The Local Index Formula
in
Noncommutative Geometry

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Abstract

These notes present a partial account of the local index theorem in noncommutative geometry discovered by Alain Connes and Henri Moscovici.

Keywords: Noncommutative Geometry, Index Theory, Cyclic Cohomology.

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Preface

Several years ago Alain Connes and Henri Moscovici discovered a quite general 'local' index formula in noncommutative geometry [10]. The formula was originally studied in relation to the transverse geometry of foliations, but more recently Connes has drawn attention to other possible areas of application, for example compact quantum groups [6] and deformations of homogeneous manifolds [8]. Moreover elaborate structures in homological algebra have been devised in the course of studying the formula [8], and these have found application in quantum field theory [7] and elsewhere [11].

These notes provide an introduction to the local index formula. They emphasize the basic, analytic aspects of the subject. This is in part because the analysis must be dealt with first, before more purely cohomological issues are tackled, and in part because the later issues are already quite well covered in survey articles by Connes and others (see for example [5]). Moreover, on the cohomological side, the final and definitive results have yet to be thoroughly investigated. I hope that the reader will be able to use these notes to introduce himself to these issues of current research interest.

The notes begin with a rapid account of the spectral theory of linear elliptic operators on manifolds, which is the launching point for the local index formula. They begin right at the beginning, and I hope that they might be accessible to students with a very modest background in analysis. Two appendices deal with still more basic issues in Hilbert space operator theory and Fourier theory.

The first result which goes beyond the totally standard canon (but which is still classical) is the theorem that the zeta functions $\mathrm{Trace}(\Delta^{-z})$ associated to elliptic operators admit meromorphic continuations to $z \in \mathbb{C}$. I shall present a proof which is more algebraic than the usual ones, and which seems to me to well adapted to Connes' noncommutative geometric point of view.

Following that, manifolds are replaced by Connes' 'noncommutative geometric spaces', and basic tools such as differential operator theory and pseudodifferential operator theory are developed in this context.

After the subject of cyclic cohomology theory is rapidly introduced, it becomes possible to formulate the basic index problem, which is the main topic of the notes. The final sections of the paper (from 6 to 8) prove the index formula.

The notes correspond very roughly to the first four of the six lectures I gave at the Trieste meeting. The remaining two lectures dealt with cyclic cohomology for Hopf algebras. The interested reader can look at the overhead transparencies from those lectures [20] to figure out more precisely what has been omitted and what has been added (note that the division of the present notes into sections does not correspond to the original division into lectures). The notes borrow (in places *verbatim*) from several preprints of mine [17, 18, 19] which will be published elsewhere. But of course they rely most of all on the work of Connes and Moscovici. If a result is given in the notes without attribution, the reader should not assume that it is original in any way. Most likely the result is due, in one form or another, to these authors.

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1 Elliptic Partial Differential Operators

We are going to develop the spectral theory of elliptic linear partial differential operators on smooth, closed manifolds. We shall approach the subject from the direction of Hilbert space theory, which is particularly well suited for the task. In fact Hilbert space theory was invented for just this purpose.

1.1 Laplace Operators

Let M be a smooth, closed, manifold of dimension n. A linear operator D mapping the vector space of smooth, complex-valued functions on M to itself is *local* if, for every smooth function ϕ, the support of $D\phi$ is contained within the support of ϕ. If D is local, then the value of $D\phi$ at a point $m \in M$ depends only on the values of ϕ near m, and as a result it makes sense to seek a local coordinate description of D.

1.1 Definition. A *linear partial differential operator* is a local operator which in every coordinate chart may be written

$$D = \sum_{|\alpha| \leq q} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}. \tag{1.1}$$

where the a_α are C^∞-functions. Here q is a non-negative integer and the sum is over non-negative integer multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ for which $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$. The *order* of D is the least q required to so represent D (in any coordinate chart).

To begin with we are mainly interested in one example. This is the *Laplace operator* Δ, also known as the *Laplace-Beltrami operator* on a closed Riemannian manifold. It is given by the compact formula $\Delta = \nabla^* \nabla$, where ∇ is the gradient operator from functions to tangent vector fields, and ∇^* is its adjoint, also called the divergence operator (up to a sign, these are direct generalizations to manifolds of the objects of the same name in vector calculus). In local coordinates the Laplace operator has the form

$$\Delta = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \text{order one operator.}$$

The order one term is a bit complicated (the exact formula is of no concern to us) but at the origin of a geodesic coordinate system all the coefficients of the order one term vanish, and we get

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \quad \text{at the origin,}$$

which is the familiar formula from ordinary vector calculus.

Our goal in Section 1 is to prove the following fundamental fact.

1.2 Theorem. *Let Δ be the Laplace operator on a closed Riemannian manifold. There is an orthonormal basis $\{\phi_j\}$ for the Hilbert space $L^2(M)$ consisting of smooth functions ϕ_j which are eigenfunctions for Δ:*

$$\Delta\phi_j = \lambda_j\phi_j, \quad \textit{for some scalar } \lambda_j.$$

The eigenvalues λ_j are non-negative and they tend to infinity as j tends to infinity.

It is possible to say a bit more. Since the functions ϕ_j constitute an orthonormal basis for $L^2(M)$, every function ϕ in $L^2(M)$ can be expanded as a series

$$\phi = \sum_{j=0}^{\infty} a_j\phi_j,$$

where the sequence of coefficients $\{a_j\}$ is square-summable. It turns out that $\phi \in L^2(M)$ is a smooth function if and only if the sequence $\{a_j\}$ is of rapid decay, which means that if $k \in \mathbb{N}$ then

$$\sup_j j^k |a_j| < \infty.$$

This should call to mind a basic fact in the theory of Fourier series: a function on the circle is smooth if and ony if its Fourier coefficient sequence is of rapid decay. Note that the basic functions in Fourier theory, the exponentials e^{inx}, constitute an orthonormal basis for $L^2(S^1)$ consisting of eigenfunctions for the Laplace operator on the circle, which is just $-\frac{d^2}{dx^2}$. So in some sense Theorem 1.2 establishes the first principles of Fourier theory on any closed Riemannian manifold.

The proof of Theorem 1.2 is more or less a resumé of a first course in functional analysis. In view of what we have said it will not surprise the reader to learn that the argument relies on one or two crucial computations in Fourier theory. But we shall also need to review various ideas from Hilbert space operator theory.

1.2 Unbounded Operators

An *unbounded operator* on a Hilbert space, H, is a linear transformation from a dense linear subspace of H into H. No continuity is assumed. When dealing with unbounded operators it is important to keep track of domains. Unbounded operators with different domains can't generally be added together in a reasonable way. Unbounded operators can't generally be composed in a reasonable way unless the range of the first is contained within the domain of the second.

An unbounded operator T is *symmetric* if

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

for all $v, w \in \operatorname{dom} T$. An unbounded operator is *self-adjoint* if it is symmetric and if in addition

$$\text{Range}(T \pm iI) = H.$$

In finite dimensions every symmetric operator is self-adjoint. In infinite dimensions self-adjointness is precisely the condition needed to get spectral theory going. Observe that if T is symmetric then

$$\|(T \pm iI)v\|^2 = \|Tv\|^2 + \|v\|^2,$$

which implies that if T is self-adjoint then the operators $T \pm iI$ map $\operatorname{dom} H$ one-to-one and onto H, so that they have well-defined inverses (which we regard as operators from H to itself).

1.3 Theorem. *Let T be a self-adjoint operator. There is a (unique) homomorphism from the algebra of bounded, continuous functions on $\mathbb{R}$ into $B(H)$ (the algebra of bounded operators on H) such that*

$$(x \pm i)^{-1} \mapsto (T \pm iI)^{-1}.$$

□

This is one version of the *Spectral Theorem*. It is proved by noting that the operators $(T \pm iI)^{-1}$ generate a commutative C^*-subalgebra of $B(H)$, and by then applying the basic theory of commutative C^*-algebras.

Note that once we have the Spectral Theorem we can define 'wave operators' e^{isT}, 'heat operators' e^{-sT^2}, and so on. Thus the result is conceptually very powerful.

Self-adjoint operators are hard to come by in nature. Typically the natural domain of an unbounded operator (e.g. the smooth, compactly supported functions in the case of a differential operator) must be enlarged, and the operator extended to this larger domain, so as to obtain a self-adjoint operator. Here is one procedure, due to Friedrichs, which we'll illustrate using the Laplace operator.

Let Δ be the Laplace operator on a Riemannian manifold M. The manifold need not be compact, or complete; it might have a boundary. Think of Δ as an unbounded operator on $H = L^2(M)$ whose domain is the space of smooth, compactly supported functions (on the interior of M, if M has boundary).

Observe that if $\phi \in \operatorname{dom}\Delta$ then

$$\langle \Delta\phi, \phi\rangle = \langle \nabla\phi, \nabla\phi\rangle \geq 0.$$

Let us exploit this to define a new inner product on $\operatorname{dom}\Delta$ by the formula

$$\langle \phi, \psi\rangle_1 = \langle (I+\Delta)\phi, \psi\rangle.$$

Denote by H_1 the Hilbert space completion of $\operatorname{dom}\Delta$ in this inner product. It is, among other things, a dense subspace of H (more about it later). Now denote by $H_2 \subseteq H_1$ the space of all ϕ for which there exists a vector $\theta \in H$ (which will be $(I+\Delta)\phi$) such that

$$\langle \theta, \psi\rangle = \langle \phi, \psi\rangle_1, \quad \forall \psi \in H_1.$$

1.4 Theorem (Friedrichs). *The operator* $I + \Delta$ *is self-adjoint on* H_2. □

The proof is a really good exercise. To get a self-adjoint extension of Δ, just subtract I from $I + \Delta$.

1.3 Sobolev Spaces

We are now going to investigate in a bit more detail the Hilbert space H_1 which appeared above. It appears as the space W_1 in the sequence of *Sobolev spaces* $W_0, W_1, W_2, \ldots$ associated to a closed manifold (and as it happens the Hilbert space W_2 is the same as the space H_2 that we defined in the last section, at least for a closed manifold, although the proof of that fact will be postponed for a while).

Although we are interested in function spaces associated to a manifold M, we shall begin not with M but with open sets in Euclidean space.

1.5 Definition. Let Ω be an open subset of $\mathbb{R}^n$ and let k be a non-negative integer. Denote by $W_k(\Omega)$ the completion of $C_c^\infty(\Omega)$ in the norm

$$\|\phi\|^2_{W_k(\Omega)} = \sum_{|\alpha|\leq k} \|\partial^\alpha \phi\|^2_{L^2(\Omega)}.$$

Thus the W_k-norm combines the L^2-norms of all the partial derivatives of ϕ of order k or less.

1.6 Definition. Let M be a manifold and let K be a compact subset of (the interior of) M. Define a Hilbertian space[1] $W_k(M|K)$ as follows.

Case 1. K is contained in a coordinate ball. Fix a diffeomorphism from a neighbourhood of K to an open set $\Omega \subseteq \mathbb{R}^n$, use the diffeomorphism to transfer the norm on $W_k(\Omega)$ to the smooth functions on M which are compactly supported within K, and then complete.

Case 2. K is any compact set. Choose smooth, compactly supported functions $\theta_1, \ldots, \theta_N$ on M, each supported in a coordinate ball, with $\sum \theta_j = 1$ on K, and let $K_j = \operatorname{supp} \theta_j$. Let $W_k(M|K)$ be the completion of the smooth functions on M which are compactly supported in K, in the norm

$$\|\phi\|^2_{W_k(M|K)} = \sum_j \|\theta_j \phi\|^2_{W_k(M|K_j)}.$$

In either case, the norms depend on coordinate choices, etc, but the underlying Hilbertian spaces do not. If we fix a smooth measure on M then all the spaces $W_k(M|K)$ can be thought of as linear subspaces of $L^2(M)$ (they are dense, if $K = M$).

The spaces $W_k(M|K)$ have the following invariance property: if Φ is a diffeomorphism carrying M onto an open subset of M', and if Φ maps K to K', then Φ carries $W_k(M'|K')$ isomorphically onto $W_k(M|K)$. Moreover pointwise multiplication by a smooth function is a bounded operator on each $W_k(M|K)$. Differential operators of order q map $W_{k+q}(M|K)$ continuously into $W_k(M|K)$.

If $K = M$, then we'll write $W_k(M)$ in place of $W_k(M|M)$. In this case (where M is compact) we can give an alternate, more concise, definition of the Sobolev spaces. The set of all order k, or less, differential operators is a finitely generated module over the ring of smooth functions on M. If $\{D_1, \ldots, D_N\}$ is a finite generating set then

$$\|\phi\|_{W_k(M)} \approx \|D_1\phi\|_{L^2(M)} + \cdots + \|D_N\phi\|_{L^2(M)}$$

(the symbol $\approx$ denotes equivalence of norms).

Recall that a bounded Hilbert space operator is *compact* if it carries the closed unit ball into a compact set (see Appendix A for a quick review of compact operator theory and related matters).

[1] A Hilbertian space is a vector space with an equivalence class of Hilbert space norms, two norms $\| \ \|_1$ and $\| \ \|_2$ being equivalent if there is a constant $C > 0$ such that $C^{-1}\| \ \|_1 \le \| \ \|_2 \le C\| \ \|_1$.

1.7 Rellich Lemma. *If* $k > 0$ *then the inclusion of* $W_k(M|K)$ *into* $L^2(M)$ *is a compact operator.*

Proof. Fix a partition of unity $\{\theta_j\}$ as in Definition 1.6, with each θ_j supported in a compact set K_j within a coordinate neighbourood U_j. The inclusion

$$W_k(M|K) \longrightarrow L^2(M)$$

can be broken down as a composition of maps

$$\begin{array}{ccc} W_k(M|K) & & \\ \downarrow & & \\ W_k(M|K_1) \oplus \cdots \oplus W_k(M|K_N) & \longrightarrow & L^2(U_1) \oplus \cdots \oplus L^2(U_N) \\ & & \downarrow \\ & & L^2(M), \end{array}$$

where the first vertical map is multiplication by θ_j in component j and the other maps are induced from the obvious inclusions. It clearly suffices to show that the inclusions $W_k(M|K_j) \to L^2(U_j)$ are compact operators. But if we embed U_j as an open set in a torus T_j then in view of the commuting diagram

$$\begin{array}{ccc} W_k(M|K_j) & \longrightarrow & L^2(U_j) \\ \downarrow & & \uparrow \\ W_k(T_j) & \longrightarrow & L^2(T_j), \end{array}$$

where the downward map is inclusion and the upward one is restriction to $U_j \subseteq T_j$, we see that it suffices to prove that the inclusion

$$W_k(T_j) \longrightarrow L^2(T_j)$$

is a compact operator. This is easily accomplished by using Fourier theory — see Appendix B. □

1.8 Remark. The same argument shows that if Ω is a bounded open set in $\mathbb{R}^n$ then the inclusion $W_k(\Omega) \subseteq L^2(M)$ is compact for all $k > 0$.

1.9 Lemma. *If* p *and* k *are non-negative integers, and if* $k > p + \frac{n}{2}$ *then* $W_k(M|K) \subseteq C^p(M|K)$*. As a result,*

$$\cap_k W_k(M|K) = C^\infty(M|K).$$

Proof. To prove that a function $\phi \in W_k(M|K)$ is in $C^p(M|K)$ it suffices to show that each $\theta_j\phi \in W_k(U_j|K_j)$ belongs to $C^p(U_j|K_j)$ (we are using the same notation as in the previous proof). After embedding U_j as an open set in a torus T_j, it suffices to show that $W_k(T_j) \subseteq C^p(T_j)$. Once again, this is easily proved using Fourier series — see Appendix B. □

1.4 Compact Resolvent

Let's return to the Laplace operator and its self-adjoint extension. Assume that the manifold M is closed. Recall that the 'intermediate' Hilbert space H_1 we constructed on the way to finding the Friedrichs extension of Δ was the completion of $C^\infty(M)$ in the norm

$$\|\phi\|_1^2 = \langle (I+\Delta)\phi, \phi\rangle = \|\phi\|_{L^2}^2 + \|d\phi\|_{L^2}^2.$$

From this it is easy to see that $H_1 = W_1(M)$. As a result, it follows from the Rellich lemma that

1.10 Theorem. *The bounded operator* $(I+\Delta)^{-1}$ *on* $L^2(M)$ *is compact.* □

Now remember from functional analysis that every compact positive-definite operator (such as $(I+\Delta)^{-1}$) has an orthonormal eigenbasis, whose corresponding eigenvalues constitute a sequence of positive numbers converging to zero (see Appendix A). Hence:

1.11 Theorem. *Let* Δ *be the self-adjoint operator on* $L^2(M)$ *obtained by the Friedrichs extension procedure from the Laplace operator on* M*. There is an orthonormal basis for* $L^2(M)$ *consisting of functions* $\phi_j \in \operatorname{dom}\Delta$ *which are eigenfunctions for* Δ*. The corresponding eigenvalues constitute a sequence of non-negative numbers converging to* ∞*.* □

1.12 Remark. We haven't yet shown that the ϕ_j are smooth functions, but at any rate we have that $\Delta\phi_j = \lambda_j\phi_j$ in the sense of distributions.

1.5 Weyl's Theorem

The solution to the problem of finding an orthonormal basis for $L^2(M)$ consisting of eigenfunctions of Δ was first great triumph of Hilbert space theory (in fact this is the problem which *began* Hilbert space theory — see [27]). Before we develop the theory any further, let us pause to prove the following very famous theorem of Weyl.

1.13 Theorem. *Let Δ be the (Friedrichs extension of the) Laplace operator on a closed Riemannian n-manifold or a smooth, bounded domain in $\mathbb{R}^n$. Let $N(\lambda)$ be the number of eigenvalues of Δ (multiplicities counted) less than λ. Then*

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\frac{n}{2}}}=\frac{\mathrm{Vol}(M)}{(4\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)}.$$

This is a little bit of a detour away from our main objectives, but it began a sequence of developments which ultimately led to the local index theory we shall be describing in these notes. For this and other reasons, Weyl's theorem is in some sense the first theorem of noncommutative geometry.

We'll deal with the case of domains in $\mathbb{R}^n$ (the case of manifolds is just a tiny bit harder), and to keep things as clear as possible we'll consider the dimension 2 case (although the case of general n is really no different). Thus for a smooth, bounded domain Ω in $\mathbb{R}^2$ we aim to prove that

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda}=\frac{\mathrm{Vol}(\Omega)}{4\pi}.$$

The first step is to check the result for some basic regions, namely rectangles.[2] This, incidentally, will fix the constant 4π.

1.14 Lemma. *Weyl's Theorem holds for rectangular domains.*

Proof. Let us work with the rectangle of width a and height b whose bottom left corner is the origin in the (x,y)-plane. For this domain an eigenbasis for the Laplace operator can be explicitly computed. The eigenfunctions are

$$u_{mn}(x,y)=\sin(m\tfrac{\pi}{a}x)\sin(n\tfrac{\pi}{b}x)$$

and the eigenvalues are $\lambda_{mn}=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$, where $m,n>0$. It follows that

$$\begin{aligned}N(\lambda)&=\#\left\{(n,m)\in\mathbb{N}\times\mathbb{N}\,\middle|\,\frac{m^2}{a^2}+\frac{n^2}{b^2}\leq\frac{\lambda}{\pi^2}\right\}\\&\sim\frac{1}{4}\left(\text{Area of Ellipse }\frac{X^2}{a^2}+\frac{Y^2}{b^2}\leq\frac{\lambda}{\pi^2}\right)\\&=\frac{ab\lambda}{4\pi}.\end{aligned}$$

[2]Weyl's Theorem holds for various non-smooth domains—as will become clear.

Thus

$$\frac{N(\lambda)}{\lambda} \sim \frac{\text{Area}(\Omega)}{4\pi},$$

as required. □

The proof of Weyl's Theorem is an eigenvalue comparison argument, based on the following simple observation.

1.15 Lemma. *Let S and T be compact and positive operators on a Hilbert space H, and denote by $\{\lambda_j(S)\}$ and $\{\lambda_j(T)\}$ the eigenvalue sequences of S and T. If*

$$\langle Sv, v\rangle \geq \langle Tv, v\rangle \geq 0,$$

for all $v \in H$, then $\lambda_j(S) \geq \lambda_j(T)$, for all j. □

Proof. This follows from Weyl's formula

$$\lambda_j(T) = \inf_{\dim(V)=j-1} \sup_{v \perp V} \frac{\|Tv\|}{\|v\|},$$

which is described in Appendix A, and the fact that if a bounded operator P is positive then

$$\|P\| = \sup_{\|v\|=1} \langle Pv, v\rangle.$$

□

The main step in the proof of Weyl's Theorem is now this:

1.16 Proposition. *Suppose that Ω_0 and Ω_1 are bounded open sets in the plane, and that $\Omega_0 \subseteq \Omega_1$. Then $N_{\Omega_0}(\lambda) \leq N_{\Omega_1}(\lambda)$, for all λ.*

Denote by Δ_{Ω_0} and Δ_{Ω_1} the Laplace operators for these two domains. The proposition (called the *Domain Dependence Inequality*) will follow if we can show that $\lambda_j(\Delta_{\Omega_0}) \geq \lambda_j(\Delta_{\Omega_1})$, for all j. This in turn will follow if we can show that $\lambda_j(\Delta_{\Omega_1}^{-1}) \geq \lambda_j(\Delta_{\Omega_0}^{-1})$, for all j. To this end we are of course going to apply Lemma 1.15, but first we have to overcome the small problem that although $\Delta_{\Omega_0}^{-1}$ and $\Delta_{\Omega_1}^{-1}$ are compact and positive operators, they are defined on different Hilbert spaces. To remedy this we regard $L^2(\Omega_0)$ as the subspace of $L^2(\Omega_1)$ consisting of functions which vanish on the complement of Ω_0 in Ω_1, and extend $\Delta_{\Omega_0}^{-1}$ to an operator on $L^2(\Omega_1)$ by definining it to be zero on the orthogonal complement of $L^2(\Omega_0)$. Having done so the proof of the Domain Dependence Inequality reduces to the following lemma.

1.17 Lemma. *Suppose that $\Omega_0 \subseteq \Omega_1$ and denote by Δ_{Ω_0} and Δ_{Ω_1} the Laplace operators for these two domains. If $\psi \in L^2(\Omega_1)$ then $\langle \Delta_{\Omega_1}^{-1}\psi, \psi\rangle \geq \langle \Delta_{\Omega_0}^{-1}\psi, \psi\rangle$.*

Proof. Let $\psi_1 \in L^2(\Omega_1)$ and denote by $\psi_0 \in L^2(\Omega_0)$ the restriction of ψ_1 to Ω_0. Write $\psi_1 = \Delta_{\Omega_1}\phi_1$ and $\psi_0 = \Delta_{\Omega_0}\phi_0$, where $\phi_0 \in \operatorname{dom}\Delta_{\Omega_0}$ and $\phi_1 \in \operatorname{dom}\Delta_{\Omega_1}$. Sorting out the notation, we see that what we need to prove is that

$$\langle \phi_1, \Delta_{\Omega_1}\phi_1\rangle \geq \langle \phi_0, \Delta_{\Omega_0}\phi_0\rangle$$

given that $\phi_0 \in \operatorname{dom}\Delta_{\Omega_0}$, that $\phi_1 \in \operatorname{dom}\Delta_{\Omega_1}$, and that the restriction of $\Delta_{\Omega_1}\phi_1$ to Ω_0 is equal to $\Delta_{\Omega_0}\phi_0$. These hypotheses certainly imply that

$$\langle \phi_0, \Delta_{\Omega_1}\phi_1\rangle = \langle \phi_0, \Delta_{\Omega_0}\phi_0\rangle.$$

Now apply the Cauchy-Schwarz inequality for the form $\langle \_ , \Delta_{\Omega_1} \_ \rangle$ on $W_1(\Omega)$ to the left hand side to complete the proof. $\square$

Proof of Weyl's Theorem. Let us first show that if Ω is any bounded open set then

$$\frac{\operatorname{Area}(\Omega)}{4\pi} \leq \liminf_{\lambda\to\infty} \frac{N_\Omega(\lambda)}{\lambda}. \tag{1.2}$$

(roughly speaking, this is 50% of Weyl's Theorem). Let I be a finite disjoint union of *open* rectangles I_k within Ω. Then $N_I(\lambda) \leq N_\Omega(\lambda)$, by the Domain Dependence Inequality. But since I is a disjoint union, we get that

$$N_I(\lambda) = \sum N_{I_k}(\lambda).$$

Moreover for each rectangle I_k it follows from Lemma 1.14 that

$$\lim_{\lambda\to\infty} \frac{N_{I_k}(\lambda)}{\lambda} = \frac{\operatorname{Area}(I_k)}{4\pi},$$

so that

$$\frac{\operatorname{Area}(I)}{4\pi} = \lim_{\lambda\to\infty}\frac{N_I(\lambda)}{\lambda} \leq \liminf_{\lambda\to\infty}\frac{N_\Omega(\lambda)}{\lambda}.$$

After approximating $\operatorname{Area}(\Omega)$ by $\operatorname{Area}(I)$ we get the required inequality (1.2).

