THE MACKEY ANALOGY AND K-THEORY

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This paper is dedicated to the memory of George Mackey.

ABSTRACT. In an interesting article from the 1970's, Mackey made a proposal to study representation theory for a semisimple group G by developing an analogy between G and an associated semidirect product group. We shall examine the connection between Mackey's proposal and C^*-algebra K-theory. In one direction, C^*-algebra theory prompts us to search for precise expressions of Mackey's analogy. In the reverse direction, we shall use Mackey's point of view to give a new proof of the complex semisimple case of the Connes-Kasparov conjecture in C^*-algebra K-theory.

1. INTRODUCTION

One of the triumphs of Mackey's theory of induced unitary representations is a simple and complete description of the unitary duals of many semidirect product groups in terms of the unitary duals of the two factors into which a semidirect product decomposes. In contrast to this, if G is a semisimple group, then although G may be in some sense assembled from various subgroups (for example a maximal compact and a Borel subgroup), a full understanding of the unitary representation theory of G, or even the tempered representation theory, has proved much harder to obtain.

In a very engaging article [Mac75], Mackey made the suggestion that there ought to be an analogy between unitary representations of a semi-simple group and unitary representations of an associated semidirect product group. Suppose for example that $G = PGL(2, \mathbb{C})$ and let G_0 be the semidirect product of $SO(3)$ and $\mathbb{R}^3$ that is constructed using the natural action of $SO(3)$ on $\mathbb{R}^3$. Then G_0 is the group of orientation-preserving isometries of three-dimensional Euclidean space, whereas G is the group of orientation-preserving isometries of three-dimensional hyperbolic space. By rescaling, G is also the group of orientation-preserving isometries of the three-dimensional space with constant curvature $-R$, and as R tends to 0 we can view G_0 as a sort of limiting case of G. Now irreducible unitary representations of G_0 have a well-known quantum-mechanical interpretation as particle states. Since hyperbolic space, particularly the scaled version with curvature close to zero, is a plausible model for physical space, Mackey was led to suppose that there ought to be corresponding particle states on this curved space:

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> . . . the physical interpretation suggests that there ought to exist a "natural" one to one correspondence between almost all of the unitary representations of G_0 and almost all the unitary representations of G—in spite of the rather different algebraic structures of these groups.[1]

Mackey made a number of calculutions in support of this proposal, the simplest of which we shall review in this paper. However he ended his article rather cautiously, as follows:

> We have not yet ventured to formulate a precise conjecture along the lines suggested by the speculations in the preceding sections because we have not studied enough examples to be reasonably sure that we have not overlooked some important phenomena that a really general result must take into account. However we feel sure that some such result exists and that a routine if somewhat lengthy investigation will tell us what it is. We also feel that a further study of the apparently rather close relationship between the representation theory of a semisimple Lie group and that of its associated semi-direct product will throw valuable light on the much more difficult semisimple case.[2]

The purpose of this note is to re-examine Mackey's proposal in the light of developments since the time of its publication in C^*-algebra theory and K-theory. Although we too shall refrain from stating a precise conjecture, the calculations presented here for complex semisimple groups, as well as others carried out for real groups but not reported here, support the proposal that there is a natural bijection between the tempered dual of a semisimple group G and the unitary dual of its associated semidirect product group G_0 that is compatible with minimal K-types and compatible with the natural deformation of G_0 into G in a way that this article should make fairly clear.

Whereas Mackey typically viewed the unitary dual of a group as a Borel space, as in [Mac57], C^*-algebra theory and especially K-theory strongly suggest viewing the dual as a topological space, and because of this a correspondence between "almost all" representations of G and G_0 is not altogether adequate. But on the other hand, the C^*-algebra point of view makes it natural to shift attention away from representation spaces and toward C^*-algebra states (or equivalently the positive definite functions on groups), since these can be used to generate representation spaces in a uniform way. This considerably simplifies the task of cataloguing the representations of G and G_0, and then comparing the results, especially because all the representations that we shall consider have canonical one-dimensional subspaces, and are therefore determined by distinguished vector states.

We shall focus on complex semisimple groups in this article. We shall begin by describing the associated semidirect product G_0 and Mackey's analogy between almost all irreducible representations of G and G_0. Then we shall make the rather simple observation that for complex semisimple groups Mackey's almost-everywhere correspondence can be elevated to a natural bijection between the unitary dual of G_0 and the tempered dual of G.

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[1] See [Mac75, p. 341]. We have replaced Mackey's $SL(2,\mathbb{C})$ with $PGL(2,\mathbb{C})$, and adjusted notation accordingly.

[2] [Mac75, p. 362].

In an effort to carry the bijection beyond the sort of "mere coincidence of parametrizations" in which Mackey was uninterested we shall review some ideas connecting representation theory to C^*-algebras and K-theory. It was observed in [BCH94, Sec. 4] that the Connes-Kasparov conjecture concerning K-theory for group C^*-algebras can be viewed as a K-theoretic counterpart of Mackey's analogy. We shall use our bijection between the duals of G_0 and G to demonstrate that the continuous field of group C^*-algebras associated to the deformation of G_0 into G is assembled from *constant* fields by Morita equivalences, extensions, and direct limits. This will lead to a new proof of the Connes-Kasparov conjecture, and also, we hope, help bring the Mackey analogy toward a more precise form.

However the computations presented here are hardly the final word on the matter, even for the special case of complex semisimple groups. What is missing is a conceptual explanation for the bijection between the reduced duals of G and G_0 (perhaps making a closer connection with the Dirac operator approach to the Connes-Kasparov conjecture). While we hope to return to this in the future, at the present time the phenomenon remains mysterious to us.

2. The Mackey Analogy

Let G be a connected Lie group and let K be a closed subgroup of G. Form the semidirect product group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$, where $\mathfrak{g}/\mathfrak{k}$ is the vector space quotient of the Lie algebra of G by the Lie algebra of K, equipped with the adjoint action of K. As a manifold, G_0 is the direct product of K and $\mathfrak{g}/\mathfrak{k}$. Its group operation is given by the formula

$$(k_1, v_1) \cdot (k_2, v_2) = (k_1 k_2, \mathrm{Ad}_{k_2^{-1}}(v_1) + v_2).$$

Following Mackey, we should like to explore what connection there is between the representation theories of G_0 and G. Although it is possible to do this in a variety of contexts, we shall confine our attention, as Mackey did, to the situation in which K is a maximal compact subgroup of G. It is here that the link with C^*-algebras and K-theory that we aim to describe seems to be the clearest and simplest.

We shall also restrict our attention to certain sorts of unitary representations:

2.1. **Definition.** If G is any Lie group G, then we shall denote by $\widehat{G}$ the *reduced unitary dual* of G. This is the set of unitary equivalence classes of those irreducible unitary representations of G that are weakly contained in the regular representation.

For further information about the reduced dual see [Dix77, Ch. 18] as well as Section 3 below. For the semidirect product group G_0 defined above, the reduced dual is the entire unitary dual; for the semisimple groups that will be the focus of our interest, the reduced dual is the same thing as the tempered dual (see [CHH88]). Our aim is to compare the reduced duals of G and G_0. To this end, we shall quickly review what these duals look like individually.

2.1. **Semidirect Product Groups.** Let K be a compact Lie group and let X be an abelian Lie group equipped with an action of K by automorphisms. Mackey's theory of induced representations describes in detail the unitary duals of a great many semidirect product groups, including the rather elementary example $K \ltimes X$. We shall quickly review Mackey's results as they apply to $K \ltimes X$ here (and we shall return to them in Section 3.1).

Let φ be a unitary character of X and let τ be an irreducible unitary representation of the isotropy group

$$K_\varphi = \{\, k \in K \mid \varphi(k(x)) = \varphi(x)\ \forall x \in X \,\},$$

on a finite-dimensional space W. The map $\tau \otimes \varphi \colon (k, x) \mapsto \varphi(x)\tau(k)$ is a representation of the semidirect product group $K_\varphi \ltimes X$ on W, which we may induce up so as to obtain a unitary representation of $K \ltimes X$.

Mackey's theory (see [Mac49, Section 7] or [Mac76, Ch. 3]) tells us that the representations of $K \ltimes X$ obtained in this way are all irreducible; that up to unitary equivalence every irreducible unitary representation of $K \ltimes X$ results from an instance of this induction construction; and that the only equivalences among these induced representations are those obtained from conjugacy in K: the induced representations obtained from the pairs (τ, φ) and (τ', φ') are equivalent to one another if and only if there is some $k \in K$ such that $\varphi'(x) = \varphi(k(x))$ for all $x \in X$, and such that $\tau' = \tau \circ \mathrm{Ad}_k$, up to unitary equivalence. Thus

$$\widehat{K \ltimes X} \cong \left\{ (\tau, \varphi) \,\middle|\, \varphi \in \widehat{X} \text{ and } \tau \in \widehat{K_\varphi} \right\} \big/ K.$$

2.2. **Complex Semisimple Groups.** Let G be a connected complex semsimple group. Let $G = KAN$ be an Iwasawa decomposition, let M be the centralizer of A in K (M is a maximal torus in K), and let $B = MAN$ (this is a Borel subgroup of G). The *unitary principal series* representations of G are constructed by associating to a pair (σ, φ) of unitary characters, one of M and one of A, the unitary character $man \mapsto \sigma(m)\varphi(a)$ of the Borel subgroup B and then unitarily inducing this character from B to a representation of G.

