2z-\ell}\operatorname{Trace}(S_z),$$

where $\{S_z\}$ is of type $-\ell$. As a result, $\operatorname{Trace}(D\Delta^{-z})$ has no pole at $z = 0$.

We can now show $\operatorname{Trace}(D\Delta_1^{z})$ has no pole at $z = 0$, no matter what the order of D. The key equation (5.1) now has more nonzero terms, but taking them all into account we get that if $\operatorname{order}(D) \leq q$ then

$$\sum_{i=1}^{N} [D\Delta_1^{-z}, A_i]B_i = (-2z + q)D\Delta_1^{-z} + Y\Delta_1^{-z} + zR_z,$$

where as before $\{R_z\}$ has type $q - 1$, and where $Y \in \mathcal{D}$ with $\operatorname{order}(Y) < q$. As a result, $\operatorname{Trace}(D\Delta_1^{-z})$ has a pole at zero if and only if $\operatorname{Trace}(Y\Delta_1^{-z})$ does.

Finally, by writing $D\Delta_1^{-z+m} = (D\Delta_1^{m})\Delta_1^{-z}$ we see that $\operatorname{Trace}(D\Delta^{-z})$ has no pole at any nonnegative integer m.

10 A Second Small Refinement

The simple local coordinate-based nature of the operators A_i and B_i in our examples suggests the following improvement of Theorem 2.1, in which for simplicity we shall work with the example of a standard elliptic operator Δ on a closed manifold M.

Let $\{T_z\}$ be a holomorphic family, as in Section 5, and assume for the moment that it is compactly supported in a single coordinate chart (thus $T_z = \phi T_z \phi$ for some smooth function ϕ which is compactly supported within a coordinate chart). Fix $h \in \mathbb{N}$. The usual Sobolev theory implies that if z is restricted to a suitable left half-plane in $\mathbb{C}$, then T_z may be represented by a kernel function $k_z \colon M \times M \to \mathbb{C}$ which is h-times continuously differentiable. Consider now the basic identity from the proof of Theorem 2.1,

$$(-2z + k + n)T_z = \sum [T_z, A_i B_i] + \sum [B_i, A_i T_z] + R_z.$$

Since T_z is supported within a single coordinate patch we can take $A_i = x_i$ and $B_i = \frac{\partial}{\partial x_i}$. The identity simplifies to

$$(-2z + k + n)T_z = \sum \Big([T_z, A_i] B_i + T_z \Big) + R_z. \tag{10.1}$$

Using local coordinates, the operator within the parentheses is given by the formula

$$f \mapsto \int k_z(x,y)(y_i - x_i)\partial_i f(y)\, dy + \int k_z(x,y) f(y)\, dy$$

If we integrate by parts we obtain

$$\int k_z(x,y)(y_i - x_i)\partial_i f(y)\, dy = -\int \Big(\partial_i k_z(x,y)(y_i - x_i) + k_z(x,y) \Big) f(y)\, dy$$

(where $\partial_i k_z(x,y)$ denotes the partial derivative with respect to the ith y-variable) and as a result, the operator within the parentheses in (10.1) is given by the formula

$$f \mapsto -\int \partial_i k_z(x,y)(y_i - x_i) f(y)\, dy.$$

We see that this operator is given by an integral kernel which vanishes pointwise along the diagonal in $M \times M$.

The proof of Theorem 2.1 now provides the well-known meromorphic continuation of trace-densities for complex powers of Δ:

Theorem 10.1. *Let Δ be a positive, elliptic operator on a smooth, closed manifold M, and let X be a differential operator on M. For $\mathrm{Re}(z) \ll 0$ let $K_z \colon M \to \mathbb{C}$ be the restriction to the diagonal in $M \times M$ of the integral kernel for the operator $X\Delta_1^{-z}$. For every $h \in \mathbb{N}$ the map $z \mapsto K_z$ extends to a meromorphic function from $\mathbb{C}$ into the h-times continuously differentiable functions on M.* □

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