To complete the proof, put Ω into a large rectangle R and denote by Ω' the complement of (the closure of) Ω in R. According to inequality (1.2),

$$\frac{\operatorname{Area}(\Omega)}{4\pi} + \frac{\operatorname{Area}(\Omega')}{4\pi} \leq \liminf_{\lambda\to\infty}\frac{N_\Omega(\lambda)}{\lambda} + \liminf_{\lambda\to\infty}\frac{N_{\Omega'}(\lambda)}{\lambda}.$$

But in addition $N_\Omega(\lambda) + N_{\Omega'}(\lambda) \le N(R)$, so that

$$\begin{aligned}\liminf_{\lambda\to\infty} \frac{N_\Omega(\lambda)}{\lambda} + \liminf_{\lambda\to\infty} \frac{N_{\Omega'}(\lambda)}{\lambda} &\le \limsup_{\lambda\to\infty} \frac{N_\Omega(\lambda)}{\lambda} + \liminf_{\lambda\to\infty} \frac{N_{\Omega'}(\lambda)}{\lambda} \\ &\le \limsup_{\lambda\to\infty} \frac{N_R(\lambda)}{\lambda} \\ &= \frac{\text{Area}(R)}{4\pi}.\end{aligned}$$

Since $\text{Area}(\Omega) + \text{Area}(\Omega') = \text{Area}(R)$ the proof is complete. □

1.6 Elliptic Operators

We shall now return to the analysis of the Laplace operator on a closed Riemannian manifold M. In the remainder of Section 1, which is a bit technical, we shall accomplish several things:

- Show that the domain H_2 of the Friedrichs extension of Δ is precisely the Sobolev space $W_2(M)$.
- Show that the eigenfunctions of Δ are in fact smooth functions on M.
- Indicate how to develop a similar eigenvalue analysis for operators more general than Δ.

The key to all this is to recognize the following local feature of the operator Δ which implies strong regularity properties for solutions of the equation $\Delta\phi = \psi$:

1.18 Definition. A linear partial differential operator D of order q is *elliptic of order* q if, in every local coordinate system, the local expression for D,

$$D = \sum_{|\alpha|\le q} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

has the property that

$$\Big|\sum_{|\alpha|=q} a_\alpha(x)\xi^\alpha\Big| \ge \varepsilon\big|\xi_1^2 + \cdots + \xi_n^2\big|^{\frac{q}{2}}$$

for every point x in the coordinate chart, some constant $\varepsilon > 0$ depending on x, and every $\xi \in \mathbb{R}^n$.

1.19 Example. If M is equipped with a Riemannian metric $[g_{ij}]$ then the associated Laplace operator Δ is elliptic of order 2. Indeed in local coordinates the formula for Δ is

$$\Delta = -\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \text{lower order terms.}$$

The required inequality is therefore

$$\sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j \geq \varepsilon \left(\xi_1^2 + \cdots + \xi_n^2\right),$$

which is an immediate consequence of the fact that the matrix $[g^{ij}(x)]$ is positive-definite.

1.7 Basic Estimate

Let M be a smooth closed manifold (equipped with a smooth measure, so we can form $L^2(M)$).

1.20 Definition. Let D be a linear partial differential operator on M of order q. We shall say that D is *basic* (this is not standard terminology) if

$$\|D\phi\|_{W_k(M)} + \|\phi\|_{L^2(M)} \approx \|\phi\|_{W_{k+q}(M)}$$

for all $k \geq 0$. Here ϕ is a smooth function on M, and the symbol $\approx$ means that the left and right hand side define equivalent norms on the space of all smooth functions on M.

1.21 Remark. The comparison $\lesssim$ holds for any order q operator, so the force of the definition is that $\gtrsim$ holds too. The latter we shall refer to as the *basic estimate* for D.

We are going to show that all elliptic operators are basic:

1.22 Theorem. *If D is an order q elliptic operator on a smooth closed manifold, then*

$$\|D\phi\|_{W_k(M)} + \|\phi\|_{L^2(M)} \gtrsim \|\phi\|_{W_{q+k}(M)}.$$

We shall prove this in the next subsection, but for motivation let us first show how the basic estimate implies a strong regularity property for elliptic operators. To keep things simple we'll focus on the Laplace operator Δ (the general case requires some small modifications which we shall mention at the end).

1.23 Lemma. *Let $\phi, \psi \in L^2(M)$ and assume that $\Delta\phi = \psi$ in the sense of distributions. If $\phi \in W_1(M)$, then in fact $\phi \in W_2(M)$.*

Sketch of the Proof. Assume first that ϕ has support in the interior of a compact set K in a coordinate neighbourhood. If κ is a compactly supported, non-negative bump function on $\mathbb{R}^n$ with total integral 1, and if K_ε is the operator of convolution with $\varepsilon^{-n}\kappa(\varepsilon^{-1}x)$ then it can be shown that

(a) If $\theta \in W_k(M|K)$ (supported in the coordinate neighbourhood) then $K_\varepsilon\theta \to \theta$ in $W_k(M|K)$ as $\varepsilon \to 0$.

(b) $[\Delta, K_\varepsilon]$ is uniformly bounded in ε as an operator from $W_{k+1}(M|K)$ to $W_k(M|K)$.

The family $\{K_\varepsilon\}$ is called a *Friedrichs mollifier*. Right now we'll only use the $k = 0$ properties of mollifiers, but later we'll consider $k > 0$. From the equation

$$\Delta K_\varepsilon \phi = K_\varepsilon \Delta\phi + [\Delta, K_\varepsilon]\phi$$

we see that $\{\Delta K_\varepsilon \phi\}_{\varepsilon>0}$ is uniformly bounded in $L^2(M)$. It therefore follows from the basic estimate that $\{K_\varepsilon\phi\}$ is uniformly bounded in $W_2(M)$. Since $K_\varepsilon\phi \to \phi$ in $L^2(M)$ it follows, after a little functional analysis, that in fact $\{K_\varepsilon\phi\}$ is actually convergent in $W_2(M)$, which implies $\phi \in W_2(M)$ as required.

In the general case, let θ be supported in a coordinate neighbourhood. Since $[\Delta, \theta]$ is a differential operator of order 1 we see that

$$\Delta\theta\phi = [\Delta, \theta]\phi + \theta\Delta\phi \in L^2(M),$$

and therefore $\theta\phi \in W_2(M)$ by the special case just considered. Varying θ, it follows that $\phi \in W_2(M)$, as required. □

1.24 Theorem. *Denote by Δ the Laplace operator on a closed Riemannian manifold. The domain of the Friedrichs extension of Δ is the Sobolev space $W_2(M)$.*

Proof. The domain of the Friedrichs extension is precisely the space of those $\phi \in W_1(M)$ for which $\Delta\phi$ (taken in the distributional sense) belongs to $L^2(M)$. So according to the lemma, if $\phi \in \operatorname{dom}\Delta$ then $\phi \in W_2(M)$. The reverse inclusion is easy. □

1.25 Theorem. *Let $\phi \in \operatorname{dom}\Delta$ and assume that $D\phi = \psi$ in the sense of distributions. If $\psi \in W_k(M)$, then $\phi \in W_{k+2}(M)$.*

Proof. This can be proved by the same Friedrichs mollifier method we used to prove Lemma 1.23. □

1.26 Theorem. *Let Δ be the Laplace operator on a closed Riemannian manifold. There is an orthonormal basis for $L^2(M)$ consisting of eigenfunctions for Δ, which are in fact smooth functions on M.*

Proof. As before, the Rellich Lemma implies that Δ has compact resolvent, and so the Spectral Theorem for compact operators applies to provide an eigenbasis. The eigenfunctions are in $\operatorname{dom}\Delta = W_2(M)$, and applying Theorem 1.25 repeatedly to the equation $\Delta\phi = \lambda\phi$ we see that $\phi \in \cap_k W_k(M)$. Hence ϕ is smooth by Lemma 1.9. □

1.27 Remark. If D is a symmetric elliptic operator of order q, then Theorems 1.24, 1.25 and 1.26 above continue to hold, although with the Sobolev space index "2" in the statements of 1.24 and 1.25 replaced by "q". The proofs are essentially the same once we introduce Sobolev spaces $W_k(M)$ with *negative* indices k (see Appendix B). Once this is done, the general version of Lemma 1.23 says that if $\psi \in W_k(M)$, and if ϕ is a distribution for which $D\phi = \psi$ in the sense of distributions, then in fact $\phi \in W_{k+q}(M)$. The proof is essentially the same, although it makes more serious use of the language of distributions.

1.8 Proof of the Basic Estimate

Before starting the proof of Theorem 1.22 we note the following fact:

1.28 Lemma. *Fix an integer $k > 0$. For every $\delta > 0$ there is a constant $C > 0$ such that*

$$\|\phi\|_{W_{k-1}(M)} \leq \delta\|\phi\|_{W_k(M)} + C\|\phi\|_{L^2(M)},$$

for all smooth functions ϕ.

Roughly speaking, this says that the W_k-norm is much stronger than the W_{k-1}-norm — only a tiny multiple of the former is needed to dominate the latter. Like just about everything else involving Sobolev spaces, the lemma is proved by reducing to the case of a torus, and doing an explicit Fourier series calculation there.

With the lemma in hand we can proceed.

Proof of Theorem 1.22. It will be helpful to introduce the following piece of terminology. We shall say that a differential operator D which is defined on some open set $U \subseteq M$ *satisfies the basic estimate over* U if for every compact subset K of U the inequality

$$\|D\phi\|_{W_k(U)} + \|\phi\|_{L^2(U)} \geq \varepsilon\|\phi\|_{W_{k+q}(U)}$$

holds, for some $\varepsilon > 0$ depending on K and k, and all ϕ supported in K.

The first step in the proof is to observe that if D_0 is a *constant coefficient* order q elliptic operator, defined in some coordinate neighbourhood U of M, then D_0 satisfies the basic estimate over U. This is an exercise in Fourier theory.

The next step is this. If D is a general order q elliptic operator, if $x \in M$, and if D_x is the constant coefficient operator obtained by freezing the coefficients of D at x, then for every $\varepsilon > 0$ there is a small neighbourhood U of x for which

$$\|D\phi - D_x\phi\|_k \leq \varepsilon\|\phi\|,$$

for every ϕ supported in U. This follows from the fact that the coefficients of $D - D_x$ vanish at x, as a result of which, $D - D_x$ can be written as a sum of terms ψE, where ψ is a smooth function vanishing at x and E is an order q operator.

From the first two steps, it follows that for every $x \in M$ there is a neighbourhood U of x such that the basic elliptic estimate holds for D over U.

Now cover M by finitely many open sets over each of which the basic elliptic estimate for D holds, and let $\{\theta_j\}$ be a smooth partition of unity which is subordinate to this cover. Write

$$\begin{aligned}
\|\phi\|_{r+k} &= \|\sum_j \theta_j\phi\|_{r+k} \\
&\leq \sum_j \|\theta_j\phi\|_{r+k} \\
&\lesssim \sum_j \|\Delta\theta_j\phi\|_k + \sum_j \|\theta_j\phi\|_0 \\
&\leq \sum_j \|\theta_j\Delta\phi\|_k + \sum_j \|[\Delta,\theta_j]\phi\|_k + \sum_j \|\theta_j\phi\|_0
\end{aligned}$$

In the middle inequality we have invoked the basic elliptic estimates over the sets in the cover; everything else is just algebra. Since multiplication by θ_j is continuous on each Sobolev space we obtain from the above sequence of inequalities the estimate

$$\|\phi\|_{q+k} \lesssim \|\Delta\phi\|_k + \|\phi\|_0 + \sum_j \|[\Delta,\theta_j]\phi\|_k$$

Finally, the operators $[\Delta,\theta_j]$ are of order $q-1$, or less, and as a result

$$\sum_j \|[\Delta,\theta_j]\phi\|_k \lesssim \|\phi\|_{k+q-1}$$

Combining this with Lemma 1.28 we get

$$\|\phi\|_{q+k} \lesssim \|\Delta\phi\|_k + K\|\phi\|_0 + \delta\|\phi\|_{k+q}$$

in which we can make δ as small as we like, say $\delta < 1$. The theorem now follows just by rearranging the terms in this inequality. □

2 Zeta Functions

In this section we shall study further the eigenvalue sequence $\{\lambda_j\}$ associated to the Laplace operator on a closed Riemannian manifold M of dimension n. The main result will be Theorem 2.2 below, although not only the result but also the proof will be important for our later purposes.

Let $z \in \mathbb{C}$. Define a sort of zeta function for M using the formula

$$\zeta_M(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-\frac{s}{2}}.$$

The definition makes sense in view of the following computation:

2.1 Lemma. *There is some* $d \in \mathbb{R}$ *such that if* $\mathrm{Re}(s) > d$ *then*

$$\sum_{\lambda_j \neq 0} |\lambda_j^{-\frac{s}{2}}| < \infty.$$

Proof. According to Weyl's Theorem we can take $d = n = \dim(M)$, which is the optimal value. However, if we are content with some value for d (not the best) then we can prove the lemma with less effort. We can take, for example, any even integer d bigger than n. It follows from the basic estimate proved in Section 1 that the operator $(I + \Delta)^{-\frac{d}{2}}$ maps $L^2(M)$ into $W_d(M)$, and since the inclusion of $W_d(M)$ into $L^2(M)$ is a trace-class operator (see Appendix B) it follows that $(I+\Delta)^{-\frac{d}{2}}$, viewed as an operator on $L^2(M)$, is trace-class. Its eigenvalue sequence is therefore summable, and it follows from this that $\sum \lambda_j^{-\frac{d}{2}} < \infty$, as required. □

Let us disregard Weyl's Theorem for a moment and refer to the smallest d with the property of the lemma as the *analytic dimension* of M. Our main result will give an independent proof that $d = n$.

Basic analysis proves that $\zeta_M(s)$ is analytic in the region $\mathrm{Re}(s) > d$. We are going to prove the following remarkable fact.

2.2 Theorem. *Let* $\{\lambda_j\}$ *be the eigenvalue sequence for the Laplace operator on a closed Riemannian* n*-manifold* M*. The zeta function*

$$\zeta_M(s) = \sum_{\lambda_j \neq 0}^{\infty} \lambda_j^{-\frac{s}{2}}$$

extends to a meromorphic function on the complex plane. The only singularities of the zeta function are simple poles, and these are located within the set of integer points $n, n-1, n-2, \ldots$.

2.3 Example. If $M = S^1$ then the zeta function ζ_M is precisely twice the famous Riemann zeta function. This explains our terminology and of course illustrates the phenomenon of meromorphic continuation.

Theorem 2.2 was discovered in the 1940's, by Minakshisundaram and Pleijel [23] in connection with attempts to refine Weyl's Theorem. The relation with Weyl's Theorem is made clear by the following *Tauberian Theorem* (see for example Hardy's book *Divergent Series* [16]):

2.4 Theorem. *Let* $\{\mu_j\}$ *be a sequence of positive real numbers and assume that it is* p*-summable for all* $p > 1$*. For* $\mu > 0$ *denote by* $M(\mu)$ *the number of* j *such that* $\mu_j > \mu$*. Then*

$$\lim_{s \searrow 1}\left((s-1)\sum_{j=1}^{\infty} \mu_j^s\right) = C \quad \Leftrightarrow \quad \lim_{\mu \to 0} \mu \cdot M(\mu) = C.$$

Thanks to the Tauberian theorem, putting $\mu_j = \lambda_j^{-\frac{n}{2}}$ we see rather easily that if ζ_M has a pole at $s = n$ then the eigenvalues of Δ satisfy the asymptotic relation

$$N(\lambda) \sim \frac{\mathrm{Res}_{s=n}\, \zeta_M(s)}{n} \cdot \lambda^{\frac{n}{2}}$$

(here $N(\lambda)$ is the counting function from Weyl's Theorem). It follows that the analytic dimension of M is equal to n, the topological dimension. Moreover Weyl's Theorem follows from the meromorphic continuation of $\zeta_M(s)$, plus a computation of the residue of the zeta function at $s = n$. Or, to put it in a better way, Weyl's Theorem, plus the Tauberian Theorem, show that the residues of the zeta function $\zeta_M(s)$ contain important geometric information about M. This is a theme we shall be developing throughout the rest of these notes.

2.1 Outline of the Proof

The proof of Theorem 2.2 will involve some Hilbert spectral theory and some algebra, notably the fundamental 'Heisenberg commutation relation'

$$[\frac{d}{dx}, x] = I$$

in the algebra of differential operators. It is closely related to Guillemin's proof of Weyl's Theorem in [15] (for a different proof based on pseudodifferential operators see [25]).

Here is the basic idea. If $x_1, \ldots, x_n$ are local coordinates on M then it follows from Heisenberg's relation that if D is *any* differential operator of order q or less then

$$qD = \sum_{i=1}^{N} [D, x_i] \frac{\partial}{\partial x_i} + R, \tag{2.1}$$

where R is a differential operator of order $q - 1$ or less. As a result of this, a little bit of algebra shows that

$$(q+n)D = \sum_{i=1}^{n} [D, x_i \frac{\partial}{\partial x_i}] + \sum_{i=1}^{n} [\frac{\partial}{\partial x_i}, x_i D] + R, \tag{2.2}$$

with the same remainder term R.

Now, we are going to show that the same sort of formula as (2.1) holds if D is replaced by a more complicated operator, roughly speaking one of the form $D\Delta^{-z}$, to which we shall assign the "order" $q - 2\,\mathrm{Re}(z)$. As for the operator Δ^{-z}, if the real part of z is positive then we can define it to be the unique bounded operator such that on eigenfunctions $\Delta^{-z}\phi_j = \lambda_j^{-z}\phi_j$ (we define the complex powers of the zero eigenvalue to be zero). We shall give a more useful description of this operator in the next subsection, but for the moment we note the key property

$$\mathrm{Trace}(\Delta^{-z}) = \sum_{\lambda_j \neq 0} \lambda_j^{-z}, \quad \text{when } \mathrm{Re}(z) > d.$$

Having found an analog of (2.1) for $D\Delta^{-z}$, it will follow that $D\Delta^{-z}$ may be substituted into (2.2) in place of D to give an equation

$$(q+n)D\Delta^{-z} = \sum_{i=1}^{n} [D\Delta^{-z}, x_i \frac{\partial}{\partial x_i}] + \sum_{i=1}^{n} [\frac{\partial}{\partial x_i}, x_i D\Delta^{-z}] + R_z,$$

The remainder term will be a combination of operators of the same general type as $D\Delta^{-z}$ but of "order" one less than $D\Delta^{-z}$.

Obtaining this formula for $D\Delta^{-z}$ is the crucial step, and from here on the rest of the proof is simple. Taking traces, and bearing in mind that the trace of a commutator is zero, we shall get

$$\mathrm{Trace}(D\Delta^{-z}) = \frac{1}{q - 2z + n} \mathrm{Trace}(R_z).$$

If we repeat the whole process, with $D\Delta^{-z}$ replaced by the remainder R_z, and then with R_z replaced by the new remainder, and so on, then we shall get

$$\mathrm{Trace}(D\Delta^{-z}) = \left(\prod_{j=0}^{J-1} \frac{1}{q - j - 2z + n}\right) \mathrm{Trace}(S_z),$$

where S_z has order $q - 2z - J$. But as J gets really large then we see from the Rellich Lemma that $\mathrm{Trace}(S_z)$ becomes well-defined and holomorphic on an increasingly large half-plane in $\mathbb{C}$. So the formula determines meromorphic extension of $\mathrm{Trace}(D\Delta^{-z})$ to any desired half-plane in $\mathbb{C}$, and hence to $\mathbb{C}$ itself.

2.2 Remark on Orders of Differential Operators

If D_1 and D_2 are differential operators then the order of $D_1 D_2$ is usually the sum of the orders of D_1 and D_2. However the order of the *commutator* $[D_1, D_2]$ is never more than the sum of the orders of D_1 and D_2 *minus* 1. This drop in degree is very important for the arguments that we are going to develop. It implies that taking the commutator of an operator D with a function lowers the degree of D by one; taking the commutator of D with a vector field does not change the degree; and taking the commutator of D with Δ raises the degree by at most one.

If we work with more general rings of differential operators (for example acting on sections of vector bundles) then the general fact about $[D_1, D_2]$ no longer holds, and one must take a little care to check that the consequences listed above hold in sufficiently generality for the arguments below to work (they *do* work).

2.3 The Actual Proof

On a closed manifold there do not exist global coordinates $x_1, \ldots, x_n$. But by using a partition of unity $\{\phi_\alpha\}$ subordinate to a cover of M by coordinate charts, we can easily find functions $A_1, \ldots, A_N$ and vector fields $B_1, \ldots, B_n$ such that

$$\sum_{j=1}^{N} [B_j, A_j] = nI$$

and

$$qD = \sum_{j=1}^{N} [D, A_j] B_j + R,$$

where as before D is an operator of order q or less and R has order less than q. (The operators A_i are of the form $\psi_\alpha \cdot x_i$, where ψ_α is supported in the α'th coordinate chart and is 1 on the support of ϕ_α, and the operators B_i are of the form $\phi_\alpha \cdot \frac{\partial}{\partial x_i}$.) For the purposes of the commutator argument sketched in the last section the A_j and B_j work just as well as the coordinates x_j and vector fields $\frac{\partial}{\partial x_j}$.

So let us begin by attempting to compute an expression of the form $[\Delta^{-z}, A]B$. For this purpose we shall need a way of looking at the operator Δ^{-z} which is better suited to computation. We shall use the Cauchy formula

$$\Delta^{-z} = \frac{1}{2\pi i}\int \lambda^{-z}(\lambda - \Delta)^{-1}\, d\lambda.$$

The integral is a contour integral along a downwards pointing vertical line in $\mathbb{C}$ which separates 0 from the eigenvalues of Δ. It is not hard to check that if $\mathrm{Re}(z) > 0$ and if $\phi \in C^\infty(M)$ then by applying the integrand to ϕ we get a convergent integral in each Sobolev space $W_k(M)$, so the integral defines an operator from $C^\infty(M)$ to $C^\infty(M)$. Cauchy's formula from complex analysis proves that this is the same as the operator Δ^{-z} we defined previously.

Now, onwards with the computation, the first part of which is straightforward:

$$\begin{aligned}
[\Delta^{-z}, A]B &= \frac{1}{2\pi i}\int \lambda^{-z}[(\lambda - \Delta)^{-1}, A]B\, d\lambda \\
&= \frac{1}{2\pi i}\int \lambda^{-z}(\lambda - \Delta)^{-1}[\Delta, A](\lambda - \Delta)^{-1}B\, d\lambda \\
&= \int \lambda^{-z}(\lambda - \Delta)^{-1}[\Delta, A]B(\lambda - \Delta)^{-1}\, d\lambda \\
&\quad + \int \lambda^{-z}(\lambda - \Delta)^{-1}[\Delta, A](\lambda - \Delta)^{-1}[\Delta, B](\lambda - \Delta)^{-1}\, d\lambda.
\end{aligned}$$

(In the last step we did two things at once: we commuted B past $(\lambda-\Delta)^{-1}$ and we then used the formula $[S^{-1}, T] = S^{-1}[T, S]S^{-1}$.) The operators $[\Delta, A]$ and $[\Delta, B]$ have orders 1 and 2, respectively.

Before going on, we shall introduce some better notation for our contour integrals.

2.5 Definition. If $D_0, \ldots, D_p$ are differential operators on the closed manifold M, then denote by $I_z(D_0, \ldots, D_p)$ the integral

$$\frac{1}{2\pi i}\int \lambda^{-z} D_0(\lambda - \Delta)^{-1} \cdots D_p(\lambda - \Delta)^{-1}\, d\lambda$$

(in the integral, copies of $(\lambda-\Delta)^{-1}$ alternate with the operators D_j). The integral converges if $\mathrm{Re}(z) < n$, in the sense we discussed above, and defines an operator on $C^\infty(M)$.

Using our new notation, we can write $D\Delta^{-z} = I_z(D)$ and by elaborating very slightly the computation we just ran through, we see that

$$[I_z(D), A]B = I_z([D,A]B) + I_z(D,[\Delta,A]B) + I_z(D,[\Delta,A],[\Delta,B]).$$

So what? Well, after replacing A and B by A_j and B_j, and summing over j, we know that

$$\sum_{j=1}^{N}[D,A_j]B_j = qD - R \quad \text{and} \quad \sum_{j=1}^{N}[\Delta,A_j]B_j = 2\Delta - S$$

where the 'remainder' S has order 1. We are going to plug these formulas into our expression for $[I_z(D), A]B$. To prepare for this, let us introduce the following terminology:

2.6 Definition. We shall say that $I_z(D_0,\dots,D_p)$ is an *integral of type* $\ell \in \mathbb{Z}$ if

$$\mathrm{order}(D_0) + \cdots \mathrm{order}(D_p) - 2p \le \ell.$$

2.7 Lemma. *If* $I_z = I_z(D_0,\dots,D_p)$ *is any integral of type* ℓ *then*

$$\sum_{j=1}^{N}[I_z,A_j]B_j = (\ell - 2z)I_z + R_z,$$

where R_z *is a finite sum of integrals of type* $\ell - 1$.