It is known that every principal series representation is irreducible; that every irreducible tempered representation of G is equivalent to a principal series representation; and that the principal series representations obtained from (σ, φ) and (σ', φ') are equivalent to one another if and only if (σ, φ) and (σ', φ') are conjugate under the Weyl group W of G (W is the quotient of the normalizer of MA by MA). Hence

$$\widehat{G} \cong \big(\widehat{M} \times \widehat{A}\big)/W.$$

In Section 3.2 we shall review some parts of this computation of the tempered dual.

2.3. **The Mackey Analogy.** Mackey's approach in [Mac75] was to find analogies between typical representations of G_0 and typical representations of G, with a view toward determining the representation theory of the latter group. Here is a sketch of his approach in the case of complex semisimple groups.

Let $\mathfrak{a}$ be the Lie algebra of the subgroup A in the given Iwasawa decomposition of G. Every unitary character of the vector space $\mathfrak{g}/\mathfrak{k}$ is conjugate to one which is determined by its restriction to the image $\mathfrak{a}$ in $\mathfrak{g}/\mathfrak{k}$ (as detailed in Definition 2.3 below), and for which the isotropy group K_φ therefore contains M. Generically K_φ is exactly equal to M, and so the data needed to specify a generic irreducible unitary representation of G_0 is a pair (σ, φ) consisting of a character of M and a generic character of $\mathfrak{g}/\mathfrak{k}$, the pair being determined up to conjugacy by an element of the Weyl group. In short, the generic irreducible unitary representations of G_0 are parametrized by an open and dense subset of $(\hat{M} \times \hat{A})/W$. Therefore the generic irreducible unitary representations of G_0 are parametrized in the same way as the (generic) irreducible tempered representations of G. More details about this parametrization will be given in a moment.

Mackey wanted to carry the analogy beyond a "mere coincidence of parameters." To do so in the case at hand, he observed that the generic representations of G_0 are induced from characters of the group $M \ltimes \mathfrak{g}/\mathfrak{k}$, while the principal series representations of G are induced from characters of the Borel subgroup B. Note that the group $M \ltimes \mathfrak{g}/\mathfrak{k}$ stands in relation to B in the same way that G_0 stands in relation to G. We find that generic representations of G and G_0 are constructed in closely analogous ways. It follows, for example, that the representation *spaces* may be identified—and not just as Hilbert spaces but as unitary representations of the common subgroup K of G and G_0.

Mackey did not continue his analysis to *all* representations of G and G_0. Presumably this was because in the non-generic case the analogy at the level of representation spaces is rather poor (for example some representations of G_0 are finite-dimensional, whereas all the tempered representations of G are infinite-dimensional), and perhaps it was also because a measure-theoretic approach to representation theory rendered the continuation unnecessary (the non-generic representations have zero Plancherel measure). However the C^*-algebra point of view prompts us to regard the reduced dual as a topological space, not a measure space, and to study representations via their matrix coefficients rather than their representation spaces. Let us therefore try to pursue Mackey's analogy a bit further.

In fact Mackey's analysis for complex semisimple groups can be completed rather simply, at least at the level of parameters, thanks to the following observation:

2.2. **Lemma.** *Let G be a connected complex semisimple group. If φ is any unitary character of $\mathfrak{g}/\mathfrak{k}$, then the isotropy group K_φ is connected.*

Proof. The Lie algebra of G may be written as $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$, from which it follows that $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{k}$ as real vector spaces equipped with representations of K. Since $\mathfrak{k}$ admits an Ad-invariant inner product, we may identify $\mathfrak{k}$ with its dual and so conclude that the dual of $\mathfrak{g}/\mathfrak{k}$ is isomorphic to $\mathfrak{k}$ as a K-space. But the centralizer of any element of $\mathfrak{k}$ is a union of maximal tori in K (see [Bou05, p. 290]). □

We shall need the Cartan-Weyl classification of irreducible representations of a compact connected Lie group K by highest weights. To review, if M and W are a maximal torus and Weyl group for K, then the set of W-orbits in $\widehat{M}$ is partially ordered by the relation

$$\mathcal{O}_1 \le \mathcal{O}_2 \quad \Leftrightarrow \quad \mathrm{Conv}(\mathcal{O}_1) \subseteq \mathrm{Conv}(\mathcal{O}_2),$$

where $\mathrm{Conv}(\mathcal{O})$ denotes the convex hull of the orbit $\mathcal{O}$ in the vector space $\widehat{M} \otimes_{\mathbb{Z}} \mathbb{R}$. The restriction of any irreducible representation τ of K to the subgroup $M \subseteq K$ decomposes into a direct sum of characters of M, called the *weights* of τ. The set of weights is a union of orbits, and there is a maximum orbit for the above partial order. The correspondence between equivalence classes of irreducible representations and maximum orbits is a bijection. The characters in the maximum orbit are called the *highest weights* of τ, and each occurs with multiplicity one in the restriction of τ to M.

2.3. **Definition.** Let $\mathfrak{a}$ be the Lie algebra of A. View it as a subspace of the vector space $\mathfrak{g}/\mathfrak{k}$ via the quotient map, and denote by $\mathfrak{a}^\perp \subseteq \mathfrak{g}/\mathfrak{k}$ the unique M-invariant complement of $\mathfrak{a}$ in $\mathfrak{g}/\mathfrak{k}$. Let us say that a character of $\mathfrak{g}/\mathfrak{k}$ is *balanced* if it is trivial on $\mathfrak{a}^\perp$, so that it factors through the projection of $\mathfrak{g}/\mathfrak{k}$ onto $\mathfrak{a}$ and can therefore be regarded as a character of $\mathfrak{a}$.

Every character of $\mathfrak{g}/\mathfrak{k}$ is conjugate to a balanced character that is unique up to conjugacy by an element of the Weyl group W (here viewed as the normalizer of M in K, divided by M). By Lemma 2.2 the isotropy subgroup K_φ of a balanced character is connected; in addition K_φ clearly contains the maximal torus M of K as a maximal torus of its own. The Weyl group of K_φ is W_φ, the isotropy group of φ in W. By the Cartan-Weyl theory, the irreducible representations of the connected group K_φ are parametrized up to equivalence by $\hat{M}/W_\varphi$. Inserting this parametrization into Mackey's classification of the irreducible unitary representations of $K \ltimes \mathfrak{g}/\mathfrak{k}$ we conclude that

$$\widehat{K \ltimes \mathfrak{g}/\mathfrak{k}} \;\cong\; \bigsqcup_{\varphi \in \hat{\mathfrak{a}}/W} \widehat{M}/W_\varphi \;\cong\; \big(\widehat{M} \times \widehat{\mathfrak{a}}\big)/W.$$

We have proved the following result:

2.4. **Theorem.** *Let $G = KAN$ be a connected complex semisimple Lie group and let M be the centralizer of A in K. View each unitary character φ of $\mathfrak{a}$ as a balanced character of $\mathfrak{g}/\mathfrak{k}$. For each pair*

$$(\sigma, \varphi) \in \widehat{M} \times \widehat{\mathfrak{a}},$$

fix an irreducible representation τ_σ of the isotropy group $K_\varphi \subseteq K$ with highest weight σ. The correspondence that associates to (σ, φ) the irreducible unitary representation of $K \ltimes \mathfrak{g}/\mathfrak{k}$ induced from the representation $(k, x) \mapsto \tau_\sigma(k)\varphi(x)$ of the subgroup $K_\varphi \ltimes \mathfrak{g}/\mathfrak{k}$ determines a bijection

$$\big(\widehat{M} \times \widehat{\mathfrak{a}}\big)/W \cong \widehat{K \ltimes \mathfrak{g}/\mathfrak{k}}.$$

Here W is the Weyl group of G. □

Of course, the abelian group $\mathfrak{a}$ is isomorphic to A via the exponential map, and we therefore find that

$$\widehat{G} \cong \big(\widehat{M} \times \widehat{A}\big)/W \cong \big(\widehat{M} \times \widehat{\mathfrak{a}}\big)/W \cong \widehat{G}_0.$$

Therefore the tempered dual of a complex semisimple group G is "the same" as the unitary dual of the associated group G_0, in the sense that we are able to exhibit a reasonably natural bijection between the two spaces.