Proof. Let us just consider the case of the integral $I_z(D)$ (thus $\ell = q = \mathrm{order}(D)$); the other cases are no harder. Using the formulas we have already obtained we get

$$\sum_{j=1}^{N}[I_z(D),A_j]B_j = qI_z(D) + 2I_z(D,\Delta) + \text{type } \ell-1 \text{ integrals.}$$

So the lemma will be proved if we can deal successfully with $I_z(D,\Delta)$. What we need to show is that

$$I_z(D,\Delta) = -zI_z(D), \tag{2.3}$$

at least modulo integrals of order $\ell - 1$. But in fact (2.3) holds *exactly*. To see why this is so, note first that from the formula

$$\Delta(\lambda-\Delta)^{-1} = \lambda(\lambda-\Delta)^{-1} - I,$$

along with our definition of the integrals I_z it follows that

$$I_z(D,\Delta) = I_{z-1}(D,I) - I_z(D)$$

So (2.3) is equivalent to the formula

$$I_{z-1}(D,I) = (1-z)I_z(D).$$

This functional equation is proved using calculus, as follows. Take the integral which defines $I_{z-1}(D)$ and differentiate the integrand with respect to λ (the integrand is of course a function of λ). We get

$$\frac{d}{d\lambda}\left(\lambda^{1-z}D(\lambda-\Delta)^{-1}\right) = (1-z)\lambda^{-z}D(\lambda-\Delta)^{-1} - \lambda^{1-z}D(\lambda-\Delta)^{-2}.$$

Using the fact that the integral of this derivative is zero we get

$$(1-z)I_z(D) - I_{z-1}(D,I) = 0,$$

as required. □

2.8 Remark. In the general case the functional equation is

$$(1-z)I_z(D_0,\ldots,D_p) = \sum_{j=0}^{p-1} I_{z-1}(D_0,\ldots,D_j,I,D_{j+1},\ldots,D_p).$$

At this stage we have almost proved our meromorphic continuation theorem. Using the algebraic tricks described earlier we can reduce the problem of computing the trace of an integral of type ℓ to the problem of computing the trace of an integral of type $\ell-1$. It only remains to relate our notion of "type" to some notion of "order" of operators, so that we can guarantee the traceability of I_z, for all z in a suitable right half plane.

2.9 Definition. Let m be an integer (positive or negative). We shall say that a linear operator $T\colon C^\infty(M) \to C^\infty(M)$ has *analytic order* m *or less* if, for every $s \in \mathbb{Z}$ such that $s \geq 0$ and $s+m \geq 0$, the operator T extends to a continuous linear operator from $W_{m+s}(M)$ to $W_s(M)$.

Thus for example every differential operator of order q or less has analytic order q or less. If $\operatorname{Re}(z) \leq -m$ then the operator Δ^{-z} has analytic order $-2m$, or less.

To prove Theorem 2.2 using our commutator strategy it remains to prove the following two results:

2.10 Lemma. *If* $I_z(D_0, \ldots, D_p)$ *is an integral of type* ℓ *then*

$$\text{analytic order}(I_z(D_0, \ldots, D_p)) \leq \ell - 2\operatorname{Re}(z).$$

2.11 Remark. The integrand which we use to define $I_z(D_0, \ldots, D_p)$ is

$$\lambda^{-z} D_0 (\lambda - \Delta)^{-1} \cdots D_p (\lambda - \Delta)^{-1}.$$

This has order $q - 2p + 2$. So when $\operatorname{Re}(z)$ is negative (recall that the integral is defined as long as $\operatorname{Re}(z) + p > 0$) the order estimate in the lemma (which is sharp) is considerably better than one would expect by looking at the integrand alone.

To understand the content of the following lemma, recall that the integral defining $I_z(D_0, \ldots, D_p)$ is convergent when $\operatorname{Re}(z) > n$, and that we have not up to this point defined the integral for other values of z. However, thanks to the previous lemma, the quantity $\operatorname{Trace}(I_z(D_0, \ldots, D_p))$ is defined in the domain $\operatorname{Re}(z) > \max\{n, \frac{n-\ell}{2}\}$ (this is where the integral makes sense and converges to an operator of order less than $-n$).

2.12 Lemma. *If* $I_z(D_0, \ldots, D_p)$ *is an integral of type* ℓ *then the function*

$$z \mapsto \operatorname{Trace}(I_z(D_0, \ldots, D_p)),$$

extends to a holomorphic function on the half-plane $\operatorname{Re}(z) > \frac{n-\ell}{2}$.

Lemmas 2.10 and 2.12 are both proved by the same explicit computation. To get the basic idea, let's pretend that the operators D_0 *commute* with the operator Δ. In this case the integral $I_z(D_0, \ldots, D_p)$ can be written as

$$\frac{1}{2\pi i} \int D_0 \ldots D_p (\lambda - \Delta)^{-(p+1)} \, d\lambda.$$

The "constant" $D_0 \ldots D_p$ can be pulled out from under the integral sign, and what is left can be evaluated by Cauchy's integral formula. We get

$$I_z(D_0, \ldots, D_p) = \binom{-z}{k} D_0 \cdots D_p \Delta^{-z-p}.$$

With this formula in hand, both lemmas are obvious.

Proof of Lemmas 2.10 and 2.12. The idea of the proof is to try to move all the terms $(\lambda - \Delta)^{-1}$ which appear in the basic quantity

$$X^0(\lambda - \Delta)^{-1} \cdots X^p(\lambda - \Delta)^{-1}$$

toward the right using the identity

$$\begin{aligned}(\lambda - \Delta)^{-1}T &= T(\lambda - \Delta)^{-1} + [(\lambda - \Delta)^{-1}, T] \\ &= T(\lambda - \Delta)^{-1} + (\lambda - \Delta)^{-1}[\Delta, T](\lambda - \Delta)^{-1}.\end{aligned}$$

The formula leads to the formal expansion

$$(\lambda - \Delta)^{-1}T \approx \sum_{k \geq 0} T^{(k)}(\lambda - \Delta)^{-1-k}$$

where we have used the notation

$$T^{(0)} = T \quad \text{and} \quad T^{(k)} = [\Delta, T^{(k-1)}] \quad \text{for } k \geq 1.$$

The series does not converge, but instead it is an asymptotic formula in the following sense: if T and T_α depend on a parameter λ, then we shall write $T \approx \sum_\alpha T_\alpha$ if, for every $m \ll 0$, every sufficiently large finite partial sum agrees with T up to an operator of analytic order m or less, whose norm as an operator from $W_{s+m}(M)$ to $W_s(M)$ is $O(|\lambda|^m)$. In our case if we truncate our series at $k = K$, then the remainder term is

$$(\lambda - \Delta)^{-1}T^{(K+1)}(\lambda - \Delta)^{-K-1}$$

and the asymptotic expansion condition is easily verified. The reason for including the $O(|\lambda|^m)$ condition is that we shall then be able to integrate with respect to λ, and obtain an asymptotic expansion for the integrated operator.

More generally one has, for any non-negative integer h, an asymptotic expansion

$$(\lambda - \Delta)^{-h}T \approx \sum_{k \geq 0} (-1)^k \binom{-h}{k} Y^{(k)}(\lambda - \Delta)^{-1-k}$$

(this can be proved by induction on h).

Before beginning the actual computation let us also define the quantities

$$c(k_1, \ldots, k_j) = \frac{(k_1 + \cdots + k_j + j)!}{k_1! \cdots k_j!(k_1 + 1) \cdots (k_1 + \cdots + k_j + j)},$$

which depend on non-negative integers $k_1, \ldots, k_j$. These have the property that $c(k_1) = 1$, for all k_1, and

$$\frac{c(k_1, \ldots, k_j)}{c(k_1, \ldots, k_{j-1})} = \binom{k_1 + \cdots + k_j + j - 1}{k_j}$$

(to be explicit, the right hand fraction is the product of the k_j successive integers from $(k_1 + \cdots + k_{j-1} + j)$ to $(k_1 + \cdots + k_j + j - 1)$, divided by $k_j!$).

Now we can begin. Using this notation we obtain an asymptotic expansion

$$(\lambda - \Delta)^{-1} D_1 \approx \sum_{k_1 \geq 0} c(k_1) D_1^{(k_1)} (\lambda - \Delta)^{-(k_1+1)},$$

and then

$$\begin{aligned} (\lambda - \Delta)^{-1} D_1 (\lambda - \Delta)^{-1} D_2 &\approx \sum_{k_1 \geq 0} c(k_1) D_1^{(k_1)} (\lambda - \Delta)^{-(k_1+2)} X^2 \\ &\approx \sum_{k_1, k_2 \geq 0} c(k_1, k_2) D_1^{(k_1)} D_2{}^{(k_2)} (\lambda - \Delta)^{-(|k|+2)}, \end{aligned}$$

where $|k| = k_1 + k_2$, and finally

$$(\lambda - \Delta)^{-1} D_1 \cdots (\lambda - \Delta)^{-1} D_p \approx \sum_{k \geq 0} c(k) D_1{}^{(k_1)} \cdots D_p{}^{(k_p)} (\lambda - \Delta)^{-(|k|+p)},$$

where we have written $k = (k_1, \ldots, k_p)$ and $|k| = k_1 + \cdots + k_p$. Premultiplying by D_0, postmultiplying by $(\lambda - \Delta)^{-1}$, and integrating with respect to λ we get

$$\begin{aligned} &\frac{1}{2\pi i} \int \lambda^{-z} D_0 (\lambda - \Delta)^{-1} \cdots D_p (\lambda - \Delta)^{-1} \, d\lambda \\ &\qquad \approx \sum_{k \geq 0} c(k) D_0 D_1^{(k_1)} \cdots D_p^{(k_p)} \binom{-z}{|k| + p} \Delta^{-z-|k|-p}. \end{aligned}$$

The terms of this expansion have analytic order

$$q - k - 2(\mathrm{Re}(z) + p) = \ell - k - 2\,\mathrm{Re}(z)$$

or less. This proves Lemma 2.10. If $\mathrm{Re}(z) > \frac{1}{2}(n - \ell)$ then all the terms in the asymptotic expansion are trace-class. This proves Lemma 2.12. □

Having proved the lemmas, the proof of Theorem 2.2 follows by using the method outlined in Subsection 2.1. Let us add one or two small remarks about the vanishing of traces of commutators. It is a fundamental property of the trace that if X and Y and bounded, and if one of them is trace-class, then $\mathrm{Trace}(XY) = \mathrm{Trace}(YX)$. The situation here is a little more complicated because we are considering the traces of commutators of possibly unbounded operators. To see that the traces still vanish, we use Sobolev spaces, as follows. First, we may assume that $\mathrm{Re}(z) \gg 0$ (if the trace of the commutator vanishes here it will vanish everywhere the commutator is defined, by unique analytic continuation). Next we note that the trace of an operator Z of analytic order $-\ell \ll -d$ is the same, whether we regard Z as an operator on $L^2(M)$ or on any Sobolev space $W_k(M)$ with $k \ll \ell$.[3] Indeed if we denote by $J\colon W_k(M) \to L^2(M)$ the inclusion, and by Z_k the operator Z acting on $W_k(M)$, then we can write

$$\mathrm{Trace}(Z_k) = \mathrm{Trace}(J^{-1}ZJ).$$

Since $J\colon W_k(M) \to L^2(M)$ is a bounded operator and $J^{-1}Z\colon L^2(M) \to W_k(M)$ is trace-class (when $\ell \gg d + k$) we get

$$\mathrm{Trace}((J^{-1}Z)J) = \mathrm{Trace}(J(J^{-1}Z)) = \mathrm{Trace}(Z).$$

Finally, if we wish to show that $\mathrm{Trace}(XY) = \mathrm{Trace}(YX)$ when say X has bounded order q and Y has order $\ell \ll -d$ we can think of XY and YX as compositions of bounded operators and trace-class

$$L^2(M) \xrightarrow{Y} W_q(M) \xrightarrow{X} L^2(M)$$

and

$$W_q(M) \xrightarrow{X} L^2(M) \xrightarrow{Y} W_q(M)$$

and apply the basic trace property together with the previous remark to $Z = YX$.

An Improvement of the Main Theorem

In this concluding subsection we shall improve a little Theorem 2.2 by proving that a number of the singularities of $\mathrm{Trace}(I_x(D_0, \ldots, D_p))$, including in particular the singularity at $z = 0$, are removable. As we shall see in Section 5, this is quite significant for index theory. Moreover the appearance of the Gamma function in the following lemma will prepare the way for our later computations in cyclic cohomology.

[3] With a bit more effort one can show that the same thing holds for all $-\ell < -d$ and all k.

2.13 Lemma. *If* $I_z(D_0, \ldots, D_p)$ *is an integral of type* k *then the function*

$$z \mapsto \Gamma(z)\,\mathrm{Trace}(I_z(D_0, \ldots, D_p)),$$

is holomorphic in the domain $\mathrm{Re}(z) > \frac{n-k}{2}$.

The content of the lemma is that $\mathrm{Trace}(I_z(D_0, \ldots, D_p))$ has zeros at the non-positive integer points in its domain $\mathrm{Re}(z) > \frac{n-k}{2}$, which cancel out the simple poles of the Γ-function. The factor $(-1)^p$ is present for tidiness; it also plays a useful role in subsequent developments within cyclic cohomology (see Section 6).

Proof. The argument used to prove Lemma 2.12 produces the formula

$$\begin{aligned}&(-1)^p\Gamma(z)\,\mathrm{Trace}(I_z(D_0, \ldots, D_p))\\ &\qquad\approx \sum_{k\geq 0}(-1)^p\Gamma(z)\binom{-z}{|k|+p}c(k)\,\mathrm{Trace}\left(D_0 D_1^{(k_1)}\cdots D_p^{(k_p)}\Delta^{-z-|k|-p}\right).\end{aligned}$$

The symbol $\approx$, which we are now applying to functions of z, means that, given any right half-plane in $\mathbb{C}$, any sufficiently large finite partial sum of the right hand side agrees with the left hand side (on the common domain of the functions involved) modulo a function of z which is holomorphic in that half-plane. It follows from the functional equation for $\Gamma(z)$ that

$$(-1)^p\Gamma(z)\binom{-z}{|k|+p} = (-1)^{|k|}\Gamma(z+p+|k|)\frac{1}{(|k|+p)!}.$$

So we get

$$\begin{aligned}&(-1)^p\Gamma(z)\,\mathrm{Trace}(I_z(D_0, \ldots, D_p))\\ &\qquad\approx \sum_{k\geq 0}(-1)^{|k|}\Gamma(z+p+|k|)\frac{1}{(|k|+p)!}c(k)\\ &\qquad\qquad\times\,\mathrm{Trace}\left(D_0 {D_1}^{(k_1)}\cdots {D_p}^{(k_p)}\Delta^{-z-|k|-p}\right).\end{aligned}$$

This completes the proof. □

Repeating the argument from the previous subsection we obtain the following result:

2.14 Theorem. *Let* $I_z(D_0, \ldots, D_p)$ *be an integral of type* k. *The function*

$$(-1)^p\Gamma(z)\,\mathrm{Trace}(I_z(D_0, \ldots, D_p))$$

extends to a meromorphic function on $\mathbb{C}$ *with only simple poles. The poles are located within the sequence* $n+k, n+k-1, \ldots$. □

3 Abstract Differential Operators

In this section we shall first introduce a more abstract notion of differential operator, and then develop a corresponding theory of pseudodifferential operators. Apart from the standard example coming from standard differential operators on a smooth, closed manifold, we shall also consider a more elaborate example related to foliation theory, and a collection of examples derived from Alain Connes' notion of spectral triple.

3.1 Algebras of Differential Operators

Let H be a complex Hilbert space. We shall assume as given an unbounded, positive, self-adjoint operator Δ on H. As the notation might suggest, the main example to keep in mind is the Laplace operator on a closed Riemannian manifold, but there are many other examples too. We shall soon introduce a notion of "order", generalizing the notion of order of a standard differential operator, and we should keep in mind that Δ need not have order 2. In fact let us now fix an integer $r > 0$, which will play the role in what follows of order(Δ).

For $k \geq 0$ denote by H^k the domain of the operator $\Delta^{\frac{k}{r}}$. In the standartd example, where Δ is the Laplace operator and $r = 2$, it follows from the basic elliptic estimate that the Hilbert space H^k may be identified with the Sobolev space $W_k(M)$.

Let $H^\infty = \cap_{k=1}^{\infty} H^k$. We shall assume as given an algebra $\mathcal{D}$ of linear operators on the vector space H^∞. In the standard example, $\mathcal{D}$ will be the algebra of all linear differential operators on M. Let us also assume that the algebra is filtered: thus it is given as an increasing union of linear subspaces

$$\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots \subseteq \mathcal{D}$$

in such a way that $\mathcal{D}_p \cdot \mathcal{D}_q \subseteq \mathcal{D}_{p+q}$. We shall write order$(X) \leq q$ if $X \in \mathcal{D}_q$.

3.1 Definition. We shall say that the pair comprised of Δ and $\mathcal{D}$ is *differential*[4] if the following conditions hold:

(i) If $X \in \mathcal{D}$, then also $[\Delta, X] \in \mathcal{D}$, and

$$\text{order}([\Delta, X]) \leq \text{order}(X) + r - 1.$$

[4] Strictly speaking we should include the integer r somewhere here.

(ii) If $X \in \mathcal{D}$, and if $\operatorname{order}(X) \leq q$, then there is a constant $\varepsilon > 0$ such that

$$\|\Delta^{\frac{q}{r}} v\| + \|v\| \geq \varepsilon \|Xv\|, \quad \forall v \in H^\infty$$

(the norm is that of the Hilbert space H).

3.2 Remark. If we introduce the natural norm on the space $H^q = \operatorname{dom} \Delta^{\frac{q}{r}}$, namely

$$\|v\|_q^2 = \|\Delta^{\frac{q}{r}} v\|^2 + \|v\|^2,$$

then the estimate in item (ii) can be rewritten as

$$\|v\|_q + \|v\| \geq \varepsilon \|Xv\|, \quad \forall v \in H^\infty$$

(for perhaps a different ε). In the standard example this is easily recognizable as the basic estimate of elliptic regularity theory.

3.3 Lemma. *If $X \in \mathcal{D}(\Delta)$, and if X has order q or less, then for every $s \geq 0$ the operator X extends to a bounded linear operator from H^{s+q} to H^s.*

Proof. If s is an integer multiple of the order r of Δ then the lemma follows immediately from the elliptic estimate above. The general case (which we shall not actually need) follows by interpolation. □

This begs us to make the following version of Definition 2.9 in our new abstract context:

3.4 Definition. A linear transformation $T\colon H^\infty \to H^\infty$ has *analytic order* $q \in \mathbb{R}$ if for all $s \geq 0$ such that $s + q \geq 0$ it extends to a bounded linear operator $T\colon H^{s+q} \to H^s$.

3.2 An Example

Let M be a smooth manifold. Assume that an *integrable* smooth vector subbundle $F \subseteq TM$ is given, along with metrics on the bundles F and TM/F (the metrics will play only a very minor role in what we are going to do here). The bundle dertermines a foliation of M by say p-dimensional submanifolds.

Let $\mathcal{D}$ be the algebra of linear partial differential operators on M with compact supports. Define a filtration on $\mathcal{D}$, which makes use of the foliation on M, as follows:

(i) If f is a C^∞-function on M then $\operatorname{order}(f) = 0$.

(ii) If X is a C^∞-vector field on M then $\operatorname{order}(X) \le 2$.

(iii) If X is a C^∞-vector field on M which is everywhere tangent to F then $\operatorname{order}(X) \le 1$.

From now onwards in this subsection we shall use the above non-standard notion of order while discussing operators in $\mathcal{D}$.

When discussing local coordinates on M we shall use coordinates which identify a neighbourhood U in M with an open set in $\mathbb{R}^p \times \mathbb{R}^q$ in such a way that the plaques of the foliation (the connected components of the interections of the leaves with the chart) are of the form $\mathbb{R}^p \times \{\mathrm{pt}\}$. Let us call these *foliation coordinates*. If $X \in \mathcal{D}$, then in local foliation coordinates we can write X as a sum

$$X = \sum_\alpha a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}.$$

If $\operatorname{order}(X) \le k$, then we can separate the sum into a part of order k, plus a part of lower order,

$$X = \sum_{\|\alpha\|=k} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} + \sum_{\|\alpha\|<k} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

where $\|\alpha\|$ is defined by the formula

$$\|\alpha\| = \alpha_1 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_d.$$

3.5 Definition. An operator $X \in \mathcal{D}$ is *elliptic of order* r, *relative to* F, if in every coordinate system, as above, and at every point x in the domain of the coordinate systems the the order r part of X has the property that

$$\left| \sum_{\|\alpha\|=r} a_\alpha(x) \xi^\alpha \right| \ge \varepsilon_x \left(|\xi_1|^2 + \cdots |\xi_p|^2 + |\xi_{p+1}|^4 + \cdots + |\xi_d|^4 \right)$$

for some $\varepsilon_x > 0$ and all ξ.

If $F = TM$ then this coincides with the usual definition of ellipticity. If we define a Sobolev norm in a foliation chart by the formula

$$\|\phi\|^2_{W_s(U,F)} = \sum_{\|\alpha\| \le s} \left\| \frac{\partial^\alpha \phi}{\partial x^\alpha} \right\|^2$$

then every order k operator is continuous from $W_{s+k}(U, F)$ to $W_k(U, F)$. Moreover the arguments used to prove the elliptic estimate in Section 1 easily adapt to show that if X is elliptic of order r relative to F, then

$$\|X\phi\|_s^2 + \|\phi\|_0^2 \geq \varepsilon_X \|\phi\|_{r+s} \tag{3.1}$$

for some $\varepsilon_X > 0$ and every smooth, compactly supported ϕ.

Passing from coordinate charts to global situation on M using partitions of unity, we obtain global Sobolev spaces $W_k(M, F)$ and the corresponding global version of the elliptic estimate (3.1). Observe also that

$$W_{2k}(M) \subseteq W_k(M, F) \subseteq W_k(M),$$

from which it follows that

$$\cap_{k \geq 0} W_k(M, F) = \cap_{k \geq 0} W_k(M) = C^\infty(M).$$

We obtain the following result:

3.6 Theorem. *Let M be a smooth manifold and let F be a smooth, integrable subbundle of TM. If Δ is a positive and elliptic operator on M (relative to F), and if Δ and its powers are essentially self-adjoint, then $(\mathcal{D}, \Delta)$ is a differential pair in the sense of Definition 3.1.* □

We can define an explicit elliptic operator

$$\Delta = \Delta_L^2 + \Delta_T,$$

composed of a "leafwise" operator Δ_L and a "transverse" operator Δ_T on M, as follows. Using the given metric on F we can define a leafwise Laplace operator Δ_L which acts just by differentiation along the leaves of the foliation. Using local foliation coordinates we can identify a foliation chart U in M with an open set in $\mathbb{R}^p \times \mathbb{R}^q$, and after having done so, we can use the given metric on TM/F to define Riemannian metrics on each transversal $\{\text{pt}\} \times \mathbb{R}^q$, which together determine a "transverse" Laplace operator on U. The operator $\Delta_{T,U}$ so constructed depends on our choice of foliation coordinates. However by covering M by charts U_α and choosing a partition of unity $\{\theta_\alpha\}$ we can form a non-canonical operator

$$\Delta_T = \sum_\alpha \theta_\alpha^{\frac{1}{2}} \Delta_{T,U_\alpha} \theta_\alpha^{\frac{1}{2}}.$$

We are requiring operators in our algebra $\mathcal{D}$ to be compactly supported, but if we put this requirement to one side for a moment and think of Δ as an element of $\mathcal{D}$

then we can say that Δ has order 4, and that up to operators of lower order, both Δ and Δ_T are independent of all the choices made in their construction.

For the particular differential pair $(\mathcal{D}, \Delta)$ we have just constructed it is a simple matter to adapt the arguments of Section 2 to prove that all the zeta functions $\mathrm{Trace}(D\Delta_1^{-z})$ admit meromorphic extensions to $\mathbb{C}$, with only simple poles. The proof begins from the basic formula

$$kD = \sum_{i=1}^{p} [D, x_i] \frac{\partial}{\partial x_i} + 2 \sum_{i=p+1}^{n} [D, x_i] \frac{\partial}{\partial x_i} + R,$$

for an order k operator, where R is a differential operator of order $k-1$ or less (as computed in the given filtration of $\mathcal{D}$). This implies that

$$\begin{aligned}(k+p+2q)D = \sum_{i=1}^{p} [D, x_i \frac{\partial}{\partial x_i}] + \sum_{i=1}^{p} [\frac{\partial}{\partial x_i}, x_i D] \\ + 2 \sum_{i=p+1}^{n} [D, x_i \frac{\partial}{\partial x_i}] + 2 \sum_{i=p+1}^{n} [\frac{\partial}{\partial x_i}, x_i D] + R,\end{aligned}$$

with the same remainder term R. From here the proof proceeds exactly as in Section 2. The result is that, if D has order k, then the zeta function $\mathrm{Trace}(D\Delta_1^{-\frac{s}{4}})$ has a meromorphic extension to $\mathbb{C}$, with at most simple poles located at the sequence of points

$$k+p+2q, k+p+2q-1, \ldots.$$

In particular the basic zeta function $\mathrm{Trace}(\Delta_1^{-\frac{s}{4}})$ has poles at $p+2q, p+2q-1, \ldots$. An interesting feature of this result is that the 'analytic dimension' of (M, F) (measured as in Weyl's Theorem by the asymptotic behavious of the eigenvalue sequences of elliptic operators) is not n, the dimension of the manifold, but $n+q = p+2q$.

An important feature of the differential pair $(\mathcal{D}, \Delta)$ is the invariance of Δ, modulo operators of lower order, under diffeomorphisms of M which preserve F and which moreover preserve the metics on F and TM/F. As Connes and Moscovici observe in [10], starting with a manifold V and any group G of diffeomorphisms of V, it is possible to build a new manifold M which fibers over M along with metrics on the vertical tangent bundle F and the quotient bundle TM/F, in such at way that the action of G lifts to M, preserving the given metrics. Starting from this observation Connes and Moscovici are able to develop elliptic operator theory and index theory on very complex spaces, for example the transverse spaces of foliated manifolds.

3.3 Pseudodifferential Operators

Let us return now to our general notion of differential pair. Starting from this concept we can reproduce many of the computations we did in Section 2, for example those used to prove Lemmas 2.10 and 2.12 in our new context (we already suggested as much at the end of the last subsection). However we shall leave this to the reader to check, and instead we shall develop the following closely notion of abstract pseudodifferential operator.

3.7 Definition. Let $(\Delta, \mathcal{D})$ be a differential pair. Fix a positive operator K of analytic order $-\infty$ (this means that K maps H into H^∞) such that the operator

$$\Delta_1 = \Delta + K$$

is invertible. A *basic pseudodifferential operator of order* $k \in \mathbb{Z}$ is a linear operator $T\colon H^\infty \to H^\infty$ with the property that for every $\ell \in \mathbb{Z}$ the operator T may be decomposed as

$$T = X\Delta_1^{\frac{m}{r}} + R,$$

where $X \in \mathcal{D}(A, D)$, $m \in \mathbb{Z}$, and $R\colon H^\infty \to H^\infty$, and where

$$\mathrm{order}(X) + m \le k \quad \text{and} \quad \mathrm{order}(R) \le \ell.$$

A *pseudodifferential operator of order* $k \in \mathbb{Z}$ is a finite linear combination of basic pseudodifferential operators of order k.

3.8 Remark. The introduction of the operator K is more or less a matter of convenience; for example we could have changed Δ_1 to $(I + \Delta)$ without changing the class of pseudodifferential operators determined by the definition. In particular the choice of K has no effect on the definition. (We should add that using spectral theory it is easy to find a suitable operator K.)

3.9 Example. If T is a pseudodifferential operator of order k, then $[\Delta, T]$ is a pseudodifferential operator of order $k + r - 1$.

3.10 Example. All of the integrals $I_z(D_0, \ldots, D_p)$ for integral z are pseudodifferential operators. This follows from the asymptotic expansion formula used in the proof of Lemma 2.10.

We are going to show that the linear space of all pseudodifferential operators is an algebra. For this purpose we shall need to develop some of the asymptotic expansions used in Section 2 in our new, abstract context.

3.11 Definition. If T and T_j are operators on H^∞, then let us write

$$T \approx \sum_{j=1}^{\infty} T_j$$

if, for every m, there exists J_0 such that if $J > J_0$, then the difference $T - \sum_{j=1}^{J} T_j$ is an operator of order m or less.

3.12 Lemma. *If T is a pseudodifferential operator, and if $z \in \mathbb{C}$, then*

$$[\Delta_1^z, T] \approx \sum_{j=1}^{\infty} \binom{z}{j} T^{(j)} \Delta_1^{z-j}.$$

3.13 Remark. We define Δ_1^{-z}, for $\mathrm{Re}(z) > 0$, by a Cauchy integral, as we did in Section 2. Since Δ_1 is invertible we can choose the contour of integration to be the (downwards pointing) imaginary axis.

Proof of the Lemma. We compute as follows:

$$\begin{aligned} [\Delta_1^z, T] &= \frac{1}{2\pi i} \int \lambda^{-z} [(\lambda - \Delta_1)^{-1}, T] \, d\lambda \\ &= \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - \Delta)^{-1} \nabla_1(T) (\lambda - \Delta)^{-1} \, d\lambda, \end{aligned}$$

where we have written $\nabla_1(T) = [\Delta_1, T]$. The integral converges as long as $\mathrm{Re}(z) > 0$ (it converges absolutely to an operator on the Frechet space H^∞), and for the moment let us confine our attention to such z. Continuing, we can write

$$\begin{aligned} [\Delta_1^z, T] &= \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - \Delta_1)^{-1} \nabla_1(T) (\lambda - \Delta_1)^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \int \lambda^{-z} \nabla_1(T) (\lambda - \Delta_1)^{-2} \, d\lambda \\ &\qquad + \frac{1}{2\pi i} \int \lambda^{-z} [(\lambda - \Delta_1)^{-1}, \nabla_1(T)] (\lambda - \Delta_1)^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \int \lambda^{-z} \nabla_1(T) (\lambda - \Delta_1)^{-2} \, d\lambda \\ &\qquad + \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - \Delta_1)^{-1} \nabla_1^2(T) (\lambda - \Delta_1)^{-2} \, d\lambda, \end{aligned}$$

and more generally

$$[\Delta_1^z, T] = \frac{1}{2\pi i}\int \lambda^{-z}\nabla_1(T)(\lambda-\Delta_1)^{-2}\,d\lambda + \frac{1}{2\pi i}\int \lambda^{-z}\nabla_1^2(T)(\lambda-\Delta_1)^{-3}\,d\lambda$$
$$+\cdots+\frac{1}{2\pi i}\int \lambda^{-z}\nabla_1^k(T)(\lambda-\Delta_1)^{-k-1}\,d\lambda$$
$$+\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta_1)^{-1}\nabla_1^{k+1}(T)(\lambda-\Delta_1)^{-(k+1)}\,d\lambda.$$

Using the Cauchy integral formula we can now compute

$$[\Delta_1^z, T] = \binom{-z}{1}\nabla_1(T)\Delta_1^{-(z+1)} + \binom{-z}{2}\nabla_1^2(T)\Delta_1^{-(z+2)}$$
$$+\cdots+\binom{-z}{k}\nabla_1^k(T)\Delta_1^{-(z+k)}$$
$$+\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta_1)^{-1}\nabla_1^{k+1}(T)(\lambda-\Delta_1)^{-(k+1)}\,d\lambda.$$

If $\mathrm{order}(T) = q$, then the remainder integral in the final display converges when $\mathrm{Re}(z) > k+1$ to an operator of order $q-k-r\,\mathrm{Re}(z)$. This proves the lemma. □

3.14 Proposition. *The set of all pseudodifferential operators is a filtered algebra.*

Proof. The set of pseudodifferential operators is a vector space. The formula

$$X\Delta_1^{\frac{m}{2}}\cdot Y\Delta_1^{\frac{n}{2}} \approx \sum_{j=0}^{\infty}\binom{\frac{m}{2}}{j}X\nabla_1^j(Y)\Delta_1^{\frac{m+n}{2}-j}$$

shows that it is closed under multiplication and moreover that the product of two pseudodifferential operators of orders k and ℓ is a pseudodifferential operator of order $k+\ell$. □

The algebra of pseudodifferential operators is a good context in which to study the residues of the zeta functions $\mathrm{Trace}(D\Delta_1^{-z})$, thanks to the following beautiful fact:

3.15 Lemma. *Assume that for every differential operator $D \in \mathcal{D}$, and all $z \in \mathbb{C}$ with sufficiently large real part, the operator $D\Delta_1^{-z}$ is trace-class. Assume that, in addition, for every $D \in \mathcal{D}$ the zeta function $\mathrm{Trace}(D\Delta_1^{-z})$ extends to a meromorphic function on $\mathbb{C}$ with only simple poles. Then the residue functional*

$$\tau(T) = \mathrm{Res}_{z=0}\,\mathrm{Trace}(T\Delta_1^{-z})$$

is a trace on the algebra of pseudodifferential operators.

3.16 Remark. If Δ itself has discrete spectrum and compact resolvent $(I+\Delta)^{-1}$, and if we define Δ^{-z} as we did in Section 2, integrating down a vertical line which separates 0 from the positive spectrum of Δ, then the residues of $\mathrm{Trace}(D\Delta^{-z})$ and $\mathrm{Trace}(T\Delta_1^{-z})$ are equal.

Proof of the Lemma. We want to show that

$$\mathrm{Res}_{z=0}\,\mathrm{Trace}(ST\Delta_1^{-z}) = \mathrm{Res}_{z=0}\,\mathrm{Trace}(TS\Delta_1^{-z}).$$

Using the trace property of the operator trace, this amounts to showing that

$$\mathrm{Res}_{z=0}\,\mathrm{Trace}\big(ST\Delta_1^{-z} - S\Delta_1^{-z}T\big) = 0.$$

Using Lemma 3.12 we get

$$ST\Delta_1^{-z} - S\Delta_1^{-z}T \approx -\sum_{j=1}^{\infty} \binom{-z}{j} ST^{(j)}\Delta_1^{z-j}$$

As a result,

$$\mathrm{Res}_{z=0}\,\mathrm{Trace}\big(ST\Delta_1^{-z} - S\Delta_1^{-z}T\big) = -\sum_{j=1}^{\infty} \mathrm{Res}_{z=0}\Big(\binom{-z}{j}\,\mathrm{Trace}(ST^{(j)}\Delta_1^{z-j})\Big).$$

This is a finite sum since all but finitely many of the residues of $\mathrm{Trace}(ST^{(j)}\Delta_1^{z-j})$ are zero. But in fact since each trace function has at worst a simple pole, *all* the residues in the sum are zero: the possible pole of $\mathrm{Trace}(ST^{(j)}\Delta_1^{z-j})$ at $z=0$ is canceled out by the factor of z in the binomial coefficient $\binom{-z}{j}$. □

3.17 Remark. This result of Wodzicki [28] was first observed in the following algebraic context (compare for example [26] for a clear account). Let A be a complex algebra and let ∂ be a derivation on A. The main example is where A is the algebra of smooth functions on unit circle and ∂ is ordinary differentiation:

$$\partial(a) = \frac{da}{dt}.$$

The space $D(A)$ of formal polynomials $\sum_{n=0}^{N} a_n\partial^n$ in ∂ with coefficients in A is an associative algebra, with multiplication law derived from

$$[\partial, a] = \partial(a).$$

In the main example this is the algebra of differential operators on the circle. Consider now the algebra $\Psi(A)$ of formal series

$$\sum_{n=-\infty}^{N} a_n \partial^n$$

in ∂ with coefficients in A. Infinitely many of the negative coefficients may be nonzero, but we require that each series contain only finitely many positive powers of ∂. This is an associative algebra with multiplication derived from the formula

$$\delta^n \cdot a = \sum_{j=0}^{\infty} \binom{n}{j} \partial^j(a)\partial^{n-j}.$$

Let $\tau\colon A \to \mathbb{C}$ be a trace functional which vanishes on the range of ∂. Thus τ is a linear functional for which

$$\tau\colon [A, A] + \partial[A] \mapsto 0.$$

In the main example, where A is the algebra of smooth functions on the circle τ is the ordinary integral:

$$\tau(a) = \int a\, dt.$$

The following is then an algebraic counterpart of Lemma 3.15:

3.18 Lemma. *The functional* $\rho\colon \Psi(A) \to \mathbb{C}$ *defined by*

$$\rho(\sum a_i \partial^i) = \tau(a_{-1}).$$

is a trace on the algebra $\Psi(A)$. □

3.4 Spectral Triples

Further examples of differential pairs $(\mathcal{D}, \Delta)$ are furnished by Connes' notion of spectral triple. In this subsection we shall briefly review the basic definitions.

3.19 Definition. A *spectral triple* is a triple (A, H, D), composed of a complex Hilbert space H, an algebra A of bounded operators on H, and a self-adjoint operator D on H with the following two properties:

(i) If $a \in A$ then the operator $a \cdot (1 + D^2)^{-1}$ is compact.

(ii) If $a \in A$ then $a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D)$ and the commutator $[D, a]$ extends to a bounded operator on H

Various examples are listed in [10]; in the standard example A is the algebra of smooth functions on a complete Riemannian manifold M, D is a Dirac-type operator on M, and H is the Hilbert space $L^2(S)$ of square-integrable sections of the vector bundle on which D acts.

3.20 Definition. Let (A, H, D) be a spectral triple. Denote by δ the unbounded derivation of $\mathcal{B}(H)$ given by commutator with $|D|$. Thus the domain of δ is the set of all bounded operators T which map the domain of $|D|$ into itself, and for which the commutator extends to a bounded operator on H.

3.21 Lemma. *Let* H^∞ *be a core of* $|D|$ *(a subspace of the domain on which the operator is essentially self-adjoint). If* T *maps* H^∞ *into itself, and if* $[|D|, T]$ *is bounded on* H^∞*, then* T *lies in the domain of* δ*.* □

3.22 Definition. A spectral triple is *regular* if A and $[D, A]$ belong to $\cap_{n=1}^{\infty} \delta^n$.

The notion of regular spectral triple (A, H, D) plays a useful role in the detailed analysis of Alain Connes' spectral triples and their Chern characters. See for example [14]. The purpose of this subsection is to show that regularity is equivalent to the basic elliptic estimate which appears in item (ii) of Definition 3.1 (the relevant pair $(\mathcal{D}, \Delta)$ will be described in a moment). This equivalence is essentially proved in [10, Appendix B], although in disguised form.

3.23 Definition. Let (A, H, D) be a spectral triple with the property that every $a \in A$ maps H^∞ into itself. Denote by Δ the operator D^2. The *algebra of differential operators* associated to (A, H, D) is the smallest algebra $\mathcal{D}$ of operators on H^∞ which contains A and $[D, A]$ and which is closed under the operation $T \mapsto [\Delta, T]$.

3.24 Remarks. If the spectral triple (A, H, D) is regular, then the condition $A \cdot H^\infty \subseteq H^\infty$ is automatically satisfied. The above description of $\mathcal{D}$ is in some sense the minimal reasonable definition of an algebra of differential operators. Note however that the operator D is not necessarily included in $\mathcal{D}$.

The algebra $\mathcal{D}$ of differential operators is filtered, as follows. We require that elements of A and $[D, A]$ have order zero, and that the operation of commutator with $\Delta = D^2$ raises order by at most one. Thus the spaces $\mathcal{D}_k$ of operators of order k or less are defined inductively as follows:

(a) $\mathcal{D}_0 =$ algebra generated by $A + [D, A]$.

(b) $\mathcal{D}_1 = [\Delta, \mathcal{D}_0] + \mathcal{D}_0[\Delta, \mathcal{D}_0]$.

(c) $\mathcal{D}_k = \sum_{j=1}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{k-j} + [\Delta, \mathcal{D}_{k-1}] + \mathcal{D}_0[\Delta, \mathcal{D}_{k-1}]$.

We want to prove the following result.

3.25 Theorem. *Let (A, H, D) be a spectral triple with the property that every $a \in A$ maps H^∞ into itself. It is regular if and only if $(\mathcal{D}, \Delta)$ is a differential pair in the sense of Definition 3.1.*

3.26 Remark. As should be clear, we assign to Δ the order $r = 2$. Condition (i) of Definition 3.1 is then automatically satisfied.

We shall begin by proving that a regular spectral triple (A, H, D) satisfies the basic estimate.

3.27 Definition. Let (A, H, D) be a regular spectral triple. Denote by $\Psi_0(A)$ the algebra of operators on H^∞ generated by all the spaces $\delta^n[A]$ and $\delta^n\big[[D, A]\big]$, for all $n \geq 0$.

Note that, according to the definition of regularity, every operator in $\Psi_0(A)$ extends to a bounded operator on H. The notation "$\Psi_0(A)$" is chosen to suggest "pseudodifferential operator of order 0" (it is indeed the case that $\Psi_0(A)$ is an algebra of order 0 pseudodifferential operators associated to the differential pair $(\mathcal{D}, \Delta)$).

3.28 Lemma. *Assume that (A, H, D) is a regular spectral triple. Every operator in $\mathcal{D}$ of order k may be written as a finite sum of operators $b|D|^\ell$, where b belongs to the algebra $\Psi_0(A)$ and where $\ell \leq k$.*

Proof. Define $\mathcal{E}$, a space of operators on H^∞, to be the linear span of the operators of the form $b|D|^k$, where $k \geq 0$ and $b \in \Psi_0(A)$. The space $\mathcal{E}$ is an algebra since $\delta[\Psi_0(A)] \subseteq \Psi_0(A)$ and since

$$b_1|D|^{k_1} \cdot b_2|D|^{k_2} = \sum_{j=0}^{k_1} \binom{k_1}{j} b_1 \delta^j(b_2)|D|^{k_1+k_2-j}.$$

Filter the algebra $\mathcal{E}$ by defining $\mathcal{E}_k$ to be the span of all operators $b|D|^\ell$ with $\ell \leq k$. The formula above shows that this does define a filtration of the algebra $\mathcal{E}$. Now the algebra $\mathcal{D}$ of differential operators is contained within $\mathcal{E}$, and the lemma we are trying to prove amounts to the assertion that $\mathcal{D}_k \subseteq \mathcal{E}_k$. Clearly $\mathcal{D}_0 \subseteq \mathcal{E}_0$. Using the formula

$$[\Delta, b|D|^{k-1}] = [|D|^2, b|D|^{k-1}] = 2\delta(b)|D|^k + \delta^2(b)|D|^{k-1},$$

along with our formula for $\mathcal{D}_k$, the inclusion $\mathcal{D}_k \subseteq \mathcal{E}_k$ is easily proved by induction. □

We can now prove that every regular spectral triple satisfies the basic estimate. According to the lemma, it suffices to prove that if $k \geq \ell$ and if $X = b|D|^\ell$, where $b \in B$, then there exists $\varepsilon > 0$ such that

$$\|D^k v\| + \|v\| \geq \varepsilon \|Xv\|,$$

for every $v \in H^\infty$. But we have

$$\|Xv\| = \|b|D|^\ell v\| \leq \|b\| \cdot \||D|^\ell v\| = \|b\| \cdot \|D^\ell v\|,$$

And since by spectral theory for every $\ell \leq k$ we have that

$$\|D^\ell v\|^2 \leq \|D^k v\|^2 + \|v\|^2 \leq \big(\|D^k v\| + \|v\|\big)^2$$

it follows that

$$\|D^k v\| + \|v\| \geq \frac{1}{\|b\| + 1}\|Xv\|,$$

as required.

We turn now to the proof of the second half of Theorem 3.25. Assume from now on that (A, H, D) is a spectral triple for which $A \cdot H^\infty \subseteq H^\infty$ and for which $(\mathcal{D}, \Delta)$ is a differential pair. Starting from the differential pair we can form the algebra of pseudodifferential operators, as in Subsection 3.3.

3.29 Lemma. *If T is a pseudodifferential operator then so is $\delta(T)$, and moreover* order$(\delta(T)) \leq$ order(T).

Proof. We compute that

$$\begin{aligned}\delta(T) = |D|T - T|D| &\approx \Delta_1^{\frac{1}{2}} T - T\Delta_1^{\frac{1}{2}} \\ &\approx \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} \nabla_1^j(T)\Delta^{\frac{1}{2}-j}.\end{aligned}$$

This computation reduces the lemma to the assertion that if T is a pseudodifferential operator of order k then $\nabla_1(T)$ is a pseudodifferential operator of order $k+1$ or less. Since $\nabla_1(T) \approx [\Delta, T]$ this in turn follows from the observation made in Example 3.9. □

Proof that (A, H, D) is regular. By the basic estimate, every pseudodifferential operator of order zero extends to a bounded operator on H. Since every operator in A or $[D, A]$ is pseudodifferential of order zero, and since $\delta(T)$ is pseudodifferential of order zero whenever T is, we see that if $b \in A$ or $b \in [D, A]$ then for every n the operator $\delta^n(b)$ extends to a bounded operator on H. Hence the spectral triple (A, H, D) is regular, as required. □

3.5 Dimension Spectrum

3.30 Definition. A spectral triple (A, H, D) is *finitely summable* if there is some $k > 0$ such that the operator $a \cdot (1 + D^2)^{-k}$ is trace-class, for every $a \in A$.

Suppose that the spectral triple (A, H, D) is regular, and denote by $\mathcal{D}$ the associated algebra of differential operators. If (A, H, D) is finitely summable then for every $X \in \mathcal{D}$ the zeta function $\mathrm{Trace}(X\Delta_1^{-z})$ is defined in a right half-plane in $\mathbb{C}$, and is holomorphic there (as before, Δ_1 is an invertible operator obtained from Δ by adding a positive, order $-\infty$ operator). The following concept has been introduced by Connes and Moscovici [10, Definition II.1].

3.31 Definition. Let (A, H, D) be a regular and finitely summable spectral triple. It has *discrete dimension spectrum* if[5] there is a discrete subset F of $\mathbb{C}$ with the following property: for every operator T in the algebra $\Psi_0(A)$ of Definition 3.27, the zeta function $\mathrm{Trace}(T\Delta_1^{-\frac{z}{2}})$ extends to a meromorphic function on $\mathbb{C}$ with all poles contained in F.

If (A, H, D) has discrete dimension spectrum then for every differential, or indeed pseudodifferential, operator X, the zeta function $\mathrm{Trace}(X\Delta_1^{-\frac{z}{2}})$ extends to a meromorphic function on $\mathbb{C}$. Moreover if X has order k then the poles of this zeta function are located in $F + q$. Conversely, if (A, H, D) is a regular spectral triple, and if, for every differential operator X of order k, the zeta function $\mathrm{Trace}(X\Delta_1^{-\frac{z}{2}})$ extends to a meromorphic function on $\mathbb{C}$ whose poles are located within $F + q$, then (A, H, D) has discrete dimension spectrum F.

A final item of terminology:

3.32 Definition. A regular and finitely summable spectral triple has *simple* dimension spectrum if it has discrete dimension spectrum and if all the zeta-type functions above have only simple poles.

[5]Connes and Moscovici add a technical condition concerning decay of zeta functions along vertical lines in $\mathbb{C}$.

It is an interesting and as yet unsolved problem to find algebraic conditions on a regular spectral triple which will imply that it has discrete or simple dimension spectrum.

4 Computation of Residues

We saw in the Section 2 that if Δ is the Laplace operator on a closed Riemannian manifold M and if D is any differential operator on M then the function

$$\mathrm{Trace}(D\Delta^{-\frac{z}{2}}) \tag{4.1}$$

is meromorphic on $\mathbb{C}$. Moreover if D has order q then the poles of this zeta function are all simple and are located at the integer points $q+n, q+n-1, \ldots$.

The purpose of this section is to explain how the residues of $\mathrm{Trace}(D\Delta^{-\frac{z}{2}})$ are given by complicated but in principal explicit and computable formulas involving the coefficients of D and Δ. This 'local computability' of residues is a very important conceptual point: in the next section we shall consider a family of globally defined index invariants of manifolds, and it will be a significant and nontrivial fact that these global invariants are given by explicit (albeit complicated) local residue formulae.

We shall not take the shortest route toward our goal of producing local formulae for residues. Instead we shall follow a method, based on commutators, which is loosely related to our proof of meromorphic continuation in Section 2. Nor shall we give a very detailed or sophisticated account of this topic. Instead, for the full story the reader is referred to [28] or [15].