2.5. **Remark.** Parallel statements can be made for real semisimple groups. The tempered dual is of course much more complicated, but so is the unitary dual of G_0; this is in part a consequence of the fact that the isotropy subgroups K_φ are no longer connected. The additional complications balance one another nicely, but even in simple cases the computations require fairly elaborate preparations just to summarize. As a result they will be presented elsewhere.

It should be noted that the bijection in the theorem is not a homeomorphism. But as we shall see later on, the reduced duals of G and G_0 can be partitioned into locally closed parts, and our bijection maps each part of the dual of G homeomorphically to a corresponding part of the dual of G_0.

3. Group C^*-Algebras

In the remainder of the paper we shall try to place the bijection between the reduced duals of G and G_0 provided by Theorem 2.4 into a context that we hope will begin to elevate it above a "mere coincidence of parameters." To do so we shall examine G and G_0 from the point of view of C^*-algebras.

3.1. **Definition.** The *reduced group C^*-algebra* of a Lie group G, denoted $C^*_\lambda(G)$, is the completion of the convolution algebra $C_c^\infty(G)$ in the norm obtained from the left regular representation of $C_c^\infty(G)$ as bounded convolution operators on the Hilbert space $L^2(G)$ (the symbol λ denotes the left regular representation).

Each unitary representation of G integrates to a representation of the algebra $C_c^\infty(G)$, and, more or less by definition, a unitary representation of G is weakly contained in the regular representation if and only if its integrated version extends from $C_c^\infty(G)$ to a representation of the C^*-algebra $C^*_\lambda(G)$. The reduced dual of G may be identified in this way with the dual of $C^*_\lambda(G)$, that is, the set of equivalence classes of irreducible representations of $C^*_\lambda(G)$.

To help orient ourselves, we pause briefly to describe the structure of the group C^*-algebras that we shall be studying. This structure should make evident the extent to which $C^*_\lambda(G)$ encodes features of the reduced dual, at least for the Lie groups we are considering. The results we shall sketch are quite well-known—certainly they are not original in any way.

3.1. **Semidirect Products.** Throughout this section we shall denote by H the Hilbert space $L^2(K)$ equipped with both the left and right regular representations of K. We shall denote by $\mathcal{K}(H)$ the C^*-algebra of compact operators on H.

As in Section 2.1 we shall denote by X any abelian Lie group that is equipped with an action of K by automorphisms. Our aim is to describe the C^*-algebra of the semidirect product $K \ltimes X$. From the point of view of C^*-algebra theory this group is hardly more difficult to handle than an abelian group.

Fix Haar measures on K and X; together they determine a Haar measure on $K \ltimes X$. Denote by Y the Pontrjagin dual group of the abelian Lie group X, and if f is a smooth, compactly supported function on $K \ltimes X$ then let us define its *Fourier transform*, a smooth function from Y into the smooth functions on $K \times K$ by the formula

$$\hat{f}(\varphi)(k_1, k_2) = \int_X f(k_1 k_2^{-1} x)\varphi(k_2^{-1}(x))\, dx.$$

Here $\varphi \in Y$, $k_1, k_2 \in K$, and $x \in X$; we are viewing both K and X as subgroups of $K \ltimes X$, so that for example $k_1 k_2^{-1} x \in K \ltimes X$.

For a fixed $\varphi \in Y$ we shall think of $\hat{f}(\varphi)$ as an integral kernel and hence as a compact operator on H. The Fourier transform $\hat{f}$ is therefore a function from Y into $\mathcal{K}(H)$. This function is equivariant for the natural action of K on Y and the conjugation action of K on $\mathcal{K}(H)$ induced from the *right* regular representation of K on H. Let us write

$$C_0\big(Y, \mathcal{K}(H)\big) = \left\{ \begin{matrix} \text{continuous functions, vanishing} \\ \text{at infinity, from } Y \text{ into } \mathcal{K}(H) \end{matrix} \right\}$$

and

$$C_0\big(Y, \mathcal{K}(H)\big)^K = \big\{ \text{K-equivariant functions in } C_0\big(Y, \mathcal{K}(H)\big) \big\}.$$

With this notation established, the following fact is easy to check (compare [Rie80, Proposition 4.3]):

3.2. **Theorem.** *Let* K *be a compact Lie group and let* X *be an abelian Lie group equipped with an action of* K *by automorphisms. Let* Y *be the Pontrjagin dual of* X*. If* $f \in C_c^\infty(K \ltimes X)$*, then the Fourier transform* $f \mapsto \hat{f}$ *extends to a* C^**-algebra isomorphism*

$$C^*_\lambda(K \ltimes X) \xrightarrow[\cong]{} C_0\big(Y, \mathcal{K}(H)\big)^K.$$

□

The theorem encodes Mackey's calculation of the irreducible unitary representations of $K \ltimes X$. The irreducible representations of $C_0(Y, \mathcal{K}(H))^K$ are obtained by evaluating functions from Y to $\mathcal{K}(H)$ at a point $\varphi \in Y$, so as to obtain an element of $\mathcal{K}(H)^{K_\varphi}$, and then projecting onto a summand of $\mathcal{K}(H)^{K_\varphi}$ (this fixed-point C^*-algebra, like any C^*-subalgebra of $\mathcal{K}(H)$, is isomorphic to a direct sum of subalgebras, each isomorphic to the C^*-algebra of compact operators on a Hilbert space).

Let us compute the fixed-point algebra. For each unitary, irreducible representation $\tau\colon K_\varphi \to \operatorname{Aut}(W_\tau)$, define a Hilbert space

$$H_{\tau\varphi} = [\, H \otimes W_\tau \,]^{K_\varphi},$$

using the right action ρ of K_φ on H. The formula

$$(f \otimes v_1) \otimes \overline{v}_2 \mapsto \dim(W_\tau)^{\frac{1}{2}} \langle v_2, v_1 \rangle f$$

gives a unitary isomorphism of Hilbert spaces

$$\bigoplus\nolimits_\tau H_{\tau\varphi} \otimes \overline{W}_\tau \longrightarrow H$$

(the direct sum is over all equivalence classes of irreducible unitary representations of K_φ). The isomorphism is K_φ-equivariant for the given actions on the representation spaces $\overline{W}_\tau$ and the right translation action of K_φ on H, and so we find that

$$\mathcal{K}(H)^{K_\varphi} \cong \bigoplus\nolimits_\tau \mathcal{K}(H_{\tau\varphi}).$$

The representation of $K \ltimes X$ corresponding to the summand $\mathcal{K}(H_{\tau\varphi})$ is the induced representation obtained from φ and τ (compare Section 5.3).

3.2. **Complex Semisimple Groups.** The reduced C^*-algebra of a complex semisimple group is not as easily determined from first principles, and instead some representation theory is needed to compute it.

The Hilbert space $H_{\sigma\varphi}$ of the principal series representation with parameter (σ, φ) may be identified with the subspace of $H = L^2(K)$ consisting of elements for which $\rho(m)f = \sigma(m)f$, for all $m \in M$ (see Section 5; ρ is the right regular representation). In particular $H_{\sigma\varphi}$ is independent of the continuous parameter φ, and so the collection of all these representation Hilbert spaces forms a locally trivial[3] Hilbert space bundle $\mathcal{H}$ over the parameter space $\hat{M} \times \hat{A}$.

Now the C^*-algebra $C^*_\lambda(G)$ acts as compact operators in each principal series representation, and if we denote by

$$C_0\big(\hat{M} \times \hat{A}, \mathcal{K}(\mathcal{H})\big)$$

[3]Here we really mean "locally trivial" as opposed to the usually intended "locally trivializable."

the algebra of norm-continuous, compact operator-valued endomorphisms of this bundle that vanish at infinity, then the totality of all principal series representations determines a $*$-homomorphism

$$\pi\colon C^*_\lambda(G) \longrightarrow C_0\big(\hat{M} \times \hat{A}, \mathcal{K}(\mathcal{H})\big).$$

According to Harish-Chandra's Plancherel theorem [HC54] the regular representation may be decomposed into principal series representations. Since $C^*_\lambda(G)$ is by definition faithfully represented in the regular representation, the $*$-homomorphism π is injective.

To determine the image of π we invoke the following nontrivial facts:

(a) Each principal series representation is irreducible [Žel68, Wal71]. This implies that if we follow π with evaluation at any point of $\hat{M} \times \hat{A}$, then the composite $*$-homomorphism into $\mathcal{K}(H_{\sigma\varphi})$ is surjective.
(b) The conjugation action of the Weyl group W on $\hat{M} \times \hat{A}$ lifts to a unitary action on the Hilbert space bundle $\mathcal{H}$ that commutes with the action of G in each fiber; in other words, the elements of W act as intertwiners of prinicipal series representations [KS67].