4.1 Computation of the Leading Residue

We are going to find a formula for the residue at $z = n + q$ of the function $\mathrm{Trace}(D\Delta^{-\frac{z}{2}})$, where D is an order q differential operator. This is the residue at the leading or rightmost pole in $\mathbb{C}$. Note that

$$\mathrm{Res}_{z=n+q}\,\mathrm{Trace}(D\Delta^{-\frac{s}{2}}) = \mathrm{Res}_{s=0}\,\mathrm{Trace}(D\Delta^{-\frac{n+q+s}{2}}) = \tau(D\Delta^{-\frac{n+q}{2}}),$$

where τ is the residue trace on the algebra of pseudodifferential operators (see Lemma 3.15). So the leading residue is the residue trace of the order $-n$ pseudodifferential operator $D\Delta^{-\frac{n+q}{2}}$. We are going to use the trace property of τ to produce a formula for the residue trace of any order $-n$ pseudodifferential operator.

In order to produce such a formula we first need to extract from a pseudodifferential operator its symbol, which is a function on the cotangent sphere bundle S^*M.

4.1 Definition. Let D be a differential operator of order q. Its *principal symbol* is the function $\sigma_D\colon S^*M \to \mathbb{C}$ defined in local coordinates by the formula

$$\sigma_D(x,\xi) = i^q \sum_{|\alpha|=q} a_\alpha(x)\xi^\alpha,$$

where

$$D = \sum_{|\alpha|\leq q} a_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}.$$

In other words, to define $\sigma_D\colon S^*M \to \mathbb{C}$ we just exchange each partial derivative $\frac{\partial}{\partial x_i}$ in the leading order terms of D for the corresponding coordinate function ξ_i on the cotangent bundle. The reason for dropping the lower order terms of D is that the principal symbol is then independent of the choice of local coordinates on M, and so well defined on all of S^*M. This would not be the case if the lower order terms of D were retained.

The overall factor i^q is conventional. It ensures that, for example, the symbol of Δ, a positive operator, is a positive function. In fact the symbol of Δ is the constant function 1. For a general operator D of order q, the symbol extends to a function on T^*M which is polynomial and homogeneous of order q in each fiber of the cotangent bundle (in the case of the Laplace operator Δ this extension is just the norm-squared function $\xi \mapsto \|\xi\|^2$ obtained from the Riemannian metric). Going in the other direction, if $\sigma\colon T^*M \to \mathbb{C}$ is polynomial and homogeneous of order q in each fiber, then it is the symbol of some order q differential operator.

4.2 Definition. Let T be an order q pseudodifferential operator. Its *principal symbol* is the function $\sigma_T\colon S^*M \to \mathbb{C}$ obtained by representing T in the form

$$T = D\Delta^{\frac{k}{2}} + R,$$

where $\mathrm{order}(R) < q$, and then setting

$$\sigma_T = \sigma_D\colon S^*M \to \mathbb{C}.$$

The symbol is well-defined. This follows in the first place from the fact that the symbol of Δ is the constant function 1 on S^*M, so that if we write $T = D\Delta \cdot \Delta^{k-1}$ then $\sigma_D = \sigma_{D\Delta}$, and in the second place from the fact that the analytic order of a differential operator is exactly equal to its differential order.

We are going to prove the following result.

4.3 Theorem. *There is a constant* c *such that if* T *is any order* $-n$ *pseudodifferential operator then*

$$\tau(T) = c \int_{S^*M} \sigma_T \, d\,\mathrm{vol}\,.$$

Here is roughly how we are going to proceed. We shall show that if the integral vanishes, then the symbol σ_T can be written as a linear combination of 'derivatives'. As we shall see, this will imply that T can be written as a linear combination of commutators of pseudodifferential operators, modulo an operator R of lower order. From the trace property of τ it will follow that $\tau(T) = \tau(R)$, and since R has order less than $-n$ it follows that $\tau(R) = 0$. All this will show that if the integral vanishes then so does $\tau(T)$. Since the integral and the trace are both linear functionas on the space of order $-n$ operators, it will follow that τ is a constant multiple of the integral, as required.

To start the argument, we consider the complex of differential forms on S^*M which are polynomial in the fiber direction (this means that the forms are local combinations of forms $p(x,\xi)dx^I d\xi^J$, where p is polynomial in the ξ-variables; here $x_1, \ldots, x_n$, together with $\xi_1, \ldots, \xi_n$, are the standard coordinate functions on T^*M). This complex computes the de Rham cohomology[6] of S^*M. The volume form on S^*M is given by the formula

$$\mathrm{vol}_{S^*M} = \sum_{j=1}^{n} (-1)^{j-1} \xi_j dx_1 \cdots dx_n d\xi_1 \cdots \widehat{d\xi_j} \cdots d\xi_n$$

and so belongs to our complex. If the integral in Theorem 4.3 is zero then $\sigma \cdot \mathrm{vol}_{S^*M}$ is exact, say

$$\sigma \cdot \mathrm{vol}_{S^*M} = d\alpha. \tag{4.2}$$

We are now going to transfer this equation to the space R^*M, obtained from T^*M by deleting the zero section. Of course, S^*M is a submanifold of R^*M. We extend σ to a function on R^*M by requiring it to be homogeneous of order $-n$ in each fiber. We extend vol_{S^*M} to the form

$$\omega = \frac{1}{r} \sum_{j=1}^{n} (-1)^{j-1} \xi_j dx_1 \cdots dx_n d\xi_1 \cdots \widehat{d\xi_j} \cdots d\xi_n.$$

[6]This part of the argument would be simpler if we used the classical notion of pseudodifferential operator from analysis, in which case the relevant class of functions on S^*M would be the class of *all* smooth functions, and the relevant complex would be the standard de Rham complex.

Here $r\colon R^*M \to \mathbb{R}$ is the function $r(\xi) = \|\xi\|$. By collapsing each positive ray in R^*M to a point we get a projection to S^*M, and using it we pull back α to a form β on R^*M. From (4.2) we get

$$d\beta = \sigma \cdot \omega.$$

Multiplying both sides by the closed form dr, and observing that $dr \cdot \omega = \mathrm{vol}_{R^*M}$ we get

$$d\gamma = \sigma \cdot \mathrm{vol}_{R^*M}$$

where $\gamma = \beta \cdot dr$. Writing this equation in local coordinates we arrive at the following:

4.4 Lemma. *If* $\int_{S^*M} \sigma d\,\mathrm{vol} = 0$, *then* σ, *viewed as a function on* R^*M, *is a sum of functions each of which is supported in a coordinate chart and is of the form*

$$\frac{\partial a}{\partial x_j} \quad \textit{or} \quad \frac{\partial b}{\partial \xi_j}.$$

The functions a *and* b *are quotients of functions which are polynomial in each fiber of* R^*M *by powers of* r. □

Proof of Theorem 4.3. Let T be an order $-n$ operator. It suffices to prove that if the integral of the symbol of T over S^*M is zero then the residue trace of T is zero. If the integral over S^*M of the symbol of T is zero then the symbol is a sum of derivatives of the type $\frac{\partial a}{\partial x_j}$ or $\frac{\partial b}{\partial \xi_j}$, as in Lemma 4.4. If we construct operators A and B with symbols a or b then we find that the commutators $[A, \frac{\partial}{\partial x_j}]$ and $[B, x_j]$ have symbols $\frac{\partial a}{\partial x_j}$ or $\frac{\partial b}{\partial \xi_j}$, respectively. Conclusion: the operator T is a sum of commutators, modulo an operator of order less than $-n$. Since the residue trace vanishes on commutators, and also on operators of order less than $-n$, it follows that the residue trace of T is zero, as required. □

4.5 Remark. It is not difficult to see that the constant c depends only on $\dim(M)$ (note that c is determined by the residue trace of an operator supported in a coordinate neighbourhood; given two different connected manifolds, apply Theorem 4.3 to a third manifold which contains coordinate neighbourhoods isometric to neighbourhoods in the first two manifolds). By checking an explicit example, like the flat torus, one can see that $c_n = (2\pi)^{-n}$.

4.2 The Lower Residues

Let D be an order q differential operator. The problem of computing the residues of $\operatorname{Trace}(D\Delta^{-\frac{z}{2}})$ (or similarly of $\Gamma(\frac{z}{2})\operatorname{Trace}(D\Delta^{-\frac{z}{2}})$) at the "lower" poles $n+q-1, n+q-2, \ldots$ can be reduced to the problem of computing the highest residue by the scheme used in our proof of meromorphic continuation.

Before starting the computation, it is useful to note that our basic meromorphic continuation theorem can be strengthened in the following way. An elaboration of the Sobolev theory that we developed in Section 1 and Appendix B shows that every p, every operator $T\colon C^\infty(M) \to C^\infty(M)$ of sufficiently large negative order may be represented by a C^p integral kernel function defined on $M \times M$:

$$T\phi(x) = \int_M k(x,y)\phi(y)\,dy.$$

It follows that if z is restricted to a suitable left half-plane in $\mathbb{C}$, then any integral I_z of the type considered in Definition 2.5 may be represented by a kernel function $k_z\colon M \times M \to \mathbb{C}$ which is p-times continuously differentiable. Consider now the basic identity from the proof of Theorem 2.2: if I_z is an integral of type k then

$$\sum_{i=1}^{N} [I_z, A_i B_i] + \sum_{i=1}^{N} [B_i, A_i I_z] = (k-2z)I_z + R_z,$$

where R_z is a finite sum of integrals of lower type. As we know, the identity is equivalent to the identity

$$\sum_{i=1}^{N} [I_z, A_i] B_i + n I_z = (k+n-2z)I_z + R_z. \tag{4.3}$$

Now, let us represent the integral I_z by an integral kernel $k_z(x,y)$, and compute the left hand side of (4.3). The vector field B_i is a skew-symmetric operator, modulo operators of lower order: this means that there is a smooth function $C_i\colon M \to \mathbb{R}$ so that

$$\int_M \psi \cdot B_i\phi \,d\,\mathrm{vol} = -\int_M B_i\psi \cdot \phi \,d\,\mathrm{vol} + \int_M C_i\psi \cdot \phi \,d\,\mathrm{vol}\,.$$

We can therefore write the left-hand side of (4.3) as an integral operator

$$
\begin{aligned}
f \mapsto \sum \int k_z(x,y)\big(A_i(y) - A_i(x)\big)B_i\phi(y)\,dy + n\int k_z(x,y)\phi(y)\,dy \\
= -\sum \int B_i\Big(k_z(x,y)\big(A_i(y)-A_i(x)\big)\Big)\phi(y)\,dy \\
+ \sum \int C_i(y)k_z(x,y)\big(A_i(y)-A_i(x)\big)\phi(y)\,dy + n\int k_z(x,y)\phi(y)\,dy
\end{aligned}
$$

(the vector field B_i acts on the y-variable). Finally,

$$B_i\Big(k_z(x,y)\big(A_i(y)-A_i(x)\big)\Big) = B_i\big(k_z(x,y)\big)\big(A_i(y)-A_i(x)\big)+k_z(x,y)B_i\big(A_i(y)\big).$$

But $B_i\big(A_i(y)\big)$ is the scalar function $[B_i, A_i]$. Setting $x = y$ in the above formulas and using the fact that $\sum[B_i, A_i] = n$ we now see that the left hand side of (4.3) is represented by an integral kernel which *vanishes identically* along the diagonal $x = y$ in $M \times M$. The proof of Theorem 2.2 now provides the following well-known meromorphic continuation of trace-densities for complex powers of Δ:

4.6 Theorem. *Let Δ be a positive, elliptic operator on a smooth, closed manifold M, and let X be a differential operator on M. For $\mathrm{Re}(z) \ll 0$ let $K_z\colon M \to \mathbb{C}$ be the restriction to the diagonal in $M \times M$ of the integral kernel $k_z(x,y)$ for the operator $X\Delta_1^{-\frac{z}{2}}$. For every $h \in \mathbb{N}$ the map $z \mapsto K_z$ extends to a meromorphic function from $\mathbb{C}$ into the h-times continuously differentiable functions on M.* □

We see that the residues we are trying to compute are the integrals over M of *residue densities* $\mathrm{Res}_{z=m}\,\mathrm{Trace}(D\Delta^{-\frac{z}{2}})(x) = \mathrm{Res}_{z=m}\,K_z(x)$. The leading residue density is given by the formula

$$\mathrm{Res}_{z=n+q}\,\mathrm{Trace}(D\Delta^{-\frac{z}{2}})(x) = \frac{1}{(2\pi)^n}\int_{S_x^*M}\sigma_{D\Delta^{-\frac{q+n}{2}}}(\xi)d\xi. \tag{4.4}$$

which integrates the symbol of $D\Delta^{-\frac{q+n}{2}}$ over the contangent sphere at x.

To compute the lower residue densities near x let us choose A_i and B_i to be of the form x_i and $\frac{\partial}{\partial x_i}$ near x. Setting $I_z = D\Delta^{-z}$ and using the formula

$$(q+n-2z)I_z = \sum_{i=1}^{n}[I_z, x_i\frac{\partial}{\partial x_i}] + \sum_{i=1}^{n}[\frac{\partial}{\partial x_i}, x_iI_z] + R_z,$$

we see as before that the trace densities of $(q+n)I_z$ and R_z are equal (since not only are the traces of the commutators zero, but their trace densities are identically zero). It follows that the residue densities of $(q+n-2z)I_z$ are equal to the residue densities of $R_z = (q-2z)I_z - \sum_{i=1}^{n}[D, x_i]\frac{\partial}{\partial x_i}$. In particular, looking at the residue just one below the leading residue we get

$$\mathrm{Res}_{z=\frac{n+q-1}{2}} \mathrm{Trace}(I_z)(x) = \mathrm{Res}_{z=\frac{n+q-1}{2}} \mathrm{Trace}(R_z)(x).$$

But on the right hand side we are computing the leading residue of R_z, so that we can invoke the explicit formula (4.4). As a result, since R_z is explicitly computable in terms of I_z, we obtain an explicit (but complicated) formula for the residue density of I_z at $q+n-1$.

Repeating this argument we get explicit formulas (which get more and more complicated) for all the residue densities of $\mathrm{Trace}(I_z)$.

5 The Index Problem

In this section we shall introduce the problem in Fredholm index theory whose solution will occupy the remained of the notes. This will require us to introduce cyclic cohomology. Since there are several good introductions to the latter subject (for example [22] or [4]) we shall do so quite rapidly.

5.1 Index of Elliptic Operators

From now on we are going to work in the $\mathbb{Z}/2$-graded situation which is standard in index theory. We shall assume that Δ, a linear partial differential operator on closed manifold M, is the square of a self-adjoint, first order, elliptic partial differential operator D. We shall assume that D acts not on scalar functions but on the sections of some smooth vector bundle S over M. We shall assume moreover that S is written as a direct sum $S = S_+ \oplus S_-$ (in other words that S is $\mathbb{Z}/2$-graded), and that, with respect to this direct sum decomposition, the operator D has the form

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

so that

$$\Delta = D^2 = \begin{pmatrix} D_-D_+ & 0 \\ 0 & D_+D_- \end{pmatrix}.$$

Denote by ε the *grading operator*

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As is customary in the $\mathbb{Z}/2$-graded world we shall call operators which commute with ε *even* and those which anticommute with ε *odd*. Even operators are diagonal in the 2×2 matrix notation and odd operators are off-diagonal.

5.1 Definition. An unbounded Hilbert space operator $T\colon H_+ \to H_-$ is *Fredholm* if it is Fredholm as a linear transformation from $\operatorname{dom} T$ into H_-. In other words, T is Fredholm if and only if its kernel is a finite dimensional subspace of $\operatorname{dom} T$ and its range has finite codimension in H_-. In this case the *index* of T is the integer

$$\operatorname{Index}(T) = \dim \ker(T) - \dim \operatorname{coker}(T).$$

5.2 Lemma. *The unbounded operator* $D_+\colon L^2(S_+) \to L^2(S_-)$ *is Fredholm.*

Proof. We want to show that, when viewed as a bounded operator from its domain into H_-, D_+ is a Fredholm operator in the usual sense, meaning that its kernel and cokernel are finite-dimensional. By the basic elliptic esitmate, the domain of D_+ is the Sobolev space $W_1(S_+)$ of W_1-sections of S_+. Denote by $Q\colon L^2(S_-) \to L^2(S_+)$ compression of the operator $(D+i)^{-1}$. Thus in matrix form we have

$$(D+i)^{-1} = \begin{pmatrix} \star & Q \\ \star & \star \end{pmatrix} : L^2(S_+) \oplus L^2(S_-) \to L^2(S_+) \oplus L^2(S_-).$$

By the basic elliptic estimate again, the range of Q is contained within $W_1(S_+)$. If we regard Q as an operator from $L^2(S_-)$ to $W_1(S_+)$ then it follows from the Rellich Lemma that Q is an inverse of D_+, modulo compact operators. As is well known, an operator which is invertible modulo the compact operators is Fredholm (this is Atkinson's Theorem), so the lemma is proved. □

5.3 Remark. By elliptic regularity theory, the kernel of D_+ consists of smooth functions. Moreover the cokernel identifies with the kernel of D_-, which again consists of smooth functions.

We can therefore pose the very famous problem of computing the Fredholm index of D_+. The full solution to the problem is provided by the Atiyah-Singer index theorem [2], and is known to involve in a very subtle way information not only about the operator D but also about the global topology of the underlying manifold M. But Atiyah and Bott [1] pointed out a very simple formula for the index involving residues of zeta functions, as follows. Fix an even, positive, order $-\infty$ operator K such that the sum

$$\Delta_1 = \Delta + K$$

is invertible.

5.4 Proposition. $\text{Index}(D_+) = \text{Res}_{z=0}\big(\Gamma(z)\,\text{Trace}(\varepsilon\Delta_1^{-z})\big)$.

Proof. It is not difficult to see that the residue is independent of the choice of K, and therefore we may take K to be the orthogonal projection on to the kernel of Δ. We shall work with this choice below.

Let $\{\phi_j\}$ be an orthonormal eigenbasis for Δ acting the orthogonal complement of $\ker(\Delta)$ in $L^2(S_+)$. Define

$$\psi_j = \frac{1}{\sqrt{\lambda_j}} D\phi_j \in L^2(S_-).$$

It is easy to check that $\Delta\psi_j = \lambda_j\psi_j$ and that the collection of all ψ_j constitutes an orthonormal basis for Δ acting on the orthogonal complement of $\ker(\Delta)$ in $L^2(S_-)$. Computing the trace in these orthonormal basis we see that

$$\mathrm{Trace}(\varepsilon\Delta_1^{-z}) = \dim\ker(\Delta|_{L^2(S_+)}) - \dim\ker(\Delta|_{L^2(S_-)}) = \mathrm{Index}(D_+).$$

The formula

$$\mathrm{Index}(D_+) = \mathrm{Res}_{z=0}\big(\Gamma(z)\,\mathrm{Trace}(\varepsilon\Delta_1^{-z})\big)$$

follows immediately from this. □

The significance of this result is that, as we saw in Subsection 2.3 and Section 4, the residue of $\Gamma(z)\,\mathrm{Trace}(\varepsilon\Delta_1^{-z})$ can in principle be determined by a completely mechanical computation, involving ultimately integrals over the cosphere bundle of M of various polynomial combinations of the symbol of Δ and its partial derivatives. This is quite remarkable since *a priori* the index problem is very global in nature, and is not at all obviously reducible to a definite sequence of computations in coordinate patches.

From this point onwards a viable approach to the index theorem is to develop means to organize the complicated computations involved in determining the residue at $z = 0$ of $\Gamma(z)\,\mathrm{Trace}(\varepsilon\Delta_1^{-z})$, so as to put the result of the computations into recognizable form. See for example [13]. But rather than carry that out, we shall spend the remaining parts of these notes developing a considerable elaboration of Proposition 5.4, in which the numerical index of an elliptic operator D is replaced by a much more detailed invariant in cyclic cohomology.

5.2 Square Root of the Laplacian

Let $(\mathcal{D}, \Delta)$ be a general differential pair, in the sense of Definition 3.1. In order to develop index theory in this context we shall now assume that Δ is the square of a self-adjoint operator D. We shall assume that the underlying Hilbert space H is $\mathbb{Z}/2$-graded; that the operator D is odd; and that the algebra $\mathcal{D}$ is stable under multiplication by the grading operator ε.

We shall also assume that an algebra $A \subseteq \mathcal{D}(\Delta)$ is specified, consisting of operators of differential order zero (the operators in A are therefore bounded operators on H) which are even with respect to the grading. We shall assign the order $\frac{r}{2}$ to D (recall that r is the order of Δ), and we shall assume that if $a \in A$, then $\mathrm{order}\big([D, a]\big) \leq \mathrm{order}(D) - 1$.

In the standard example of a smooth manifold, A will be the ring of smooth, compactly supported functions on M.

5.5 Example. In the case of a differential pair which is generated from a regular spectral triple (A, H, D), we shall assume that the spectral triple is *even*, which means that H is $Z/2$-graded, A is comprised of even operators, and D is odd. We enlarge the algebra $\mathcal{D}$ of Definition 3.23 by guaranteeing it to be closed under multiplication by the grading operator ε. Then we can let D itself be our square root of Δ, and take A to be the algebra of order zero operators.

5.3 Cyclic Cohomology Theory

In this subsection we shall establish some notation and terminology related to cyclic cohomology theory. We shall follow Connes' approach to cyclic cohomology, which is described for example in his book [4, Chapter 3], to which we refer the reader for more details.

Let A be an associative algebra over $\mathbb{C}$ and for the moment let us assume that A has a multiplicative unit. If V is a complex vector space and p is a non-negative integer, then let us denote by $C^p(V)$ space of $(p+1)$-multi-linear maps from A into V. Usually one is interested in the case where $V = \mathbb{C}$, but for our purposes it is useful to consider other cases too.

We are going to define the periodic cyclic cohomology of A with coefficients in V, and to do so we introduce the operators

$$b\colon C^p(V) \to C^{p+1}(V) \quad \text{and} \quad B\colon C^{p+1}(V) \to C^p(V),$$

which are defined by the formulas

$$b\phi(a^0, \dots, a^{p+1}) = \sum_{j=0}^{p} (-1)^j \phi(a^0, \dots, a^j a^{j+1}, \dots, a^{p+1}) + (-1)^{p+1}\phi(a^{p+1}a^0, \dots, a^p) \tag{5.1}$$

and

$$B\phi(a^0, \dots, a^p) = \sum_{j=0}^{p} (-1)^{pj}\phi(1, a^j, a^{j+1}, \dots, a^{j-1}) + \sum_{j=0}^{p} (-1)^{p(j-1)}\phi(a^j, a^{j+1}, \dots, a^{j-1}, 1). \tag{5.2}$$

5.6 Lemma. $b^2 = 0$, $B^2 = 0$ *and* $bB + Bb = 0$. □

As a result of the lemma, we can assemble from the spaces $C^p(V)$ the following double complex, which is continued indefinitely to the left and to the top.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} \\
\cdots & \xrightarrow{B} & C^3(V) & \xrightarrow{B} & C^2(V) & \xrightarrow{B} & C^1(V) & \xrightarrow{B} & C^0(V) \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
\cdots & \xrightarrow{B} & C^2(V) & \xrightarrow{B} & C^1(V) & \xrightarrow{B} & C^0(V) & & \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & & & \\
\cdots & \xrightarrow{B} & C^1(V) & \xrightarrow{B} & C^0(V) & & & & \\
 & & \uparrow{\scriptstyle b} & & & & & & \\
\cdots & \xrightarrow{B} & C^0(V) & & & & & &
\end{array}
$$

5.7 Definition. The *periodic cyclic cohomology of* A*, with coefficients in* V is the cohomology of the totalization of this complex. Thanks to the symmetry inherent in the complex, all even periodic cyclic cohomology groups are the same, as are all the odd groups. So we shall use the notations $PHC^{\text{even}}(A, V)$ and $PHC^{\text{odd}}(A, V)$.

A cocycle for $PHC^{\text{even}}(A, V)$ is a sequence

$$(\phi_0, \phi_2, \phi_4, \dots),$$

where $\phi_{2k} \in C^{2k}(V)$, $\phi_{2k} = 0$ for all but finitely many k, and

$$b\phi_{2k} + B\phi_{2k+2} = 0$$

for all $k \geq 0$. A cocycle for $PHC^{\text{odd}}(A, V)$ is a sequence

$$(\phi_1, \phi_3, \phi_5, \dots),$$

where $\phi_{2k+1} \in C^{2k+1}(V)$, $\phi_{2k+1} = 0$ for all but finitely many k, and

$$b\phi_{2k+1} + B\phi_{2k+3} = 0$$

for all $k \geq 0$ (and in addition $B\phi_1 = 0$).

The periodic cyclic cohomology groups of A can be computed from a variety of complexes, so we shall refer to cocycles of the above sort as (b, B)-*cocycles*, with coefficients in V.

If we totalize the (b, B)-bicomplex by taking a direct *product* of cochain groups along the diagonals instead of a direct sum, then we obtain a complex with zero cohomology. We shall refer to cocycles for this complex (consisting in the even case of sequences $(\phi_0, \phi_2, \phi_4, \dots)$, all of whose terms may be nonzero) as *improper* (b, B)-cocycles. On their own, improper periodic (b, B)-cocycles have no cohomological significance, but nevertheless the concept will be a convenient one for us.

If the algebra A has no multiplicative unit then by a (b, B)-cocycle for A we shall mean a (b, B)-cocycle $\{\phi_{2k}\}$ or $\{\phi_{2k+1}\}$ for the algebra $\tilde{A}$ obtained from A by adjoining a unit, which gives the value zero when the value $1 \in \mathbb{C}$ is placed in any but the first argument of any of the multilinear maps ϕ_j (in the even case one also requires that $\phi_0(1) = 0$). This vanishing condition defines a subcomplex of the (b, B)-bicomplex.

5.8 Example. Let M be a smooth, closed manifold and denote by $C^\infty(M)$ the algebra of smooth, complex-valued functions on M. For $p \geq 0$ denote by Ω_p the space of p-dimensional de Rham currents (dual to the space Ω^p of smooth p-forms). Each current $c \in \Omega_p$ determines a cochain $\phi_c \in C^p(\mathbb{C})$ for the algebra $C^\infty(M)$ by the formula

$$\phi_c(f^0, \dots, f^p) = \int_c f^0 df^1 \cdots df^p.$$

One has that

$$b\phi_c = 0 \quad \text{and} \quad B\phi_c = p \cdot \phi_{d^* c},$$

where $d^*\colon \Omega_p \to \Omega_{p-1}$ is the operator adjoint to the de Rham differential. This leads one to consider the following bicomplex:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \\
\cdots & \xrightarrow{4d^*} & \Omega_3 & \xrightarrow{3d^*} & \Omega_2 & \xrightarrow{2d^*} & \Omega_1 & \xrightarrow{d^*} & \Omega_0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \\
\cdots & \xrightarrow{3d^*} & \Omega_2 & \xrightarrow{2d^*} & \Omega_1 & \xrightarrow{d^*} & \Omega_0 & & \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & & & \\
\cdots & \xrightarrow{2d^*} & \Omega_1 & \xrightarrow{d^*} & \Omega_0 & & & & \\
 & & \uparrow{\scriptstyle 0} & & & & & & \\
\cdots & \xrightarrow{d^*} & \Omega_0 & & & & & &
\end{array}$$

A fundamental result of Connes [3, Theorem 46] asserts that this complex computes periodic cyclic cohomology for $A = C^\infty(M)$:

5.9 Theorem. *The inclusion* $c \mapsto \phi_c$ *of the above double complex into the* (b, B)*-bicomplex induces isomorphisms*

$$\mathrm{HCP}^{\text{even}}_{\text{cont}}(C^\infty(M)) \cong H_0(M) \oplus H_2(M) \oplus \cdots$$

and

$$\mathrm{HCP}^{\text{odd}}_{\text{cont}}(C^\infty(M)) \cong H_1(M) \oplus H_3(M) \oplus \cdots$$

Here $\mathrm{HCP}^*_{\text{cont}}(C^\infty(M))$ *denotes the periodic cyclic cohomology of* M*, computed from the bicomplex of* continuous *multi-linear functionals on* $C^\infty(M)$. □

It follows that an even/odd (b, B)-cocycle for $C^\infty(M)$ is something very like a family of closed currents on M of even/odd degrees. This close connection with de Rham theory makes the (b, B)-description of cyclic cohomology particularly well suited to index theory problems.

5.10 Definition. A multi-linear functional $\phi_p \in C^p(V)$ is said to be *cyclic* if

$$\phi_p(a^0, a^1, \ldots, a^p) = (-1)^p \phi_p(a^p, a^0, \ldots, a^{p-1}),$$

for all $a^0, \ldots, a^p$ in A.

If ϕ_p is cyclic then it is clear from the formula (5.2) that $B\phi_p = 0$. As a result, if in addition $b\phi_p = 0$, then we obtain a (b, B)-cocycle

$$(0, \ldots, 0, \phi_p, 0, \ldots)$$

by placing ϕ_p in position p and 0 everywhere else. These are the *cyclic cocycles* of Connes [3], using which Connes first formulated the definition of cyclic cohomology.

5.11 Lemma. *Every* (b, B)*-cocycle is cohomologous to a cyclic cocycle of some degree* p. □

5.4 Chern Character and Pairings with K-Theory

One of the most important cyclic cocycles is defined as follows. Let A be an algebra of bounded operators on a Hilbert space H and let F be a bounded operator on H such that $F^2 = 1$. Assume in addition that the Hilbert space H is $\mathbb{Z}/2$-graded, and that A consists of even operators, while F is odd.

5.12 Theorem. *Let* n *be an* even *integer and assume that for all* $a^0, \dots, a^n$ *in* A *the product* $[F, a^0][F, a^1]\cdots[F, a^n]$ *lies in the trace ideal. The formula*

$$\operatorname{ch}^F_n\big(a^0, \dots, a^n\big) = \frac{\Gamma(\frac{n}{2}+1)}{2\cdot n!}\operatorname{Trace}\big(\varepsilon F[F, a^0][F, a^1]\dots[F, a^n]\big) \tag{5.3}$$

defines a cyclic n*-cocycle whose class in periodic cyclic cohomology is independent of* n *(as can be seen by inserting* ε *at the front of the formula above for* ψ_{n+1}*).* □

This is Connes' *(even) cyclic Chern character* of F. The constant in front of the trace is chosen in such a way that the periodic cyclic cohomology class of the (b, B)-cocycle determined by ch^F_n is independent of n. To see that this is so, one can define

$$\psi_{n+1}(a^0, \dots, a^{n+1}) = \frac{\Gamma(\frac{n}{2}+2)}{(n+2)!}\operatorname{Trace}\big(\varepsilon a^0 F[F, a^1][F, a^2]\dots[F, a^{n+1}]\big).$$

and then compute that $b\psi_{n+1} = -\operatorname{ch}^F_{n+2}$ while $B\psi_{n+1} = \operatorname{ch}^F_n$.[7]

Each even (b, B)-cocycle determines a homomorphism from the algebraic K-theory group $K_0(A)$ to $\mathbb{C}$, depending only on the periodic cyclic cohomology class of the cocycle. If e is an idempotent in A then we can form the element $[e] \in K_0(A)$. Under the pairing between cyclic theory and K-theory the class $[e]$ is mapped by an even (b, B)-cocycle $\phi = (\phi_0, \phi_2, \dots)$ to the scalar

$$\phi\big([e]\big) = \phi_0(e) + \sum_{k=1}^{\infty}(-1)^k\frac{(2k)!}{k!}\phi_{2k}(e - \frac{1}{2}, e, e, \dots, e). \tag{5.4}$$

Compare [12]. In the case of the even cyclic Chern character defined in the last section, the pairing is

$$\operatorname{ch}^F\big([e]\big) = \operatorname{Index}(eFe\colon eH_0 \to eH_1), \tag{5.5}$$

where H_0 and H_1 are the degree zero and degree one parts of the $\mathbb{Z}/2$-graded Hilbert space H.

This connection with index theory makes it a very interesting problem to compute the cyclic Chern character in various instances, and it is this problem to which

[7] This formula actually proves Theorem 5.12 since $b^2 = 0$ and the image of the differential B is comprised entirely of cyclic multi-linear functionals.

we want to turn our attention. For example, in the case of an ordinary elliptic operator on a closed manifold, where the cyclic cohomology of $A = C^\infty(M)$ identifies with the de Rham homology of M, the identification of the class of the Chern character with a specific homology class on M is equivalent to the Atiyah-Singer Index Theorem.

Our goal will be not so much to compute the Chern character in this (or any other) specific instance. Instead we aim to show that in general the Chern character is cohomologous to a cocycle constructed entirely out of residues of zeta functions. As we saw in Section 4, in at least the classical case this leads to complicated but explicit formulas from which Fredholm indexes may in principle be computed. The problem of actually organizing and simplifying these formulas in various cases is both interesting and important, but we shall not consider it in these notes.

5.5 Zeta Functions

Let us continue to assume as given an differential pair $(\mathcal{D}, \Delta)$ of generalized differential operators, along with a square-root decomposition $\Delta = D^2$.

We are now going to define certain zeta-type functions associated with the algebra. To simplify matters we shall now assume that the operator Δ is *invertible*. This assumption will remain in force until Section 8, where we shall consider the general case.

5.13 Definition. The differential pair $(\mathcal{D}, \Delta)$ has *finite analytic dimension* if there is some $d \geq 0$ with the property that if $X \in \mathcal{D}$ has order q or less, then for every $z \in \mathbb{C}$ with real part greater than $\frac{q+d}{r}$ the operator $X\Delta^{-z}$ extends by continuity to a trace-class operator on H (here r is the order of Δ, as described in Section 3.1).

Assume that $(\mathcal{D}, \Delta)$ has finite analytic dimension d. If $X \in \mathcal{D}(\Delta)$ and if $\text{order}(X) \leq q$ then the complex function $\text{Trace}(X\Delta^{-z})$ is holomorphic in the right half-plane $\text{Re}(z) > \frac{q+d}{r}$.

5.14 Definition. An differential pair $(\mathcal{D}, \Delta)$ which has finite analytic dimension has the *meromorphic continuation property* if for every $X \in \mathcal{D}(\Delta)$ the analytic function $\text{Trace}(X\Delta^{-z})$, defined initially on a half-plane in $\mathbb{C}$, extends to a meromorphic function on the full complex plane.

Actually, for what follows it would be sufficient to assume that $\text{Trace}(X\Delta^{-s})$ has an analytic continuation to $\mathbb{C}$ with only isolated singularities, which could perhaps be essential singularities.

5.15 Definition. Let $(\mathcal{D}, \Delta)$ be a differential pair which has finite analytic dimension. Define, for $\mathrm{Re}(z) > 0$ and $X^0, \ldots, X^p \in \mathcal{D}$,[8] the quantity

$$(5.6) \quad \langle X^0, X^1, \ldots, X^p \rangle_z = (-1)^p \frac{\Gamma(z)}{2\pi i} \operatorname{Trace}\left(\int \lambda^{-z} \varepsilon X^0 (\lambda - \Delta)^{-1} X^1 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \, d\lambda \right)$$

(the factors in the integral alternate between the X^j and copies of $(\lambda - \Delta)^{-1}$). The contour integral is evaluated down a vertical line in $\mathbb{C}$ which separates 0 and $\mathrm{Spectrum}(\Delta)$.

5.16 Remark. If $\mathrm{order}(X^0) + \cdots + \mathrm{order}(X^p) \le q$ and if the integrand in equation (5.6) is viewed as a bounded operator from H^{s+q} to H^s, then the integral converges absolutely in the operator norm whenever $\mathrm{Re}(z) + p > 0$. In particular, if $\mathrm{Re}(z) > 0$ then the integral converges to a well defined operator on H^∞.

Of course, apart from the insertion of the grading operator ε, this is precisely the sort of integral we encountered in our discussion of meromorphic continuation in Section 2. In our former notation,

$$\langle X^0, X^1, \ldots, X^p \rangle_z = (-1)^p \Gamma(z) I_z(\varepsilon X^0, X^1, \ldots, X^p).$$

Using the arguments we developed in Section 2 we obtain the following results:

5.17 Proposition. *Let $(\mathcal{D}, \Delta)$ be a differential pair and let $X^0, \ldots, X^p \in \mathcal{D}$. Assume that*

$$\mathrm{order}(X^0) + \cdots + \mathrm{order}(X^p) \le q.$$

If $(\mathcal{D}, \Delta)$ has finite analytic dimension d, and if $\mathrm{Re}(z) + p > \frac{1}{r}(q + d)$, then the integral in Equation (5.6) extends by continuity to a trace-class operator on H, and the quantity $\langle X^0, \ldots, X^p \rangle_z$ defined by Equation (5.6) is a holomorphic function of z in this half-plane. If in addition the algebra $(\mathcal{D}, \Delta)$ has the meromorphic continuation property then the quantity $\langle X^0, \ldots, X^p \rangle_z$ extends to a meromorphic function on $\mathbb{C}$. □

5.18 Definition. Let $k = (k_1, \ldots, k_p)$ be a multi-index with non-negative integer entries. Define a constant $c(k)$ by the formula

$$c(k) = \frac{(k_1 + \cdots + k_p + p)!}{k_1! \cdots k_p! (k_1 + 1) \cdots (k_1 + \cdots + k_p + p)}.$$

[8] Occasionally we shall take one or more of the X^j to lie within a larger algebra, for example the algebra generated by $\mathcal{D}$, I and D.

5.19 Proposition. *Let $(\mathcal{D}, \Delta)$ be a differential pair with the meromorphic continuation property and let $X^0, \ldots, X^p \in \mathcal{D}$. There is an asymptotic expansion*

$$\langle X^0, \ldots, X^p \rangle_z \approx \sum_{k \geq 0} (-1)^{|k|} \Gamma(z + p + |k|) \frac{1}{(|k| + p)!} c(k) \times \operatorname{Trace}\left(\varepsilon X^0 X^{1^{(k_1)}} \cdots X^{p^{(k_p)}} \Delta^{-z-|k|-p} \right),$$

where the symbol $\approx$ means that, given any right half-plane in $\mathbb{C}$, any sufficiently large finite partial sum of the right hand side agrees with the left hand side modulo a function of z which is holomorphic in that half-plane. □

5.6 Formulation of the Local Index Theorem

The following result is the local index formula of Connes and Moscovici:

5.20 Theorem. *Let $(\mathcal{D}, \Delta)$ be a differential pair with the meromorphic continuation property and let D be a square root of Δ. The formula*

$$\psi_p(a^0, \ldots, a^p) = \sum_{k \geq 0} \frac{(-1)^{|k|} c(k)}{(|k| + p)!} \times \operatorname{Res}_{s=0}\left(\Gamma(s + \frac{p}{2} + |k|) \operatorname{Tr}\left(\varepsilon a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} \Delta^{-\frac{p}{2}-|k|-s} \right) \right)$$

defines an periodic (b, B)-cocycle $\{\psi_{2k}\}$ for A which is cohomologous to the cyclic Chern character of the operator $F = D|D|^{-1}$.

5.21 Remark. If $|k| + p > d$ then the (p, k)-contribution to the above sum of residues is actually zero. Hence for every p the sum is in fact finite (and the sum is 0 when $p > d$).

5.22 Remark. If all the poles of the zeta functions $\operatorname{Trace}(X\Delta^{-z})$ are simple then the above cocycle can be rewritten as

$$\psi_p(a^0, \ldots, a^p) = \sum_{k \geq 0} C_{p,k} \operatorname{Res}_{s=0} \operatorname{Tr}\left(\varepsilon a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} \Delta^{-\frac{p}{2}-|k|-s} \right),$$

where

$$C_{pk} = \frac{(-1)^k}{k!} \frac{\Gamma(|k| + \frac{p}{2})}{(k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots+k_p+p)}.$$

(Note: the constant $C_{00} = \Gamma(0)$ is not well defined in our formula since 0 is a pole of the Γ-function. To cope with this problem we must treat the $p = 0$, $k = 0$ term separately and replace $C_{00} \operatorname{Res}_{s=0} \left(\operatorname{Tr}(\varepsilon a^0 \Delta^{-s})\right)$ with $\operatorname{Res}_{s=0} \left(\Gamma(s) \operatorname{Tr}(\varepsilon a^0 \Delta^{-s})\right)$.)

6 The Residue Cocycle

6.1 Improper Cocycle

In this section we shall assume as given a differential pair $(\mathcal{D}, \Delta)$ with the meromorphic continuation property, a square root D of Δ, and an algebra $\mathcal{A} \subseteq \mathcal{D}$, as in the previous section.

We are going to define a periodic cyclic cocycle $\Psi = (\Psi_0, \Psi_2, \dots)$ for the algebra $\mathcal{A}$. The cocycle will be *improper*—all the Ψ_p will be nonzero. Moreover the cocycle will assume values in the field of meromorphic functions on $\mathbb{C}$. But in the next section we shall convert it into a proper cocycle with values in $\mathbb{C}$ itself.

We are going to assemble Ψ from the quantities $\langle X^0, \dots, X^p \rangle_z$ defined in Subsection 5.5.[9] We begin by establishing some 'functional equations' for the quantities $\langle \cdots \rangle_z$. In order to keep the formulas reasonably compact, if $X \in \mathcal{D}$ then we shall write $(-1)^X$ to denote either $+1$ or -1, according as X is an even or odd operator on the $\mathbb{Z}/2$-graded Hilbert space H.

6.1 Lemma. *The meromorphic functions $\langle X^0, \dots, X^p \rangle_z$ satisfy the following functional equations:*

$$\langle X^0, \dots, X^{p-1}, X^p \rangle_{z+1} = \sum_{j=0}^{p} \langle X^0, \dots, X^{j-1}, 1, X^j, \dots, X^p \rangle_z \tag{6.2}$$

$$\langle X^0, \dots, X^{p-1}, X^p \rangle_z = (-1)^{X^p} \langle X^p, X^0, \dots, X^{p-1} \rangle_z \tag{6.3}$$

Proof. The first identity follows from the fact that

$$\begin{aligned} \frac{d}{d\lambda} \big(\lambda^{-z} X^0 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1}\big) \\ &= (-z)\lambda^{-z-1} X^0 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \\ &\quad - \sum_{j=0}^{p} \lambda^{-z} X^0 (\lambda - \Delta)^{-1} \cdots X^j (\lambda - \Delta)^{-2} X^{j+1} \cdots X^p (\lambda - \Delta)^{-1} \end{aligned}$$

[9] In doing so we shall follow quite closely the construction of the so-called JLO cocycle in entire cyclic cohomology (see [21] and [12]), which is assembled from the quantities

$$\langle X^0, \dots, X^p \rangle^{\mathrm{JLO}} = \mathrm{Trace}\left(\int_{\Sigma^p} \varepsilon X^0 e^{-t_0 \Delta} \dots X^p e^{-t_p \Delta} \, dt \right) \tag{6.1}$$

(the integral is over the standard p-simplex). The computations which follow in this section are more or less direct copies of computations already carried out for the JLO cocycle in [21] and [12].

and the fact that the integral of the derivative is zero. As for the second identity, if $p \gg 0$ then the integrand in Equation (5.6) is a trace-class operator, and Equation (6.3) is an immediate consequence of the trace-property. In general we can repeatedly apply Equation (6.2) to reduce to the case where $p \gg 0$. □

6.2 Lemma.

$$
\begin{aligned}
(6.4)\quad & \langle X^0, \dots, [D^2, X^j], \dots, X^p \rangle_z = \\
& \qquad \langle X^0, \dots, X^{j-1}X^j, \dots X^p \rangle_z - \langle X^0, \dots, X^j X^{j+1}, \dots X^p \rangle_z
\end{aligned}
$$

Proof. This follows from the identity

$$
\begin{aligned}
& X^{j-1}(\lambda - \Delta)^{-1}[D^2, X^j](\lambda - \Delta)^{-1}X^{j+1} \\
& \qquad = X^{j-1}(\lambda - \Delta)^{-1}X^j X^{j+1} - X^{j-1}X^j(\lambda - \Delta)^{-1}X^{j+1}.
\end{aligned}
$$

□

6.3 Lemma.

$$
(6.5)\qquad \sum_{j=0}^{p} (-1)^{X^0 \cdots X^{j-1}} \langle X^0, \dots, [D, X^j], \dots, X^p \rangle_z = 0
$$

Proof. The identity is equivalent to the formula

$$
\text{Trace}\left(\varepsilon \left[D, \int \lambda^{-z} X^0 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1}\, d\lambda \right] \right) = 0,
$$

which holds since the supertrace of any (graded) commutator is zero. □

With these preliminaries out of the way we can obtain very quickly our improper (b, B)-cocycle.

6.4 Definition. If p is a non-negative and even integer then define a $(p+1)$-multi-linear functional on A with values in the meromorphic functions on $\mathbb{C}$ by the formula

$$
\Psi_p(a^0, \dots, a^p) = \langle a^0, [D, a^1], \dots, [D, a^p] \rangle_{s - \frac{p}{2}}
$$

6.5 Theorem. *The even* (b, B)*-cochain* $\Psi = (\Psi_0, \Psi_2, \Psi_4 \cdots)$ *is an improper* (b, B)*-cocycle with coefficients in the space of meromorphic functions on* $\mathbb{C}$*.*

Proof. First of all, it follows from the definition of B and Lemma 6.1 that

$$\begin{aligned} B\Psi_{p+2}(a^0,\dots,a^{p+1}) &= \sum_{j=0}^{p+1}(-1)^j\langle 1,[D,a^j],\dots,[D,a^{j-1}]\rangle_{s-\frac{p+2}{2}} \\ &= \sum_{j=0}^{p+1}\langle [D,a^0],\dots,[D,a^{j-1}],1,[D,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p+2}{2}} \\ &= \langle [D,a^0],[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}. \end{aligned}$$

Next, it follows from the definition of b and the Leibniz rule $[D,a^ja^{j+1}] = a^j[D,a^{j+1}]+[D,a^j]a^{j+1}$ that

$$\begin{aligned} b\Psi_p(a^0,\dots,a^{p+1}) &= \big(\langle a^0a^1,[D,a^2],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}} \\ &\qquad - \langle a^0,a^1[D,a^2],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big) \\ &\quad - \big(\langle a^0,[D,a^1]a^2,[D,a^3],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}} \\ &\qquad - \langle a^0,[D,a^1],a^2[D,a^3],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big) \\ &\quad + \cdots \\ &\quad + \big(\langle a^0,[D,a^1],\dots,[D,a^p]a^{p+1}\rangle_{s-\frac{p}{2}} \\ &\qquad - \langle a^{p+1}a^0,[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big). \end{aligned}$$

Applying Lemma 6.2 we get

$$b\Psi_p(a^0,\dots,a^{p+1}) = \sum_{j=1}^{p+1}(-1)^{j-1}\langle a^0,[D,a^1],\dots,[D^2,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}$$

Setting $X^0 = a^0$ and $X^j = [D,a^j]$ for $j \geq 1$, and applying Lemma 6.3 we get

$$\begin{aligned} &B\Psi_{p+2}(a^0,\dots,a^{p+1}) + b\Psi_p(a^0,\dots,a^{p+1}) \\ &\qquad = \sum_{j=0}^{p+1}(-1)^{X^0\dots X^{j-1}}\langle X^0,\dots,[D,X^j],\dots,X^{p+1}\rangle_{s-\frac{p}{2}} = 0. \end{aligned}$$

□

6.2 Residue Cocycle

By taking residues at $s = 0$ we map the space of meromorphic functions on $\mathbb{C}$ to the scalar field $\mathbb{C}$, and we obtain from any (b,B)-cocycle with coefficients in the space of meromorphic functions a (b, B)-cocycle with coefficients in $\mathbb{C}$. This operation transforms the improper cocycle Ψ that we constructed in the last section into a *proper* cocycle $\mathrm{Res}_{s=0}\Psi$. Indeed, it follows from Proposition 5.17 that if p is greater than the analytic dimension d of $(\mathcal{D}, \Delta)$ then the function

$$\Psi_p(a^0, \dots, a^p)_s = \langle a^0, [D, a^1], \dots, [D, a^p] \rangle_{s-\frac{p}{2}}$$

is holomorphic at $s = 0$.

The following proposition identifies the proper (b, B)-cocycle $\mathrm{Res}_{s=0}\Psi$ with the residue cocycle studied by Connes and Moscovici. The proof follows immediately from our computations in Section 2, as summarized in Subsection 5.5.

6.6 Theorem. *For all* $p \geq 0$ *and all* $a^0, \dots, a^p \in A$,

$$\mathrm{Res}_{s=0}\Psi_p(a^0, \dots, a^p) = \sum_{k \geq 0} \frac{(-1)^{|k|} c(k)}{(|k| + p)!}$$
$$\times \mathrm{Res}_{s=0}\left(\Gamma(s + \frac{p}{2} + |k|) \,\mathrm{Tr}\left(\varepsilon a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} \Delta^{-\frac{p}{2} - |k| - s} \right) \right),$$

where

$$c(k) = \frac{(k_1 + \cdots + k_p + p)!}{k_1! \cdots k_p! (k_1 + 1) \cdots (k_1 + \cdots + k_p + p)}.$$

□

6.3 Complex Powers in a Differential Algebra

In this subsection we shall try to sketch out a more conceptual view of the improper cocycle which was constructed in Section 6.1. This involves Quillen's cochain picture of cyclic cohomology [24], and in fact it was Quillen's account of the JLO cocycle from this perspective which first led to the formula for the quantity $\langle X^0, \dots, X^p \rangle_z$ given in Definition 5.15. Since our purpose is only to view the cocycle Ψ in a more conceptual way we shall not carefully keep track of analytic details.