These and some relatively straightforward calculations like those in [Lip70] and [Fel60] lead to the following description of $C^*_\lambda(G)$:

3.3. **Theorem.** *Let G be a connected complex semisimple group, and let $\mathcal{H}$ be the associated Hilbert space bundle of principal series representations over $\widehat{M} \times \widehat{A}$. The $*$-homomorphism*

$$\pi\colon C^*_\lambda(G) \to C_0\big(\hat{M} \times \hat{A}, \mathcal{K}(\mathcal{H})\big)^W$$

is an isomorphism. □

4. The Connes-Kasparov Conjecture

Our aim is to relate Mackey's analogy to the Connes-Kasparov conjecture in C^*-algebra K-theory. We shall give a quick account of the Connes-Kasparov conjecture here, but we shall not go into much detail about K-theory. This is because when we later give a new proof of the Connes-Kasparov conjecture (for complex semisimple groups) we shall need only the simplest K-theoretic ideas.

4.1. **Dirac Operators.** Let G be a Lie group and let K be a compact and connected subgroup. Fix a K-invariant complement $\mathfrak{p}$ to $\mathfrak{k} \subseteq \mathfrak{g}$ and equip the vector space $\mathfrak{p}$ with a K-invariant inner product. The inner product determines a complete Riemannian metric on the homogeneous right G-space $M = K\backslash G$. An equivariant *Dirac-type operator* on M is given by the following *symbol data*: a linear map and unitary representation

$$c\colon \mathfrak{p} \to \operatorname{End}(S) \quad \text{and} \quad \tau\colon K \to \operatorname{Aut}(S).$$

such that

$$c(P)^* = -c(P), \quad c(P)^2 = -\|P\|^2 I, \quad \text{and} \quad c(\operatorname{Ad}_k(P)) = \tau(k)c(P)\tau(k)^{-1}$$

for every $P \in \mathfrak{p}$ and every $k \in K$. The usual induced bundle construction (see for example [Bot65, Sec. 2]) associates to the representation τ an equivariant bundle on M that we shall also denote by S. Its sections may be identified as follows:

$$C^\infty(M, S) \cong \big\{\, \xi\colon G \xrightarrow{\text{smooth}} S \,\big|\, \xi(kg) = \tau(k)\xi(g) \ \ \forall k \in K,\ \forall g \in G \,\big\}.$$

In other words,

$$C^\infty(M,S) \cong [S \otimes C^\infty(G)]^K$$

where K acts on $C^\infty(G)$ via the left regular representation. Using this isomorphism, the Dirac operator acting on sections of the bundle S is defined by

$$D = \sum_{j=1}^{n} c(P_j) \otimes \lambda(P_j),$$

where $\{P_i\}$ is any orthonormal basis for $\mathfrak{p}$. It is a linear, G-equivariant, formally self-adjoint,[4] first-order elliptic partial differential operator.

If $\mathfrak{p}$ is even-dimensional and oriented, then the operator

$$\gamma = i^{n/2} c(P_1) \cdots c(P_n)$$

decomposes S into positive and negative eigenbundles, whose sections D exchanges. In this situation one can develop index theory for D.

Bott showed in [Bot65] that if G is compact and connected, and if K is its maximal torus, then every irreducible representation of G, viewed as a generator of the representation ring $R(G)$, is the index of a Dirac-type operator. Atiyah and Schmid [AS77] (following Parthasarathy [Par72]) showed that every discrete series representation of a connected semisimple group (with finite center) is the L^2-index of a Dirac-type operator, and recovered Harish-Chandra's parametrization of the discrete series.

4.2. **The Connes-Kasparov Conjecture.** Especially for K-theoretic purposes, it is convenient to take a more C^*-algebraic view of the Dirac operator that involves passing from the space of smooth sections of the bundle S to the object

$$C^*_\lambda(M,S) = [S \otimes C^*_\lambda(G)]^K$$

where, as with $C^\infty(M,S)$, the group K is made to act via left translation on G. This is a right $C^*_\lambda(G)$-module, and in fact a so-called Hilbert $C^*_\lambda(G)$-module [Lan95]. The Dirac operator is, in an appropriate sense, a densely defined, essentially self-adjoint, Fredholm operator on the Hilbert module $C^*_\lambda(M,S)$ (see [Lan95, Ch. 9]). It has an index in the K-theory group $K(C^*_\lambda(G))$ (we are assuming for simplicity that $\mathfrak{p}$, and hence M, is even-dimensional).

Building on the Atiyah-Schmid work, Connes and Kasparov conjectured that the Dirac operator can be used to account for the full reduced dual of any connected Lie group, at least at the level of K-theory (for discussion and references, see [Con94, pp. 142–150]).

The *spin module* $R_{\text{spin}}(G,K)$ is the Grothendieck group generated by isomorphism classes of the symbol data (c,τ) described above. By associating to a symbol the corresponding Dirac-type operator and then forming the index of the Dirac operator we obtain a homomorphism from the abelian group $R_{\text{spin}}(G,K)$ into $K(C^*_\lambda(G))$.

4.1. **Conjecture** (Connes and Kasparov)**.** *Let G be a connected Lie group, let K be a maximal compact subgroup of G, and assume (for simplicity) that $\dim(G) \equiv \dim(K)$, modulo 2. The index homomorphism*

$$\mu\colon R_{\text{spin}}(G,K) \longrightarrow K(C^*_\lambda(G))$$

is an isomorphism of abelian groups.

[4] It is formally self-adjoint as written if G is unimodular; otherwise a correction term is required.

4.2. **Remark.** There is an elaboration involving in addition the odd K-group K_1, which also takes care of the case where $K\backslash G$ is odd-dimensional.

The index map μ may be analyzed, even without going into details about C^*-algebra K-theory, as follows. Suppose that G is a connected, complex semisimple group (and that K is a maximal compact subgroup). For any principal series representation $\pi_{\sigma\varphi}$ on $H_{\sigma\varphi}$ we can form the Hilbert module tensor product

$$C^*_\lambda(M,S) \otimes_{C^*_\lambda(G)} H_{\sigma\varphi} = [S \otimes C^*_\lambda(G)]^K \otimes_{C^*_\lambda(G)} H_{\sigma\varphi} = [S \otimes H_{\sigma\varphi}]^K$$

on which the tensor product operator $D \otimes I$ acts. We obtain a self-adjoint endomorphism of a *finite-dimensional* Hilbert space.

If we carry out this localization construction for each fiber of the bundle $\mathcal{H}$ of principal series representations that we introduced in Section 3.2, then we obtain from the Dirac operator an endomorphism of a finite-dimensional, W-equivariant vector bundle over the parameter space $\widehat{M} \times \widehat{A}$:

$$\begin{tikzcd} C^*_\lambda(M,S) \otimes_{C^*_\lambda(G)} \mathcal{H} \arrow[r, "D\otimes I"] \arrow[d] & C^*_\lambda(M,S) \otimes_{C^*_\lambda(G)} \mathcal{H} \arrow[d] \\ \widehat{M} \times \widehat{A} \arrow[r, equal] & \widehat{M} \times \widehat{A}. \end{tikzcd}$$

It can be viewed as the index of the Dirac operator: the vector bundle, plus endomorphism, is a cycle for the topological K-theory group of the locally compact space $(\widehat{M} \times \widehat{A})/W$, and this K-theory group turns out to be the same as the K-theory of $C^*_\lambda(G)$.

With some work the vector bundle and endomorphism can be computed explicitly. Penington and Plymen did so in [PP83] and discovered the remarkable fact that the Dirac operators with irreducible symbols determine canonical generating cycles (Bott generators) for the K-theory of the space $(\widehat{M} \times \widehat{A})/W$, thus verifying the Connes-Kasparov conjecture for complex semisimple groups. The more complicated case of real groups was subsequently worked out in a conceptually similar way by Wassermann [Was87], and eventually the Connes-Kasparov conjecture was proved for all Lie groups [CEN03].

5. Matrix Coefficients and Induced Representations

We are going to use the Mackey analogy to show how the C^*-algebras $C^*_\lambda(G)$ and $C^*_\lambda(G_0)$, which do not appear to be especially alike from their descriptions in Sections 3.2 and 3.1, are assembled from an identical set of building blocks. We shall then use this information to prove the Connes-Kasparov conjecture.

First we shall need to carry out some simple matrix coefficient calculations. These will be used to identify the constituent pieces of $C^*_\lambda(G)$ and of $C^*_\lambda(G_0)$.