As we did when we looked at cyclic cohomology in Subsection 5.3, let us fix an algebra A. But let us now also fix a second algebra L. For $n \geq 0$ denote by $\mathrm{Hom}^n(A, L)$ the vector space of n-linear maps from A to L. By a 0-linear map from A to L we shall mean a linear map from $\mathbb{C}$ to L, or in other words just an element of L. Let $\mathrm{Hom}^{**}(A, L)$ be the direct product

$$\mathrm{Hom}^{**}(A, L) = \prod_{n=0}^{\infty} \mathrm{Hom}^n(A, L).$$

Thus an element ϕ of $\mathrm{Hom}^{**}(A, L)$ is a sequence of multi-linear maps from A to L. We shall denote by $\phi(a^1, \dots, a^n)$ the value of the n-th component of ϕ on the n-tuple $(a^1, \dots, a^n)$.

The vector space $\mathrm{Hom}^{**}(A, L)$ is $\mathbb{Z}/2$-graded in the following way: an element ϕ is even (resp. odd) if $\phi(a^1, \dots, a^n) = 0$ for all odd n (resp. for all even n). We shall denote by $\deg_M(\phi) \in \{0, 1\}$ the grading-degree of ϕ. (The letter 'M' stands for 'multi-linear'; a second grading-degree will be introduced below.)

6.7 Lemma. *If* $\phi, \psi \in \mathrm{Hom}^{**}(A, L)$*, then define*

$$\phi \vee \psi(a^1, \dots, a^n) = \sum_{p+q=n} \phi(a^1, \dots, a^p)\psi(a^{p+1}, \dots, a^n)$$

and

$$d_M\phi(a^1, \dots, a^{n+1}) = \sum_{i=1}^{n} (-1)^{i+1}\phi(a^1, \dots, a^i a^{i+1}, \dots, a^{n+1}).$$

The vector space $\mathrm{Hom}^{**}(A, L)$*, so equipped with a multiplication and differential, is a* $\mathbb{Z}/2$*-graded differential algebra.* □

Let us now suppose that the algebra L is $\mathbb{Z}/2$-graded. If $\phi \in \mathrm{Hom}^{**}(A, L)$ then let us write $\deg_L(\phi) = 0$ if $\phi(a^1, \dots, a^n)$ belongs to the degree-zero part of L for every n and every n-tuple $(a^1, \dots, a^n)$. Similarly, if $\phi \in \mathrm{Hom}^{**}(A, L)$ then let us write $\deg_L(\phi) = 1$ if $\phi(a^1, \dots, a^n)$ belongs to the degree-one part of L for every n and every n-tuple $(a^1, \dots, a^n)$. This is a new $\mathbb{Z}/2$-grading on the vector space $\mathrm{Hom}^{**}(A, L)$. The formula

$$\deg(\phi) = \deg_M(\phi) + \deg_L(\phi)$$

defines a third $\mathbb{Z}/2$-grading—the one we are really interested in. Using this last $\mathbb{Z}/2$-grading, we have the following result:

6.8 Lemma. *If* $\phi, \psi \in \mathrm{Hom}^{**}(A, L)$*, then define*

$$\phi \diamond \psi = (-1)^{\deg_M(\phi)\deg_L(\psi)}\phi \vee \psi$$

and

$$d\phi = (-1)^{\deg_L(\phi)} d'\phi$$

These new operations once again provide $\mathrm{Hom}^{**}(A, L)$ *with the structure of a* $\mathbb{Z}/2$*-graded differential algebra (for the total* $\mathbb{Z}/2$*-grading* $\deg(\phi) = \deg_M(\phi) + \deg_L(\phi)$*).* □

We shall now specialize to the situation in which A and $\Delta = D^2$ are as in previous sections, and L is the algebra of all operators on the $\mathbb{Z}/2$-graded vector space $H^\infty \subseteq H$.

Denote by ρ the inclusion of A into L. This is of course a 1-linear map from A to L, and we can therefore think of ρ as an element of $\mathrm{Hom}^{**}(A, L)$ (all of whose n-linear components are zero, except for $n = 1$). In addition, let us think of D as a 0-linear map from A to L, and therefore as an element of $\mathrm{Hom}^{**}(A, L)$ too. Combining D and ρ let us define the 'superconnection form'

$$\theta = D - \rho \in \mathrm{Hom}^{**}(A, L)$$

This has odd $\mathbb{Z}/2$-grading degree (that is, $\deg(\theta) = 1$). Let K be its 'curvature':

$$K = d\theta + \theta^2,$$

which has even $\mathbb{Z}/2$-grading degree. Using the formulas in Lemma 6.8 the element K may be calculated, as follows:

6.9 Lemma. *One has*

$$K = \Delta - E \in \mathrm{Hom}^{**}(A, L),$$

where $E \colon A \to L$ *is the* 1*-linear map defined by the formula*

$$E(a) = [D, \rho(a)]. \qquad \square$$

In all of the above we are following Quillen, who then proceeds to make the following definition, which is motivated by the well-known Banach algebra formula

$$e^{b-a} = \sum_{n=0}^{\infty} \int_{\Sigma^n} e^{-t_0 a} b e^{-t_1 a} \cdots b e^{-t_n a}\, dt.$$

6.10 Definition. Denote by $e^{-K} \in \mathrm{Hom}^{**}(A, L)$ the element

$$e^{-K} = \sum_{n=0}^{\infty} \int_{\Sigma^n} e^{-t_0 \Delta} E e^{-t_1 \Delta} \dots E e^{-t_n \Delta}\, dt.$$

The n-th term in the sum is an n-linear map from A to L, and the series should be regarded as defining an element of $\mathrm{Hom}^{**}(A, L)$ whose n-linear component is this term. As Quillen observes, in [24, Section 8] the exponential e^{-K} defined in this way has the following two crucial properties:

6.11 Lemma (Bianchi Identity). $d(e^{-K}) + [e^{-K}, \theta] = 0$. □

6.12 Lemma (Differential Equation). *Suppose that* δ *is a derivation of* $\mathrm{Hom}^{**}(A, L)$ *into a bimodule. Then*

$$\delta(e^{-K}) = -\delta(K)e^{-K},$$

modulo (limits of) commutators. □

Both lemmas follow from the 'Duhamel formula'

$$\delta(e^{-K}) = \int_0^1 e^{-tK}\delta(K)e^{-(1-t)K}\,dt,$$

which is familiar from semigroup theory and which may be verified for the notion of exponential now being considered. (Once more, we remind the reader that we are disregarding analytic details.)

Suppose we now introduce the 'supertrace' $\mathrm{Trace}_\varepsilon(X) = \mathrm{Trace}(\varepsilon X)$ (which is of course defined only on a subalgebra of L). Quillen reinterprets the Bianchi Identity and the Differential Equation above as coboundary computations in a complex which computes periodic cyclic cohomology (using improper cocycles, in our terminology here). As a result he is able to recover the following basic fact about the JLO cocycle — namely that it really is a cocycle:

6.13 Theorem (Quillen). *The formula*

$$\Phi_{2n}(a^0, \ldots, a^{2n}) = \int_{\Sigma^n} \mathrm{Trace}\left(\varepsilon a^0 e^{-t_0\Delta}[D, a^1]e^{-t_1\Delta}[D, a^2]\ldots[D, a^n]e^{-t_n\Delta}\right)\,dt$$

defines a (b, B)*-cocycle.* □

With this in mind, let us consider other functions of the curvature operator K, beginning with resolvents.

6.14 Lemma. *If* $\lambda \notin \mathrm{Spectrum}(\Delta)$ *then the element* $(\lambda - K) \in \mathrm{Hom}^{**}(A, L)$ *is invertible.*

Proof. Since $(\lambda - K) = (\lambda - \Delta) + E$ we can write

$$(\lambda - K)^{-1} = (\lambda - \Delta)^{-1} - (\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1} + (\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1} - \cdots$$

This is a series whose nth term is an n-linear map from A to L, and so the sum has an obvious meaning in $\mathrm{Hom}^{**}(A, L)$. One can then check that the sum defines $(\lambda - K)^{-1}$, as required. □

6.15 Definition. For any $z \in \mathbb{C}$ with positive real part define $K^{-z} \in \mathrm{Hom}^{**}(A, L)$ by the formula

$$K^{-z} = \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - K)^{-1} \, d\lambda,$$

in which the integral is a contour integral along a downward vertical line in $\mathbb{C}$ separating 0 from $\mathrm{Spectrum}(\Delta)$.

The assumption that $\mathrm{Re}(z) > 0$ guarantees convergence of the integral (in each component within $\mathrm{Hom}^{**}(A, L)$ the integral converges in the pointwise norm topology of n-linear maps from A to the algebra of bounded operators on H; the limit is also an operator from H^∞ to H^∞, as required). The complex powers K^{-z} so defined satisfy the following key identities:

6.16 Lemma (Bianchi Identity). $d(K^{-z}) + [K^{-z}, \theta] = 0$. □

6.17 Lemma (Differential Equation). *If δ is a derivation of* $\mathrm{Hom}^{**}(A, L)$ *into a bimodule, then*

$$\delta(K^{-z}) = -z\delta(K)K^{-z-1},$$

modulo (limits of) commutators. □

These follow from the derivation formula

$$\delta(K^{-z}) = \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - K)^{-1} \delta(K) (\lambda - K)^{-1} \, d\lambda.$$

In order to simplify the Differential Equation it is convenient to introduce the Gamma function, using which we can write

$$\delta\Big(\Gamma(z)K^{-z}\Big) = -\delta(K)\Gamma(z+1)K^{-(z+1)}$$

(modulo limits of commutators, as before). Except for the appearance of $z+1$ in place of z in the right hand side of the equation, this is exactly the same as the differential equation for e^{-K}. Meanwhile even after introducing the Gamma function we still have available the Bianchi identity:

$$d\Big(\Gamma(z)K^{-z}\Big) + \Big[\Gamma(z)K^{-z}, \theta\Big] = 0.$$

The degree n component of $\Gamma(z)K^{-z}$ is the multi-linear function

$$(a^1, \ldots, a^n) \mapsto \frac{(-1)^n}{2\pi i} \Gamma(z) \int \lambda^{-z} (\lambda - \Delta)^{-1} [D, a^1] \ldots [D, a^n] (\lambda - \Delta)^{-1} \, d\lambda,$$

Quillen's approach to JLO therefore suggests (and in fact upon closer inspection proves) the following result:

6.18 Theorem. *If we define*

$$\Psi_p^s(a^0, \ldots, a^p) = \frac{(-1)^p \Gamma(s - \frac{p}{2})}{2\pi i} \operatorname{Trace}\left(\int \lambda^{\frac{p}{2} - s} \varepsilon a^0 (\lambda - \Delta)^{-1} [D, a^1] \ldots [D, a^p] (\lambda - \Delta)^{-1} \, d\lambda \right),$$

then $b\Psi_p^s + B\Psi_{p+2}^s = 0$. □

This is of course precisely the conclusion that we reached in Section 6.1.

7 Comparison with the Chern Character

Our goal in this section is to identify the cohomology class of the residue cocycle $\mathrm{Res}_{s=0}\Psi$ with the cohomology class of the Chern character cocycle ch_n^F associated to the operator $F = D|D|^{-1}$ (see Section 5.3). Here n is any even integer greater than or equal to the analytic dimension d. It follows from the definition of analytic dimension and some simple manipulations that

$$[F, a^0] \cdots [F, a^n] \in \mathcal{L}^1(H),$$

for such n, so that the Chern character cocycle is well-defined.

We shall reach the goal in two steps. First we shall identify the cohomology class of $\mathrm{Res}_{s=0}\Psi$ with the class of a certain specific cyclic cocycle, which involves no residues. Secondly we shall show that this cyclic cocycle is cohomologous to the Chern character ch_n^F.

The following result summarizes step one.

7.1 Theorem. *Fix an even integer n strictly greater than $d-1$. The multi-linear functional*

$$(a^0, \ldots, a^n) \mapsto \frac{1}{2}\sum_{j=0}^{n} (-1)^{j+1} \langle [D, a^0], \ldots, [D, a^j], D, [D, a^{j+1}], \ldots, [D, a^n] \rangle_{-\frac{n}{2}}.$$

is a cyclic n-cocycle which, when considered as a (b, B)-cocycle, is cohomologous to the residue cocycle $\mathrm{Res}_{s=0}\Psi$.

7.2 Remark. It follows from Proposition 5.17 that the quantities

$$\langle [D, a^0], \ldots, [D, a^j], D, [D, a^{j+1}], \ldots, [D, a^n] \rangle_z$$

which appear in the theorem are holomorphic in the half-plane $\mathrm{Re}(z) > -\frac{n}{2} + \frac{1}{r}(d - (n+1))$. Therefore it makes sense to evaluate them at $z = -\frac{n}{2}$, as we have done. Appearances might suggest otherwise, because the term $\Gamma(z)$ which appears in the definition of $\langle \ldots \rangle_z$ has poles at the non-positive integers (and in particular at $z = -\frac{n}{2}$ if n is even). However these poles are canceled by zeroes of the contour integral in the given half-plane.

Theorem 7.1 and its proof have a simple conceptual explanation, which we shall give in a little while (after Lemma 7.7). However a certain amount of elementary, if laborious, computation is also involved in the proof, and we shall get to work on this first. For this purpose it is useful to introduce the following notation.

7.3 Definition. If $X^0, \ldots, X^p$ are operators in the algebra generated by $\mathcal{D}$, then define

$$\langle\langle X^0, \ldots, X^p \rangle\rangle_z = \sum_{k=0}^{p} (-1)^{X^0 \cdots X^k} \langle X^0, \ldots, X^k, D, X^{k+1}, \ldots, X^p \rangle_z,$$

which is a meromorphic function of $z \in \mathbb{C}$.

The new notation allows us to write a compact formula for the cyclic cocycle appearing in Theorem 7.1:

$$(a^0, \ldots, a^n) \mapsto \tfrac{1}{2} \langle\langle [D, a^0], \ldots, [D, a^n] \rangle\rangle_{-\frac{n}{2}}.$$

We shall now list some properties of the quantities $\langle\langle \cdots \rangle\rangle_z$ which are analogous to the properties of the quantities $\langle \cdots \rangle_z$ that we verified in Section 6.1. The following lemma may be proved using the formulas in Lemmas 6.1 and 6.2.

7.4 Lemma. *The quantity $\langle\langle X^0, \ldots, X^p \rangle\rangle_z$ satisfies the following identities:*

$$\langle\langle X^0, \ldots, X^p \rangle\rangle_z = \langle\langle X^p, X^0, \ldots, X^{p-1} \rangle\rangle_z \tag{7.1}$$

$$\sum_{j=0}^{p} \langle\langle X^0, \ldots, X^j, 1, X^{j+1}, \ldots, X^p \rangle\rangle_{z+1} = \langle\langle X^0, \ldots, X^p \rangle\rangle_z \tag{7.2}$$

In addition,

$$\begin{aligned} &\langle\langle X^0, \ldots, X^{j-1}X^j, \ldots, X^p \rangle\rangle_z - \langle\langle X^0, \ldots, X^jX^{j+1}, \ldots, X^p \rangle\rangle_z \\ &\quad = \langle\langle X^0, \ldots, [D^2, X^j], \ldots, X^p \rangle\rangle_z - (-1)^{X^0 \cdots X^{j-1}} \langle X^0, \ldots, [D, X^j], \ldots, X^p \rangle_z. \end{aligned} \tag{7.3}$$

(In both instances within this last formula the commutators are graded commutators.) □

We shall also need a version of Lemma 6.3, as follows.

7.5 Lemma.

$$\sum_{j=0}^{p} (-1)^{X^0 \cdots X^{j-1}} \langle\langle X^0, \ldots, [D, X^j], \ldots, X^p \rangle\rangle_z = 2 \sum_{k=0}^{p} \langle X^0, \ldots, X^{k-1}, D^2, X^k, \ldots, X^p \rangle_z. \tag{7.4}$$

□

Proof. This follows from Lemma 6.3. Note that $[D, D] = 2D^2$, which helps explain the factor of 2 in the formula. □

The formula in Lemma 7.5 can be simplified by means of the following computation:

7.6 Lemma.

$$\sum_{j=0}^{p} \langle X^0, \dots, X^j, D^2, X^{j+1}, \dots, X^p \rangle_z = (z+p)\langle X^0, \dots, X^p \rangle_z$$

Proof. If we substitute into the integral which defines $\langle X^0, \dots, D^2, \dots, X^p \rangle_z$ the formula

$$D^2 = \lambda - (\lambda - \Delta)$$

we obtain the (supertrace of the) terms

$$(-1)^{p+1}\frac{\Gamma(z)}{2\pi i}\int \lambda^{-z+1} X^0(\lambda-\Delta)^{-1}\cdots 1(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda \\ - (-1)^{p+1}\frac{\Gamma(z)}{2\pi i}\int \lambda^{-z} X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda$$

Using the functional equation $\Gamma(z) = (z-1)\Gamma(z-1)$ we therefore obtain the quantity

$$(z-1)\langle X^0, \dots, X^j, 1, X^{j+1}, \dots, X^p \rangle_{z-1} + \langle X^0, \dots, X^p \rangle_z$$

(the change in the sign preceding the second bracket comes from the fact that the bracket contains one fewer term, and the fact that $(-1)^{p+1} = -(-1)^p$). Adding up the terms for each j, and using Lemma 6.1 we therefore obtain

$$\begin{aligned}\sum_{j=0}^{p} \langle X^0, \dots, X^j, D^2, X^{j+1}, \dots, X^p \rangle_z &= (z-1)\langle X^0, \dots, X^p \rangle_z + (p+1)\langle X^0, \dots, X^p \rangle_z \\ &= (z+p)\langle X^0, \dots, X^p \rangle_z\end{aligned}$$

as required. □

Putting together the last two lemmas we obtain the formula

$$(7.5) \qquad \sum_{j=0}^{p} (-1)^{X^0 \cdots X^{j-1}} \langle\!\langle X^0, \dots, [D, X^j], \dots, X^p \rangle\!\rangle_z = 2(z+p)\langle X^0, \dots, X^p \rangle_z.$$

With this in hand we can proceed to the following computation:

7.7 Lemma. *Define multi-linear functionals Θ_p on A, with values in the space of meromorphic functions on $\mathbb{C}$, by the formulas*

$$\Theta_p(a^0, \dots, a^p) = \langle\!\langle a^0, [D, a^1], \dots, [D, a^p] \rangle\!\rangle_{s-\frac{p+1}{2}}.$$

Then

$$B\Theta_{p+1}(a^0, \dots, a^p) = \langle\!\langle [D, a^0], \dots, [D, a^p] \rangle\!\rangle_{s-\frac{p}{2}}.$$

and in addition

$$b\Theta_{p-1}(a^0, \dots, a^p) + B\Theta_{p+1}(a^0, \dots, a^p) = 2s\Psi_p(a^0, \dots, a^p)$$

for all $s \in \mathbb{C}$ and all $a^0, \dots, a^p \in A$.

Proof. The formula for $B\Theta_{p+1}(a^0, \dots, a^p)$ is a simple consequence of Lemma 7.4. The computation of $b\Theta_{p-1}(a^0, \dots, a^p)$ is a little more cumbersome, although still elementary. The reader who wants to see it carried out (rather than do it himself) is referred to [19]. □

7.8 Remark. The statement of Lemma 7.7 can be explained as follows. If we replace D by tD and Δ by $t^2\Delta$ in the definitions of $\langle \cdots \rangle_z$ and Ψ_p, so as to obtain a new improper (b, B)-cocycle $\Psi^t = (\Psi_0^t, \Psi_2^t, \dots)$, then it is easy to check from the definitions that

$$\Psi_p^t(a^0, \dots, a^p) = t^{-2s}\Psi_p(a^0, \dots, a^p).$$

Now, we expect that as t varies the cohomology class of the cocycle Ψ^t should not change. And indeed, by borrowing known formulas from the theory of the JLO cocycle (see for example [12], or [14, Section 10.2], or Section 7.1 below) we can construct a (b, B)-cochain Θ such that

$$B\Theta + b\Theta + \frac{d}{dt}\Psi^t = 0.$$

This is the same Θ as that which appears in the lemma.

The proof of Theorem 7.1 is now very straightforward:

Proof of Theorem 7.1. According to Lemma 7.7 the (b, B)-cochain

$$\Big(\mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_1\big), \mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_3\big), \dots, \mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_{n-1}\big), 0, 0, \dots\Big)$$

cobounds the difference of $\mathrm{Res}_{s=0}\,\Psi$ and the cyclic n-cocycle $\mathrm{Res}_{s=0}\big(\frac{1}{2s}B\Theta_{n+1}\big)$. Since

$$\mathrm{Res}_{s=0}\Big(\frac{1}{2s}B\Theta_{n+1}\Big)(a^0,\ldots,a^n)=\frac{1}{2}\langle\!\langle [D,a^0],\ldots,[D,a^n]\rangle\!\rangle_{-\frac{n}{2}}$$

the theorem is proved. □

We turn now to the second step. We are going to alter D by means of the following homotopy:

$$D_t = D|D|^{-t} \qquad (0\le t\le 1)$$

(the same strategy is employed by Connes and Moscovici in [9]). We shall similarly replace Δ with $\Delta_t = D_t^2$, and we shall use Δ_t in place of Δ in the definitions of $\langle\cdots\rangle_z$ and of $\langle\!\langle\cdots\rangle\!\rangle_z$.

To simplify the notation we shall drop the subscript t in the following computation and denote by $\dot{D} = -D_t\cdot\log|D|$ the derivative of the operator D_t with respect to t.

7.9 Lemma. *Define a multi-linear functional on* A, *with values in the analytic functions on the half-plane* $\mathrm{Re}(z)+n>\frac{d-1}{2}$, *by the formula*

$$\Phi_n^t(a^0,\ldots,a^n)=\langle\!\langle a^0\dot{D},[D,a^1],\ldots,[D,a^n]\rangle\!\rangle_z.$$

Then $B\Phi_n^t$ *is a cyclic* $(n-1)$*-cochain and*

$$\begin{aligned} bB\Phi_n^t&(a^0,\ldots,a^n)\\ &=\frac{d}{dt}\langle\!\langle [D,a^0],\ldots,[D,a^n]\rangle\!\rangle_z+(2z+n)\sum_{j=0}^{n}\langle \dot{D},[D,a^j],\ldots,[D^{j-1}]\rangle_z. \end{aligned}$$

7.10 Remark. Observe that the operator $\log|D|$ has analytic order δ or less, for every $\delta>0$. As a result, the proof of Proposition 5.17 shows that the quantity is a holomorphic function of z in the half-plane $\mathrm{Re}(z)+n>\frac{d-1}{2}$. But we shall not be concerned with any possible meromorphic continuation to $\mathbb{C}$.

Proof. See [19]. □

We can now complete the second step, and with it the proof of the Connes-Moscovici Residue Index Theorem:

7.11 Theorem (Connes and Moscovici). *The residue cocycle* $\operatorname{Res}_{s=0}\Psi$ *is cohomologous, as a* (b, B)*-cocycle, to the Chern character cocycle of Connes.*

Proof. Thanks to Theorem 7.1 it suffices to show that the cyclic cocycle

$$\frac{1}{2}\langle\!\langle [D, a^0], \ldots, [D, a^n] \rangle\!\rangle_{-\frac{n}{2}} \tag{7.6}$$

is cohomologous to the Chern character. To do this we use the homotopy D_t above. Thanks to Lemma 7.9 the coboundary of the cyclic cochain

$$\int_0^1 B\Psi_n^t(a^0, \ldots, a^{n-1})\, dt$$

is the difference of the cocycles (7.6) associated to $D_0 = D$ and $D_1 = F$. For D_1 we have $D_1^2 = \Delta_1 = I$ and so

$$\begin{aligned}
&\frac{1}{2}\langle\!\langle [D_1, a^0], \ldots, [D_1, a^n] \rangle\!\rangle_z \\
&\qquad = \frac{1}{2}\sum_{j=1}^{n} (-1)^{j+1} \frac{(-1)^{n+1}\Gamma(z)}{2\pi i} \times \\
&\qquad\qquad \text{Trace}\left(\int \lambda^{-z} \varepsilon [F, a^0] \cdots [F, a^j] F \cdots [F, a^n] (\lambda - I)^{-(n+2)}\, d\lambda \right)
\end{aligned}$$

Since F anticommutes with each operator $[F, a^j]$ this simplifies to

$$\frac{1}{2}\sum_{j=1}^{n} \frac{(-1)^{n+1}\Gamma(z)}{2\pi i} \text{Trace}\left(\int \lambda^{-z} \varepsilon F [F, a^0] \cdots [F, a^n] (\lambda - I)^{-(n+2)}\, d\lambda \right).$$

The terms in the sum are now all the same, and after applying Cauchy's formula we get

$$\frac{n+1}{2}(-1)^{n+1}\Gamma(z) \cdot \text{Trace}\left(\varepsilon F [F, a^0] \cdots [F, a^n]\right) \cdot \binom{-z}{n+1}.$$

Using the functional equation for the Γ-function this reduces to

$$\frac{\Gamma(z+n+1)}{2 \cdot n!} \text{Trace}\left(\varepsilon F [F, a^0] \cdots [F, a^n]\right)$$

and evaluating at $z = -\frac{n}{2}$ we obtain the Chern character of Connes. □

7.1 Homotopy Invariance and Index Formula

By combining the Theorem 7.11 with the formula (5.5) for the pairing between cyclic theory and K-theory we obtain the index formula

$$\operatorname{Index}\big(eDe\colon eH_0 \to eH_1\big) = \big(\operatorname{Res}_{s=0}\Psi\big)\big([e]\big)$$

for a projection p in A (there is a similar equation for projections in matrix algebras over A). In this section we shall very briefly indicate a shorter route to this formula.

The starting point is the following transgression formula for the basic improper cocycle Ψ that we have been studying: informally, if $\Delta_t = D_t^2$ is a smooth family of operators satisfying our basic hypotheses (for a fixed algebra A) then

$$B\Theta^t_{p+1} + b\Theta^t_{p-1} + \frac{d}{dt}\Psi^t_p = 0,$$

where
(7.7)

$$\Theta_p(a^0,\dots,a^p) = \sum_{j=0}^{p}(-1)^{j-1}\langle a^0,\dots,[D,a^j],\dot{D},[D,a^{j+1}],\dots,[D,a^p]\rangle_{s-\frac{p+1}{2}}.$$

It is a little tedious to precisely formulate and prove this result in any generality (one problem is to understand the analytic continuation property for algebras which contain the operators $\frac{dD}{dt}$). But fortunately we are only interested in a very easy special case, where

$$D_t = D + tX,$$

and where X is a differential order-zero operator in the algebra $\mathcal{D}$. The formula (7.7) can be proved without any real difficulty in this case by following the methods used in the proofs of Lemmas 7.7 and 7.9.

With the transgression formula (7.7) in hand the proof of the index formula can be finished rather quickly, using a trick due to Connes. Given a projection p, define

$$D^e = eDe + e^{\perp}De^{\perp} = D + X,$$

where X is of course an order zero operator in $\mathcal{D}$, and let $D_t = D + tX$, as above. Thanks to the transgression formula, it suffices to show that the residue cocycle $\operatorname{Res}_{s=0}\Psi^e$ of D^e, paired with the K-theory class of the projection e, gives the Fredholm index of eDe (considered as an operator from eH_0 to eH_1). Now, by

Equation (5.4),

$$\mathrm{Res}_{s=0}\,\Psi^{e}\big([e]\big) = \mathrm{Res}_{s=0}\,\Psi^{e}_{0}(e) + \sum_{k\geq 1}(-1)^{k}\frac{(2k)!}{k!}\,\mathrm{Res}_{s=0}\,\Psi^{e}_{2k}(e-\frac{1}{2}, e, \ldots, e).$$

But the terms in the series are all zero since they all involve the commutator of e with D^{e}, which is zero. Hence

$$\begin{aligned}\mathrm{Res}_{s=0}\,\Psi^{e}\big([e]\big) &= \mathrm{Res}_{s=0}\,\Psi^{e}_{0}(e)\\ &= \mathrm{Res}_{s=0}\,\big(\Gamma(s)\,\mathrm{Trace}(\varepsilon e\,(\Delta^{e}_{K})^{-s})\big)\\ &= \mathrm{Index}\big(eDe\colon eH_{0}\to eH_{1}\big),\end{aligned}$$

as required (the last step is the index computation made by Atiyah and Bott that we mentioned in the Subsection 5.1).

8 The General Case

Up to now we have been assuming that the self-adjoint operator Δ is invertible (in the sense of Hilbert space operator theory, meaning that Δ is a bijection from its domain to the Hilbert space H). We shall now remove this hypothesis.

To do so we shall begin with an operator D which is not necessarily invertible (with $D^2 = \Delta$). We shall assume that all our assumptions from Subsection 5.2 concerning the differential pair $(\mathcal{D}, \Delta)$, the the square root D, and the algebra A hold. Fix a bounded self-adjoint operator K with the following properties:

(i) K commutes with D.

(ii) K has analytic order $-\infty$ (in other words, $K \cdot H \subseteq H^\infty$).

(iii) The operator $\Delta + K^2$ is invertible.

Having done so, let us construct the operator

$$D_K = \begin{pmatrix} D & K \\ K & -D \end{pmatrix}$$

acting on the Hilbert space $H \oplus H^{\mathrm{opp}}$, where H^{opp} is the $\mathbb{Z}/2$-graded Hilbert space H but with the grading reversed. It is invertible.

8.1 Example. If D is a Fredholm operator then we can choose for K the projection onto the kernel of D.

Let $\Delta_K = (D_K)^2$ and denote by $\mathcal{D}_K$ the smallest algebra of operators on $H \oplus H^{\mathrm{opp}}$ which contains the 2×2 matrices over $\mathcal{D}$ and which is closed under multiplication by operators of analytic order $-\infty$.

The conditions set forth in Subsection 5.2 for the pair $(\mathcal{D}_K, \Delta_K)$, the square root D_K and the algebra A, which we embed into $\mathcal{D}_K$ as matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

8.2 Lemma. *Assume that the operators* K_1 *and* K_2 *both have the properties* (i)-(iii) *listed above. Then* $\mathcal{D}_{K_1} = \mathcal{D}_{K_1}$. *Moreover the algebra has finite analytic dimension* d *and has the analytic continuation property with respect to* Δ_{K_1} *if and only if it has the same with respect to* Δ_{K_2}. *If these properties do hold then the quantities* $\langle X^0, \ldots, X^p \rangle_z$ *associated to* Δ_{K_1} *and* Δ_{K_2} *differ by a function which is analytic in the half-plane* $\mathrm{Re}(z) > -p$.

Proof. It is clear that $\mathcal{D}_{K_1} = \mathcal{D}_{K_2}$. To investigate the analytic continuation property it suffices to consider the case where K_1 is a fixed function of Δ, in which case K_1 and K_2 commute. Let us write

$$X\Delta^{-z} = \frac{1}{2\pi i}\int \lambda^{-z} X(\lambda-\Delta)^{-1}\, d\lambda$$

for $\mathrm{Re}(z) > 0$. Observe now that

$$(\lambda-\Delta_{K_1})^{-1} - (\lambda-\Delta_{K_2})^{-1} \sim M(\lambda-\Delta_{K_1})^{-2} - M(\lambda-\Delta_{K_1})^{-3} + \cdots ,$$

where $M = \Delta_{K_1} - \Delta_{K_2}$ (this is an asymptotic expansion in the sense described prior to the proof of Proposition 5.17). Integrating and taking traces we see that

$$\text{(8.1)}\quad \mathrm{Trace}\big(X\Delta_{K_1}^{-z}\big) - \mathrm{Trace}\big(X\Delta_{K_2}^{-z}\big) \approx \sum_{k\geq 1} (-1)^{k-1}\binom{-z}{k}\,\mathrm{Trace}\big(XM\Delta_{K_1}^{-z-k}\big),$$

which shows that the difference $\mathrm{Trace}\big(X\Delta_{K_1}^{-z}\big) - \mathrm{Trace}\big(X\Delta_{K_2}^{-z}\big)$ has an analytic continuation to an entire function. Therefore Δ_{K_1} has the analytic continuation property if and only if Δ_{K_2} does (and moreover the analytic dimensions are equal).

The remaining part of the lemma follows from the asymptotic formula

$$\langle X^0,\ldots,X^p\rangle_z \approx \sum_{k\geq 0} (-1)^{|k|}\Gamma(z+p+|k|)\frac{1}{(|k|+p)!}c(k) \times \mathrm{Trace}\left(\varepsilon X^0 {X^1}^{(k_1)}\cdots {X^p}^{(k_p)}\Delta^{-z-|k|-p}\right)$$

that we proved earlier. □

8.3 Definition. The *residue cocycle* associated to the possibly non-invertible operator D is the residue cocycle $\mathrm{Res}_{s=0}\Psi$ associated to the invertible operator D_K, as above.

Lemma 8.2 shows that if $p > 0$ then the residue cocycle given by Definition 8.3 is independent of the choice of the operator K. In fact this is true when $p = 0$ too. Indeed Equation (8.1) shows that not only is the difference $\mathrm{Trace}(\varepsilon a^0\Delta_{K_1}^{-s}) - \mathrm{Trace}(\varepsilon a^0\Delta_{K_2}^{-s})$ analytic at $s = 0$, but it vanishes there too. Therefore

$$\begin{aligned}\mathrm{Res}_{s=0}\Psi_0^{K_1}(a^0) - \mathrm{Res}_{s=0}\Psi_0^{K_2}(a^0) \\ = \mathrm{Res}_{s=0}\Gamma(s)\left(\mathrm{Trace}(\varepsilon a^0\Delta_{K_1}^{-s}) - \mathrm{Trace}(\varepsilon a^0\Delta_{K_2}^{-s})\right) = 0.\end{aligned}$$

8.4 Example. If D happens to be invertible already then we obtain the same residue cocycle as before.

8.5 Example. In the case where D is Fredholm, the residue cocycle is given by the same formula that we saw in Theorem 5.20:

$$\begin{aligned}\operatorname{Res}_{s=0}\Psi_p(a^0,\dots,a^p)\\ &= \sum_{k\geq 0} c_{p,k}\operatorname{Res}_{s=0}\operatorname{Tr}\left(\varepsilon a^0[D,a^1]^{(k_1)}\cdots[D,a^p]^{(k_p)}\Delta^{-\frac{p}{2}-|k|-s}\right).\end{aligned}$$

The complex powers Δ^{-z} are defined to be zero on the kernel of D (which is also the kernel of Δ). When $p=0$ the residue cocycle is

$$\operatorname{Res}_{s=0}\big(\Gamma(s)\operatorname{Trace}(\varepsilon a^o\Delta^{-s})\big)+\operatorname{Trace}(\varepsilon a^0 P),$$

where the complex power Δ^{-s} is defined as above and P is the orthogonal projection onto the kernel of D.

Now Connes' Chern character cocycle is defined for a not necessarily invertible operator D by forming first D_K, then $F_K = D_K|D_K|^{-1}$, then $\mathrm{ch}_n^{F_K}$. See Appendix 2, and also Section 5, of [3, Part I]. The following result therefore follows immediately from our calculations in the invertible case.

8.6 Theorem. *For any operator D, invertible or not, the class in periodic cyclic cohomology of the residue cocycle $\operatorname{Res}_{s=0}\Psi$ is equal to the class of the Chern character cocycle of Connes.* □

8.7 Remark. There is another way that the index theorem can be generalized — by considering the 'odd-dimensional' case instead of the even-dimensional one that we have been examining. This involves the construction of an odd cyclic cocycle starting from data the same as we have been using, except that all assumptions about the $\mathbb{Z}/2$-grading of the Hilbert space H are dropped. There is a completely analogous local index formula in this case (indeed it was the odd case that Connes and Moscovici originally considered). For remarks on how to adapt our approach to the odd case see [19].

A Appendix: Compact and Trace-Class Operators

In this appendix we shall take a rapid walk through the elementary theory of compact Hilbert space operators.

A.1 Definition. A bounded linear operator on a Hilbert space H is *compact* (or *completely continuous*, in old-fashioned terms) if it maps the closed unit ball of H to a (pre)compact set in usual norm topology.

We write '(pre)compact' because it turns out that if the image of the closed unit ball has compact closure, then it is in fact already closed, and therefore compact. Here are various ways of using the compactness of a bounded Hilbert space operator T:

(i) If $\{v_j\}$ is a bounded sequence of vectors, then the sequence $\{Tv_j\}$ contains a norm-convergent subsequence.

(ii) If $\{v_j\}$ is a bounded sequence of vectors, and if it converges weakly to v (this means that $\langle v_j, w\rangle$ converges to $\langle v, w\rangle$, for every w), then $\{Tv_j\}$ converges in the norm topology to Tv.

(iii) The quadratic functional $v \mapsto \langle Tv, v\rangle$ is continuous from the closed unit ball with its weak topology into $\mathbb{C}$. Since the closed unit ball is compact in the weak topology, the functional has extreme values.

The first two items are actually equivalent formulations of compactness. The last item, has a very important consequence:

A.2 Lemma. *If T is a compact and self-adjoint operator (which means that $\langle Tv, w\rangle = \langle v, Tw\rangle$, for all v and w), then T has a non-zero eigenvector.*

Proof. Let v be a unit vector which is an extreme point of the functional in item (iii). If w is a unit vector orthogonal to v, then by differentiating the function

$$s \mapsto \Big\langle T\big(\cos(s)v + \sin(s)w\big), \cos(s)v + \sin(s)w\big)\Big\rangle$$

at $s = 0$ (which is an extreme point) we find that Tv is orthogonal to w. Hence Tv must be a scalar multiple of v, which is to say an eigenvector. □

We can now restrict the operator T of the lemma to the orthogonal complement of v, and then apply the lemma again to get a second eigenvector. Continuing in this way we get Hilbert's Spectral Theorem for compact, self-adjoint operators:

A.3 Spectral Theorem. *If T is a compact and self-adjoint operator on a Hilbert space H, then there is an orthonormal basis for H consisting of eigenvectors for T. The corresponding eigenvalues are all real, and converge to zero.* □

Conversely, if a bounded operator T has such an eigenbasis, then it is readily checked that T must be compact. Examples of compact operators tend to come either from this source, or from one of the following two observations:

(i) If T is a norm limit of finite-rank operators, then T is compact (moreover every compact operator is a norm limit of finite-rank operators).

(ii) If T is an operator on $L^2(X)$, and if T can be represented as an integral operator

$$Tf(x) = \int_X k(x, y) f(y)\, dy,$$

where the kernel $k(x, y)$ is square-integrable on $X \times X$, then T is compact (these are the *Hilbert-Schmidt operators*; not every compact operator on $L^2(X)$ is of this type).

It follows from the Spectral Theorem that the theory of compact self-adjoint operators has much in common with the theory of real sequences which converge to 0. It is therefore quite natural to consider subclasses of compact operators for which the eigenvalue sequence is summable, p-summable, and so on, and to develop, for example, Holder inequalities, and so on. This program has in fact been carried out very far.

We can apply many of the same ideas to non-self-adjoint compact operators by means of the following device.

A.4 Definition. Let T be a bounded operator on a Hilbert space H. The *singular values* $\mu_1(T), \mu_2(T), \ldots$ of T are the non-negative scalars defined by the formula

$$\mu_j(T) = \inf_{\dim(V)=j-1} \sup_{v \perp V} \frac{\|Tv\|}{\|v\|}.$$

Thus $\mu_1(T)$ is the norm of T, and $\mu_j(T)$ measures the norm of T acting on all codimension $j-1$ subspaces of H. Observe that $\mu_1(T) \geq \mu_2(T) \geq \ldots$ and that

$$T \text{ is compact} \quad \Leftrightarrow \quad \lim_{j \to \infty} \mu_j(T) = 0.$$

(If T is not compact, then the singular value sequence is typically not very interesting — often it is constant.)

A.5 Lemma. *Let* T *be compact and* positive *Hilbert space operator (this means* $\langle Tv, v\rangle \geq 0$ *for all* v*, which implies that* T *is self-adjoint). Let* $\{\lambda_j\}$ *be the eigenvalue sequence for* T*, arranged in decreasing order, and with multiplicities counted. Then* $\mu_j(T) = \lambda_j(T)$*, for all* j.

Proof. This follows readily from the Spectral Theorem, which gives a concrete representation for T as a diagonal matrix:

$$T = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{pmatrix}.$$

□

Apart from being quite meaningful for arbitrary compact operators, the advantage of the singular values over the eigenvalues is that by virtue of their definition it is rather easy to prove inequalities involving them. For example:

A.6 Lemma. *Let* T_1 *and* T_2 *be compact operators on a Hilbert space and let* S *be a bounded operator. Then*

$$\mu_j(T_1 + T_2) \leq \mu_j(T_1) + \mu_j(T_2) \leq \mu_{2j}(T_1 + T_2).$$

and

$$\mu_j(ST), \mu_j(TS) \leq \|S\| \mu_j(T).$$

□

With these inequalities to hand we can make the following definition:

A.7 Definition. Let H be a Hilbert space and denote by $\mathcal{B}(H)$ the algebra of bounded operators on H. The *trace ideal* in $\mathcal{B}(H)$ is

$$\mathcal{L}^1(H) = \left\{ T \mid \sum \mu_j(T) < \infty \right\}.$$

Every trace-class operator is compact. Thanks to the inequalities in Lemma A.6 the trace ideal really is a two-sided ideal in the algebra $\mathcal{B}(H)$. It is not closed in the norm-topology, in fact its closure is the ideal of all compact operators.

From the definition of the singular values $\mu_j(T)$ it follows that if $\{v_1, \dots, v_N\}$ is any orthonormal set in H, then

$$\sum_{j=1}^{N} |\langle v_j, Tv_j\rangle| \leq \sum_{j=1}^{N} \mu_j(T).$$

As a result of this new inequality we can make the following definition.

A.8 Definition. If $T \in \mathcal{L}^1(H)$, then the *trace* of T is the scalar

$$\text{Trace}(T) = \sum_{j=1}^{\infty} \langle v_j, Tv_j \rangle,$$

where the sum is over an orthonormal basis of H.

The series converges absolutely, so our definition makes some sense. Simple algebra (reinforced by the guarantee of absolute convergence of all the series involved in the argument) shows that $\text{Trace}(T)$ does not depend on the choice of orthonormal basis, and that

$$S \in \mathcal{B}(H),\ T \in \mathcal{L}^1(H) \quad \Rightarrow \quad \text{Trace}(ST) = \text{Trace}(TS).$$

Thus the operator-trace has the fundamental property of the trace on matrices, to the fullest extent it *can* have it.

If S and T are Hilbert-Schmidt operators, then it may be shown that ST is a trace-class operator (incidentally, an operator T on $L^2(X)$ belongs to the Hilbert-Schmidt class if and only if $\sum \mu_j(T)^2 < \infty$). The trace of many integral operators may be computed using the following result:

A.9 Lemma. *Let M be a closed manifold which equipped with a smooth measure. If k is a smooth function on $M \times M$, then the operator T defined by the formula*

$$Tf(x) = \int_M k(x, y) f(y)\, dy,$$

is a trace-class operator. Moreover

$$\text{Trace}(T) = \int_M k(x, x)\, dx.$$

□

A.10 Remark. One can replace ‘smooth’ by ‘differentiable sufficiently many times’, but the order of differentiability depends on the dimension of the manifold (assuming that the kernel k is merely continuous is not enough).

B Appendix: Fourier Theory

If $\phi\colon \mathbb{T}^n \to \mathbb{C}$ is a smooth function on the n-torus $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$ then its *Fourier Transform* is the function $\hat{\phi}\colon \mathbb{Z}^n \to \mathbb{C}$ defined by

$$\hat{\phi}(j) = \int_{\mathbb{T}^n} \phi(x) e^{-2\pi i\, j\cdot x}\, dx.$$

The Fourier transform $\phi \mapsto \hat{\phi}$ extends to an isometric isomorphism of Hilbert spaces

$$L^2(\mathbb{T}^n) \longrightarrow \ell^2(\mathbb{Z}^n).$$

This is the *Plancherel Theorem*. If $\phi\colon \mathbb{T}^n \to \mathbb{C}$ is smooth then the Fourier transforms of its partial derivatives $\partial^\alpha \phi$ may be computed from the formula

$$\widehat{\partial^\alpha \phi}(j) = \frac{j^\alpha}{(2\pi i)^\alpha} \hat{\phi}(j).$$

Thanks to this and the Plancherel Theorem, the norm in the Sobolev space $W_k(\mathbb{T}^n)$ may be computed from the formula

$$\|\phi\|^2_{W_k(\mathbb{T}^n)} \approx \sum_{j \in \mathbb{Z}^n} (1 + j^2)^{\frac{k}{2}} |\hat{\phi}(j)|^2.$$

B.1 Lemma. *If* $k > 0$ *then the inclusion of* $W_k(\mathbb{T}^n)$ *into* $L^2(\mathbb{T}^n)$ *is a compact operator.*

Proof. If $j \in \mathbb{Z}^n$ then denote by e_j the function $e^{2\pi i\, j\cdot x}$ on T. Using our formula for the norm in $W_k(\mathbb{T}^n)$ we see that the Hilbert spaces $L^2(\mathbb{T}^n)$ and $W_k(T^n)$ have an orthonormal bases

$$\{e_j\} \quad \text{and} \quad \{f_j = (1 + j^2)^{-\frac{k}{2}} e_j\},$$

respectively. Using these bases, the inclusion of $W_k(\mathbb{T}^n)$ into $L^2(\mathbb{T}^n)$ takes the form

$$f_j \mapsto (1 + j^2)^{-\frac{k}{2}} e_j.$$

If $k > 0$ then the scalar coefficient sequence converges to zero, and so the inclusion operator is compact. □

B.2 Remark. If $k > n$ then the coefficient sequence is summable, and therefore the inclusion is a trace-class operator.

If ϕ is a smooth function on $\mathbb{T}^n$ then according to the Plancherel Theorem,

$$\phi = \sum_{j\in\mathbb{Z}^n} \widehat{\phi}(j)e_j,$$

where as above $e_j(x) = e^{2\pi i\, j\cdot x}$. To begin with, the series converges in $L^2(\mathbb{T}^n)$, so that the coefficient family $\{\widehat{\phi}(j)\}$ is square-summable, but from the formula

$$\partial^\alpha \phi = \sum_{j\in\mathbb{Z}^n} \widehat{\partial^\alpha\phi}(j)e_j = \sum_{j\in\mathbb{Z}^n} \frac{j^\alpha}{(2\pi i)^\alpha}\widehat{\phi}(j)e_j$$

we see that the coefficient family $\{\widehat{\phi}(j)\}$ remains square-summable after multiplication by any polynomial in j. So by the Cauchy-Schwarz inequality the series $\sum_{j\in\mathbb{Z}^n} \widehat{\phi}(j)e_j$ is in fact absolutely summable, and therefore converegent in the uniform norm. A refinement of this computation proves the following lemma:

B.3 Lemma. *If* p *and* k *are non-negative integers, and if* $k > p + \frac{n}{2}$ *then* $W_k(\mathbb{T}^n) \subseteq C^p(\mathbb{T}^n)$.

Proof. Let ϕ be a smooth function on $\mathbb{T}^n$. We have that

$$\|\phi\|_{C^p(\mathbb{T}^n)} = \max_{|\alpha|\le p} \sup_{x\in\mathbb{T}^n} |\partial^\alpha\phi(x)|.$$

Since $\partial^\alpha\phi(x) = \sum_{j\in\mathbb{Z}^n} \widehat{\partial^\alpha\phi}(j)e_j(x)$ we get

$$|\partial^\alpha\phi(x)| \le \sum_{j\in\mathbb{Z}^n} |\widehat{\partial^\alpha\phi(j)}| \lesssim \sum_{j\in\mathbb{Z}^n} |j|^p \cdot |\widehat{\phi(j)}|.$$

If $k > p + \frac{n}{2}$ then the Cauchy Schwarz inequality implies that

$$\sum_{j\in\mathbb{Z}^n} |j|^p \cdot |\widehat{\phi(j)}| \lesssim \|\phi\|_{W_k(\mathbb{T}^n)},$$

and therefore $\|\phi\|_{C^p(\mathbb{T}^n)} \lesssim \|\phi\|_{W_k(\mathbb{T}^n)}$, as required. □

The Fourier Transform of a smooth, compactly supported function $\phi\colon \mathbb{R}^n \to \mathbb{C}$ is the function $\widehat{\phi}\colon \mathbb{R}^n \to \mathbb{C}$ given by the formula

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x)e^{2\pi i\, \xi\cdot x}\, dx.$$

Once again, the Fourier transform extends to an isometric isomorphism of Hilbert spaces, but this time from $L^2(\mathbb{R}^n)$ to itself. If Ω is an open set in R^n then the Sobolev norm $\| \ \|_{W_k(\Omega)}$ of Definition 1.5 can be given, up to equivalence, by the formula

$$\|\phi\|^2_{W_k(\Omega)} = \int_{\mathbb{R}^n} (1+\xi^2)^{\frac{k}{2}} |\widehat{\phi}(\xi)|^2 \, d\xi.$$

With this formula available we can obviously now define Sobolev spaces $W_k(\Omega)$ for any real $k \in \mathbb{R}$ just by completing the smooth, compactly supported functions in the above norm. Using partitions of unity and local coordinates we can now define Sobolev spaces $W_k(M)$ for any $k \in \mathbb{R}$ and any closed manifold, just as we did in the case where k was a non-negative integer. These are the spaces we briefly referred to in Remark 1.27.

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