5.1. **Orthogonality Relations.** We begin by reviewing some well-known facts about matrix coefficients on compact groups. Let G be a Lie group and let K be a compact subgroup of G. If s is a smooth function on K, and if f is a smooth, compactly supported function on G, then we define

$$(s * f)(g) = \frac{1}{\operatorname{vol}(K)} \int_K s(k) f(k^{-1}g)\, dk,$$

which is again a smooth, compactly supported function on G. We similarly define the convolution product $f * s$. The products are associative with respect to both convolution on G (for any given left Haar measure) and convolution on K (which we shall normalize, as above, by dividing out $\mathrm{vol}(K)$).

We are interested in convolutions with matrix coefficient functions. Recall that if $\tau\colon K \to \mathrm{Aut}(V)$ is an irreducible unitary representation of a compact group, and if $v_1, v_2, w_1, w_2 \in V$, then

$$\frac{1}{\mathrm{vol}(K)} \int_K \langle v_1, \tau(k) v_2\rangle \langle \tau(k) w_1, w_2\rangle \, dk = \frac{1}{\dim(V)} \langle w_1, v_2\rangle \langle v_1, w_2\rangle.$$

As a result, if $\{v_\alpha\}$ is an orthonormal basis for V, and if

$$e_{\alpha\beta}(k) = \dim(V) \langle \tau(k) v_\beta, v_\alpha\rangle,$$

then $e_{\alpha\beta} * e_{\beta\gamma} = e_{\alpha\gamma}$ for all α, β and γ (in particular, $e_{\alpha\alpha}$ is a projection). We shall also use the following small extension of these basic orthogonality relations:

5.1. **Lemma.** *If K_1 is a compact subgroup of K, if τ_1 is the restriction of an irreducible unitary representation $\tau\colon K \to \mathrm{Aut}(V)$ to an irreducible K_1-invariant subspace W of V, and if we define*

$$d_{\alpha\beta}(k) = \dim(W) \langle \tau_1(k) v_\beta, v_\alpha\rangle,$$

then $d_{\alpha\beta} * e_{\beta\gamma} = e_{\alpha\gamma} = e_{\alpha\beta} * d_{\beta\gamma}$ whenever $v_\alpha, v_\beta, v_\gamma \in W$ (the convolution products are over K_1). □

Finally, we note that if

$$\xi_{\alpha\beta}(k) = \frac{\dim(V)^{\frac{1}{2}}}{\mathrm{vol}(K)^{\frac{1}{2}}} \langle \tau(k) v_\beta, v_\alpha\rangle,$$

then $\|\xi_{\alpha\beta}\|^2_{L^2(K)} = 1$. This follows from the convolution relations by evaluation at the identity element of K.

5.2. **Complex Semisimple Groups.** Let $G = KAN$ be a connected complex semisimple group. If we define a group homomorphism $\delta\colon A \to (0,\infty)$ by the change of variables formula

$$\int_N f(ana^{-1})\, dn = \delta(a)^{-2} \int_N f(n)\, dn,$$

then the formula

$$\int_G f(g)\, dg = \int_K \int_A \int_N f(kan)\delta(a)^2 \, dk da dn$$

defines a Haar measure on the unimodular group G (because the right-hand integral is left K-invariant and right AN-invariant). If σ and φ are unitary characters of M and A, then by definition the Hilbert space of the principal series representation $\pi_{\sigma\varphi}$ associated to the pair (σ, φ) is the completion of the vector space

$$\left\{ \xi\colon G \xrightarrow{\text{smooth}} \mathbb{C} \;\middle|\; \begin{array}{c} \xi(gman) = \sigma(m)^{-1}\varphi(a)^{-1}\delta(a)^{-1} \\ \forall g \in G \;\; \forall a \in A \;\; \forall n \in N \end{array} \right\}$$

in the norm associated to the inner product

$$\langle \xi_1, \xi_2\rangle = \int_K \overline{\xi_1(k)} \xi_2(k)\, dk.$$

The inclusion of the factor $\delta(a)$ assures that the natural left translation action of G on this Hilbert space is unitary.

5.2. **Lemma.** *The principal series representation $\pi_{\sigma\varphi}$ contains the K-type τ having highest weight σ, and contains it with multiplicity one. Every other K-type of $\pi_{\sigma\varphi}$ is greater than τ in the highest weight partial order.*

Proof. Restricting functions in the representation space for $\pi_{\sigma\varphi}$ to K and using the Peter-Weyl theorem we see that

$$H_{\sigma\varphi} \cong \bigoplus V_\tau \otimes \overline{V_\tau^\sigma}$$

where the sum is over the equivalence classes of irreducible representations of K and V_τ^σ denotes the σ-weight space of the representation space V_τ. The result follows. $\square$

5.3. **Remark.** This is the simplest possible instance of the very general multiplicity one principle for minimal K-types described in [Vog79].

5.4. **Definition.** Let $\tau \colon K \to \mathrm{Aut}(V)$ be an irreducible unitary representation of K with highest weight σ, and let $v_\sigma \in V$ be a unit vector in the σ-weight space of V. Define a smooth, norm-one function $\xi_{\sigma\varphi} \colon G \to \mathbb{C}$ in the representation space for the induced representation $\pi_{\sigma\varphi}$ by the formula

$$\xi_{\sigma\varphi}(kan) = \frac{\dim(V)^{\frac{1}{2}}}{\mathrm{vol}(K)^{\frac{1}{2}}} \big\langle \tau(k)v_\sigma, v_\sigma \big\rangle \varphi(a)^{-1}\delta(a)^{-1}.$$

5.5. **Lemma.** *If $f \in C_c^\infty(G)$ and if $f = e_{\sigma\sigma} * f * e_{\sigma\sigma}$, then*

$$\int_G f(g)\langle \xi_{\sigma\varphi}, \pi_{\sigma\varphi}(g)\xi_{\sigma\varphi}\rangle \, dg = \frac{\mathrm{vol}(K)}{\dim(V)} \int_A \int_N f(an)\varphi(a)\delta(a)\, da dn.$$

Proof. Using the definition of the inner product on $H_{\sigma\varphi}$ and then making the change of variables $g := g^{-1}k$ we find that

$$\begin{aligned}\int_G f(g)\langle \xi_{\sigma\varphi}, \pi_{\sigma\varphi}(g)\xi_{\sigma\varphi}\rangle \, dg &= \int_G \int_K f(g)\overline{\xi_{\sigma\varphi}(k)}\xi_{\sigma\varphi}(g^{-1}k)\, dg dk \\ &= \int_G \int_K f(kg^{-1})\overline{\xi_{\sigma\varphi}(k)}\xi_{\sigma\varphi}(g)\, dg dk.\end{aligned}$$

If we now insert the definition of $\xi_{\sigma\varphi}$ into the last integral, then we obtain the quantity

$$\frac{\dim(V)^{\frac{1}{2}}}{\mathrm{vol}(K)^{\frac{1}{2}}} \int_G \left(\int_K f(kg^{-1})\langle \tau(k^{-1})v_\sigma, v_\sigma\rangle \, dk \right) \xi_{\sigma\varphi}(g)\, dg.$$

The inner integral is, up to a factor of $\dim(V)/\mathrm{vol}(K)$, the convolution of the projection $e_{\sigma\sigma}$ with f, evaluated at g^{-1}. Since $e_{\sigma\sigma} * f = f$, we obtain the quantity

$$\frac{\mathrm{vol}(K)^{\frac{1}{2}}}{\dim(V)^{\frac{1}{2}}} \int_G f(g^{-1})\xi_{\sigma\varphi}(g)\, dg.$$

We now insert the definition of $\xi_{\sigma\varphi}$, and use the formula $dg = \delta(a)^2 dk da dn$ to obtain

$$\int_K \int_A \int_N f(n^{-1}a^{-1}k^{-1})\langle \tau(k)v_\sigma, v_\sigma\rangle \varphi(a)^{-1}\delta(a)\, dk da dn.$$

Once again, we recognize the integral over K as a convolution, giving

$$\frac{\mathrm{vol}(K)}{\dim(V)}\int_A\int_N f(n^{-1}a^{-1})\varphi(a)^{-1}\delta(a)\,da\,dn.$$

The changes of variables $n := an^{-1}a^{-1}$ and then $a := a^{-1}$ give the required result. $\square$

5.3. **Semidirect Product Groups.** We turn now to the semidirect product $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ associated to a complex semisimple group G. The results about K-types and matrix coefficients that we shall need are closely analogous to those for G, although the calculations are different and in fact a bit more involved.

Fix characters σ of M and φ of $\mathfrak{a}$, and view the latter as a balanced character of $\mathfrak{g}/\mathfrak{k}$. Let $\tau_\sigma\colon K_\varphi \to \mathrm{Aut}(W)$ be an irreducible unitary representation of K_φ with highest weight σ. The representation space for the induced representation

$$\pi_{\sigma\varphi} = \mathrm{Ind}_{K_\varphi\ltimes\mathfrak{g}/\mathfrak{k}}^{K\ltimes\mathfrak{g}/\mathfrak{k}}\,\tau_\sigma\otimes\varphi$$

is then the completion of the space

$$\left\{\,\xi\colon G_0\to W\;\middle|\;\begin{array}{l}\xi(gkx) = \tau(k)^{-1}\varphi(x)^{-1}\xi(g)\\ \forall g\in G_0\;\;\forall k\in K_\varphi\;\;\forall x\in\mathfrak{g}/\mathfrak{k}\end{array}\right\}$$

of smooth vector-valued functions in the norm associated to the inner product

$$\langle\xi_1,\xi_2\rangle = \int_K \big\langle \xi_1(k),\xi_2(k)\big\rangle\,dk.$$

5.6. **Lemma.** *The induced representation $\pi_{\sigma\varphi}$ of G_0 contains the K-type with highest weight σ with multiplicity one. Every other K-type is greater than this one in the highest weight partial order.*

Proof. Using Peter-Weyl, we can write the Hilbert space of $\pi_{\sigma\varphi}$ as

$$H_{\sigma\varphi} = \bigoplus V_\tau\otimes[\overline{V_\tau}\otimes W]^{K_\varphi}.$$

The lemma follows from this. $\square$

To construct a distinguished unit vector in the representation space of $\pi_{\sigma\varphi}$ we need to fix an irreducible representation $\tau\colon K\to\mathrm{Aut}(V)$ with highest weight σ and realize the representation space W for τ_σ as the smallest K_φ-invariant subspace of V that contains the σ-weight space of V.

5.7. **Definition.** Define a smooth, norm-one function $\xi_{\sigma\varphi}$ in the representation space for $\pi_{\sigma\varphi}$ by the formula

$$\xi_{\sigma\varphi}(kx) = \frac{\dim(V)^{\frac12}}{\mathrm{vol}(K)^{\frac12}\dim(W)^{\frac12}}\varphi(x)^{-1}\sum_\alpha\big\langle\tau(k)v_\alpha,v_\sigma\big\rangle v_\alpha,$$

where $\{v_\alpha\}$ is an orthonormal basis for the K_φ-invariant subspace $W\subseteq V$.

5.8. **Lemma.** *If $f\in C_c^\infty(G_0)$ and if $f = e_{\sigma\sigma}*f*e_{\sigma\sigma}$, then*

$$\int_{G_0} f(g)\langle\xi_{\sigma\varphi},\pi_{\sigma\varphi}(g)\xi_{\sigma\varphi}\rangle\,dg = \frac{\mathrm{vol}(K)}{\dim(V)}\int_{\mathfrak{g}/\mathfrak{k}} f(x)\varphi(x)\,dx.$$

Proof. Most of this calculation is very similar to the one carried out in the proof of Lemma 5.5. Using the definition of $\xi_{\sigma\varphi}$, the substitution $g := g^{-1}k$, and the fact that $dk\,dl$ is an invariant measure on $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$, we find that

$$\int_{G_0} f(g)\langle \xi_{\sigma\varphi}, \pi_{\sigma\varphi}(g)\xi_{\sigma\varphi}\rangle\, dg = \frac{\mathrm{vol}(K)}{\dim(V)\dim(W)} \sum_\alpha \int_{\mathfrak{g}/\mathfrak{k}} (e_{\alpha\sigma} * f * e_{\sigma\alpha})(x)\varphi(x)\, dx.$$

At this point we invoke Lemma 5.1, which implies that

$$\begin{aligned} e_{\alpha\sigma} * f * e_{\sigma\alpha} &= e_{\alpha\sigma} * e_{\sigma\sigma} * f * e_{\sigma\sigma} * e_{\sigma\alpha} \\ &= d_{\alpha\sigma} * e_{\sigma\sigma} * f * e_{\sigma\sigma} * d_{\sigma\alpha} = d_{\alpha\sigma} * f * d_{\sigma\alpha}. \end{aligned}$$

But now using the substitution $x := k_1^{-1}xk_1$ and the fact that φ is K_φ-invariant, we find that

$$\begin{aligned} &\int_{\mathfrak{g}/\mathfrak{k}} (d_{\alpha\sigma} * f * d_{\sigma\alpha})(x)\varphi(x)\, dx \\ &\quad= \frac{1}{\mathrm{vol}(K_\varphi)^2} \int_{K_\varphi}\int_{K_\varphi}\int_{\mathfrak{g}/\mathfrak{k}} d_{\alpha\sigma}(k_1)f(k_1^{-1}xk_2)d_{\sigma\alpha}(k_2^{-1})\varphi(x)\, dk_1\, dk_2\, dx \\ &\quad= \frac{1}{\mathrm{vol}(K_\varphi)^2} \int_{K_\varphi}\int_{K_\varphi}\int_{\mathfrak{g}/\mathfrak{k}} d_{\alpha\sigma}(k_1)f(xk_1^{-1}k_2)d_{\sigma\alpha}(k_2^{-1})\varphi(x)\, dk_1\, dk_2\, dx \\ &\quad= \int_{\mathfrak{g}/\mathfrak{k}} (f * d_{\sigma\alpha} * d_{\alpha\sigma})(x)\varphi(x)\, dx. \end{aligned}$$

Since by Lemma 5.1 again $f * d_{\sigma\alpha} * d_{\alpha\sigma} = f$, we conclude that

$$\frac{\mathrm{vol}(K)}{\dim(V)\dim(W)} \sum_\alpha \int_{\mathfrak{g}/\mathfrak{k}} (e_{\alpha\sigma} * f * e_{\sigma\alpha})(x)\varphi(x)\, dx = \frac{\mathrm{vol}(K)}{\dim(V)} \int_{\mathfrak{g}/\mathfrak{k}} f(x)\varphi(x)\, dx,$$

as required. □

6. THE MACKEY ANALOGY AND C*-ALGEBRAS

As we noted earlier, the *dual* of a C^*-algebra C is the set $\hat{C}$ of equivalence classes of irreducible representations of C. The dual carries a topology in which closed sets correspond bijectively to the closed ideals of C: to a closed ideal J one associates the set of all equivalence classes of irreducible representations that vanish on J.

Actually for our purposes it will be a bit more convenient to associate to an ideal J the complementary open set, which may be identified homeomorphically with the dual of J (note that each irreducible representation of C either restricts to zero on J or restricts to an irreducible representation of J, and each irreducible representation of J extends uniquely to an irreducible representation of C). We shall always make this identification.

If $I \subseteq J \subseteq C$ are nested closed ideals in a C^*-algebra, then the duals of I and J identify homeomorphically with the nested open subsets of the dual of C, and we shall identify the dual of J/I with the dual of J minus the dual of I.

We shall reduce most of our computations to one simple C^*-algebra lemma, which it is convenient to state using the language of multiplier algebras. A *multiplier* of a C^*-algebra C is a map $m\colon C \to C$ for which there exists a map $m^*\colon C \to C$ such that

$$\langle mc_1, c_2\rangle = \langle c_1, m^*c_2\rangle \qquad (c_1, c_2 \in C),$$

where $\langle c_1, c_2\rangle = c_1^*c_2$. The *multiplier algebra* of C is the algebra of all multipliers. It is a C^*-algebra in the operator norm on C and it contains C as an ideal since C can be viewed as acting on itself by left multiplication. See [Lan95, Ch. 2].

As with any C^*-algebra containing C as an ideal, an irreducible representation of C extends uniquely to an irreducible representation of the multiplier algebra.

6.1. **Lemma.** *Let C be a C^*-algebra and let p be a projection in the multiplier algebra of C. If for every irreducible representation π of C the operator $\pi(p)$ is a rank-one projection, then:*

(a) $CpC = C$, *while pCp is a commutative C^*-algebra;*
(b) *the dual $\widehat{C}$ is a Hausdorff locally compact space; and*
(c) *the map $a \mapsto \hat{a}$ from pCp to $C_0(\hat{C})$ that is defined by the formula*

$$\pi(a) = \hat{a}(\pi)\pi(p)$$

is an isomorphism of C^*-algebras . □

6.2. **Remark.** The bimodule Cp implements a *Morita equivalence* between CpC and pCp. We shall later use the fact that the inclusion of pCp into CpC induces an isomorphism in K-theory; this is a consequence of the Morita invariance of K-theory.

6.1. Subquotients of Group C*-Algebras.

6.3. **Definition.** Let K be the maximal compact subgroup of a connected Lie group, and fix for the remainder of the paper a linear order on the equivalence classes of irreducible representations of K,

$$\widehat{K} = \{\tau_1, \tau_2, \tau_3, \dots\}$$

with the property that if τ_i precedes τ_j in the natural partial order coming from highest weights, as in Section 2.3, then $i < j$.

Such an ordering is possible since each irreducible representation τ is preceded by only finitely many other irreducible representations in the highest weight partial order, up to equivalence.

6.4. **Definition.** Let M be a maximal torus for K. Associate to each irreducible representation τ_n of K a highest weight σ_n. Let ν be a unit vector in the σ_n-weight space and define a smooth function p_n on K by

$$p_n(k) = \dim(\tau_n)\langle\tau_n(k)\nu, \nu\rangle.$$

The action of p_n on $C_c^\infty(G)$ by convolution extends to an action on $C^*_\lambda(G)$ as a multiplier.

6.5. **Lemma.** *If $\pi\colon C^*_\lambda(G) \to \mathcal{B}(H)$ is any representation, then $\pi(p_n)$ is the orthogonal projection onto the σ_n-weight space of the τ_n-isotypical component of the unitary representation of K determined by π.* □

6.6. **Definition.** Denote by $\widehat{G}_n$ the subset of the reduced dual of G determined by those irreducible representations of G whose restrictions to K contain τ_n as a subrepresentation.

6.7. **Remark.** Lemma 6.5 implies that each set $\hat{G}_n$ is an open subset of the reduced dual, since in fact it is the dual of the closed ideal

$$J_n = C^*_\lambda(G) \cdot p_n \cdot C^*_\lambda(G)$$

of $C^*_\lambda(G)$ generated by the multiplier p_n.

6.8. **Definition.** Define ideals J_n as above, and define subquotients C_n by $C_1 = J_1$ and

$$C_n = (J_1 + \cdots + J_n)\big/(J_1 + \cdots + J_{n-1})$$

for $n > 1$.

The dual of the C^*-algebra C_n is naturally homeomorphic to the locally closed subspace

$$\widehat{G}_n \big\backslash \big(\widehat{G}_1 \cup \cdots \cup \widehat{G}_{n-1}\big) \subseteq \widehat{G}.$$

This is precisely the subspace of $\hat{G}$ determined by irreducible representations of G whose lowest K-type (for the linear ordering of Definition 6.3) is τ_n. We therefore obtain a partition of $\hat{G}$ into locally closed subsets

$$\widehat{G} = \widehat{C}_1 \sqcup \widehat{C}_2 \sqcup \widehat{C}_3 \sqcup \cdots .$$

In the case where G is complex semisimple we shall use this and the similar decomposition for G_0 to compare $C^*_\lambda(G)$ and $C^*_\lambda(G_0)$.

6.9. **Definition.** Let $G = KAN$ be a connected complex semisimple group. If f is a smooth, compactly supported function on G, then define a function $\hat{f} \colon \hat{A} \to \mathbb{C}$ by the formula

$$\hat{f}(\varphi) = \operatorname{vol}(K) \int_A \int_N f(an)\varphi(a)\delta(a)\, da\, dn$$

(see Section 5.2 for the definition of δ).

6.10. **Proposition.** *Let $G = KAN$ be a connected complex semisimple group. The correspondence that associates to each smooth compactly supported function f on G the function $\dim(\tau_n)^{-1}\hat{f}$ determines an isomorphism of C^*-algebras*

$$p_n C_n p_n \xrightarrow[\cong]{} C_0(\hat{A}/W_n),$$

where W_n is the subgroup of the Weyl group W that fixes the highest weight σ_n of the representation τ_n.

Proof. The irreducible representations of C_n correspond to the irreducible representations of G in the reduced dual with lowest K-type τ_n. By Lemma 5.2, these are precisely the principal series representations $\pi_{\sigma\varphi}$ with discrete parameter $\sigma = \sigma_n$; and for these, the projection $\pi_{\sigma\varphi}(p_n)$ has rank one. Moreover it was shown in Lemma 5.5 that

$$\pi_{\sigma\varphi}(f) = \frac{\hat{f}(\varphi)}{\dim(\tau_n)}\pi_{\sigma\varphi}(p_n)$$

for every smooth, compactly supported function f on G such that $f = p_n f p_n$. Because $\hat{f}$ is a continuous function of φ, and because the Riemann-Lebesgue lemma shows that $\hat{f}$ vanishes at infinity, the correspondence $\varphi \mapsto \pi_{\sigma\varphi}$ defines a continuous and proper bijection from $\widehat{A}/W_n$ to $\widehat{C}_n$. Since by Lemma 6.1 the space $\widehat{C}_n$ is locally compact and Hausdorff, this bijection must in fact be a homeomorphism. The proposition now follows from the rest of Lemma 6.1. $\square$

6.11. **Definition.** Let G be a connected complex semisimple group and let G_0 be its associated semidirect product group. If f is a smooth, compactly supported function on G_0, then define a function $\hat{f}\colon \hat{\mathfrak{a}} \to \mathbb{C}$ by the formula

$$\hat{f}(\varphi) = \mathrm{vol}(K) \int_{\mathfrak{g}/\mathfrak{k}} f(x)\varphi(x)\, dx,$$

in which the characters φ of $\mathfrak{a}$ are viewed as balanced characters of $\mathfrak{g}/\mathfrak{k}$.

Of course, this $\hat{f}$ is not the same as the Fourier transform that was considered in Section 3.1.

6.12. **Proposition.** *Let G be a connected complex semisimple group and let G_0 be its associated semidirect product group. The correspondence that associates to a smooth compactly supported function f on G the function $\dim(\tau_n)^{-1}\hat{f}$ determines an isomorphism of C^*-algebras*

$$p_n C_n p_n \underset{\cong}{\longrightarrow} C_0(\hat{\mathfrak{a}}/W_n).$$

Proof. The proof is exactly the same as that of Proposition 6.10, but this time using Lemma 5.6 and 5.8. $\square$

6.2. **Continuous Fields of Group C*-Algebras.** In the previous section we showed how the C^*-algebra of a complex semisimple group G and its associated semidirect product G_0 decompose into subquotients, each Morita equivalent to the commutative C^*-algebra of C_0-functions on the space $\widehat{A}/W_n \cong \hat{\mathfrak{a}}/W_n$. We are going to show that these decompositions for G and G_0 are compatible with one another in a very strong sense. To this end we are going to fit $C^*_\lambda(G)$ and $C^*_\lambda(G_0)$ into a continuous field of C^*-algebras.

The first stage of the construction involves only smooth manifolds. Let N be a closed submanifold of a smooth manifold M and let νN be the normal bundle for $N \subseteq M$; that is, the quotient of the tangent bundle of M, restricted to N, by the tangent bundle of N. Then let

$$\nu(N, M) = \nu N \times \{0\} \ \sqcup \ M \times \mathbb{R}^\times$$

(disjoint union). There are unique smooth manifold structures on the sets $\nu(N, M)$ such that:

(a) If U is an open subset of M and if $V = U \cap N$, then $\nu(V, U)$ is an open submanifold of $\nu(N, M)$.
(b) Every diffeomorphism from M to M' that carries N onto N' induces a diffeomorphism from $\nu(N, M)$ onto $\nu(N', M')$.
(c) If $N = \mathbb{R}^n$ and $M = \mathbb{R}^{n+k}$, then the map from $\nu(N, M)$ onto $\mathbb{R}^{n+k+1}$ defined by
$$(x, y, t) \mapsto (x, t^{-1}y, t) \qquad (x \in \mathbb{R}^n,\ y \in \mathbb{R}^k,\ t \neq 0)$$
and
$$(x, X, 0) \mapsto (x, X, 0) \qquad (x, \in \mathbb{R}^n,\ X \in \nu_x N)$$
is a diffeomorphism (we identify the fibers of the normal bundle with $\mathbb{R}^k$ in the natural way).

The spaces $\nu(N, M)$ are the smooth manifold analogues of the *deformation to the normal cone* construction in algebraic geometry [Ful98, Ch 5].

The construction of $\nu(N, M)$ is functorial, and if it is applied to a closed subgroup K of a Lie group G, then it produces a smooth family of Lie groups over $\mathbb{R}$ (in a sense that the reader will readily make precise). All the fibers in this family are copies of the Lie group G, except for the fiber over $t = 0$, which is the Lie group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ that we have been studying.

Let K be a maximal compact subgroup of a connected Lie group G and denote the fibers of $\nu(K, G)$ by G_t. Choose a smoothly varying family of Haar measures on the groups G_t and form the C^*-algebras $C^*_\lambda(G_t)$. The following fact is easy to check using the fact that the group G_0 is amenable.

6.13. **Lemma.** *Let f be a smooth, compactly supported function on $\nu(K, G)$. Let f_t be its restriction to a compactly supported function on G_t, and view f_t as an element of $C^*_\lambda(G_t)$. The function $t \mapsto \|f_t\|$ is continuous on $\mathbb{R}$.*

Proof. The function is certainly continuous at each $t \neq 0$ because all the C^*-algebras are isomorphic via scalar multiplication with the smooth function of t that rescales our choices of Haar measures. The lower semicontinuity of the norm function is a simple feature of the reduced C^*-algebra construction (compare [Rie89, Thm. 2.5] where the different but related case of twisted group C^*-algebras is considered), whereas upper semicontinuity is a feature of the *full* group C^*-algebra construction (compare [Rie89, Prop. 2.2]). Since G_0 is amenable, its full group C^*-algebra is the same as its reduced group C^*-algebra, and continuity at $t = 0$ follows (compare [Rie89, Cor. 2.7]). □

We therefore obtain a *continuous field of C^*-algebras* $\{C^*_\lambda(G_t)\}$, as in [Dix77, Ch. 10] whose continuous sections are generated by the smooth, compactly supported functions on $\nu(K, G)$.

6.3. **Analysis of the Continuous Field.** We shall now specialize the construction of the previous section to the case where G is a connected complex semisimple group with maximal compact subgroup K. We are going to analyze the continuous field $\{C^*_\lambda(G_t)\}$ in the same way that we analyzed the individual C^*-algebras $C^*_\lambda(G)$ and $C^*_\lambda(G_0)$ in Section 6.1.

It will be convenient to restrict the continuous field $\{C^*_\lambda(G_t)\}$ to the closed interval $[0, 1]$ in $\mathbb{R}$. Thus we shall introduce the following notation:

6.14. **Definition.** Denote by $\mathcal{C}$ the C^*-algebra of continuous sections of the restriction of the continuous field $\{C^*_\lambda(G_t)\}$ to the interval $[0, 1]$. Thus $\mathcal{C}$ is the completion of the fiberwise convolution algebra of smooth, compactly supported functions on $\nu(K, G)|_{[0,1]}$ in the norm

$$\|f\|_{\mathcal{C}} = \sup\left\{ \|f_t\| \,\middle|\, t \in [0, 1] \right\}.$$

Let us retain the ordering $\{\tau_1, \tau_2, \dots\}$ on equivalence classes of irreducible representations of K that we introduced in Definition 6.3. The functions p_n on K from Definition 6.4 act by fiberwise convolution on smooth compactly supported functions on $\nu(K, G)$, so that

$$(s \cdot f)_t(g) = \frac{1}{\mathrm{vol}(K)} \int_K s(k) f_t(k^{-1}g)\, dk.$$

They define multipliers of $\mathcal{C}$.

6.15. **Definition.** Let $\mathcal{J}_n$ be the ideal $\mathcal{J}_n = \mathcal{C}p_n\mathcal{C}$ of $\mathcal{C}$. Define subquotient C^*-algebas $\mathcal{C}_n$ by the formulas $\mathcal{C}_1 = \mathcal{J}_1$ and

$$\mathcal{C}_n = (\mathcal{J}_1 + \cdots + \mathcal{J}_n)/(\mathcal{J}_1 + \cdots + \mathcal{J}_{n-1}).$$

In the following definition we shall identify the nonunitary character δ of the Lie group A with a nonunitary character of $\mathfrak{a}$ by means of the exponential map.

6.16. **Definition.** Let G be a connected complex semisimple group. If f is a smooth, compactly supported function on the manifold $\nu(K, G)$, then define a function $\hat{f}\colon \hat{\mathfrak{a}} \times [0,1] \to \mathbb{C}$ by

$$\hat{f}(\varphi, t) = \begin{cases} \displaystyle\mathrm{vol}(K)\int_{\mathfrak{g}/\mathfrak{k}} f(x,0)\varphi(x)\,dx & t = 0 \\ \displaystyle t^{-n}\,\mathrm{vol}(K)\int_{\mathfrak{a}}\int_{\mathfrak{n}} f(\exp(a)\exp(n), t)\varphi(t^{-1}a)\delta(a)\,da\,dn & t \neq 0. \end{cases}$$

6.17. **Lemma.** *Let f be a smooth, compactly supported function on $\nu(K, G)$. Its transform $\hat{f}$ is a smooth function on $\hat{\mathfrak{a}} \times [0,1]$ that vanishes at infinity.*

Proof. The product of K with the Lie algebra $\mathfrak{a} + \mathfrak{n}$ is diffeomorphic to G via the map that sends $(k, a + n)$ to $k\exp(a)\exp(n)$. It follows from the properties (a), (b) and (c) that we used to characterize $\nu(K, G)$ in the previous section that the bijection

$$K \times \mathfrak{a} \times \mathfrak{n} \times \mathbb{R} \longrightarrow \nu(K, G)$$

defined by the formula

$$(k, a, n, t) \mapsto \begin{cases} (k, a + n, 0) & (t = 0) \\ (k\exp(ta)\exp(tn)), t) & (t \neq 0) \end{cases}$$

is a diffeomorphism (on the right we are identifying $a + n \in \mathfrak{a} + \mathfrak{n}$ with its image in the quotient $\mathfrak{g}/\mathfrak{k}$). As a result, every smooth, compactly supported function f on $\nu(K, G)$ has the form

$$\begin{cases} f(k, a + n, 0) = F(k, a, n, 0) \\ f(k\exp(ta)\exp(tn), t) = F(k, a, n, t) \quad t \neq 0 \end{cases}$$

for some smooth and compactly supported function F on $K \times \mathfrak{a} \times \mathfrak{n} \times \mathbb{R}$. We see that the transform of f is given by the formula

$$\hat{f}(\varphi, t) = \mathrm{vol}(K)\int_{\mathfrak{a}}\int_{\mathfrak{n}} F(e, a, n, 0)\varphi(a)\delta(ta)\,da\,dn$$

for all t, from which the result follows. $\square$

The algebra $\mathcal{Z} = C[0,1]$ of continuous functions on the interval $[0,1]$ lies in the center of the multiplier algebra of $\mathcal{C}_n$. If $t \in [0,1]$, then let

$$\mathcal{Z}_t = \{\, h \in C[0,1] \mid h(t) = 0 \,\}.$$

The product $\mathcal{Z}_t\mathcal{C}_n$ is a closed ideal in $\mathcal{C}_n$, and the evaluation map induces an isomorphism from $\mathcal{C}_n/\mathcal{Z}_t\mathcal{C}_n$ to the algebra C_n, as in Definition 6.8, associated to the group G_t (this is a consequence of the upper semicontinuity of the field $\{C^*_\lambda(G_t\}$). Since by Schur's lemma the algebra of multipliers $\mathcal{Z}$ acts as multiples of the identity in any irreducible representation of $\mathcal{C}$, we see that the dual of $\mathcal{C}_n$ is the disjoint union over $t \in [0,1]$ of the sets $\widehat{G}_{t,n}$ determined by the irreducible representations

$\pi_{\sigma\varphi,t}$ of G_t with lowest K-type τ_n (here $\sigma = \sigma_n$). As a set, the dual of $\mathcal{C}_n$ therefore identifies with $\hat{\mathfrak{a}}/W_n \times [0,1]$.

6.18. **Theorem.** *Let* $G = KAN$ *be a connected complex semisimple group. Then* $\mathcal{C}_n p_n \mathcal{C}_n = \mathcal{C}_n$*. Moreover the correspondence that associates to a smooth compactly supported function* f *on* $\nu(K,G)$ *the function* $\dim(\tau_n)^{-1}\hat{f}$ *determines an isomorphism of* C^**-algebras*

$$p_n \mathcal{C}_n p_n \underset{\cong}{\longrightarrow} C_0\big(\hat{\mathfrak{a}}/W_n \times [0,1]\big),$$

where W_n *is the subgroup of the Weyl group* W *that fixes the highest weight* σ_n *of the representation* τ_n*.*

Proof. The multiplier p_n acts as a rank-one projection in each irreducible representation of $\mathcal{C}_n$. Exactly as in the proof of Proposition 6.10, we find that the correspondence $(\varphi, t) \mapsto \pi_{\sigma\varphi,t}$ defines a homeomorphism from $\hat{A}/W_n \times [0,1]$ to $\hat{\mathcal{C}}_n$, and the theorem follows from Lemma 6.1. □

7. The Mackey Analogy and K-Theory

Finally we are ready to use the Mackey analogy in the form given in the last section to prove the Connes-Kasparov conjecture for complex semisimple groups. Aside from the novelty of the argument, our reason for giving a new proof is to promote the idea that the Connes-Kasparov isomorphism should be viewed as only a K-theoretic reflection of a much more precise statement in representation theory.[5]

If we extend each $f \in C^*_\lambda(G_0)$ to a continuous section of the continuous field $\{C^*_\lambda(G_t)\}$ and denote by $\mu_t(f)$ its value at t, then from continuity of the field (in fact from upper continuity of the field) it follows that

$$\lim_{t\to 0} \left\{ \begin{array}{c} \mu_t(\alpha_1 f_1 + \alpha_2 f_2) - \alpha_1 \mu_t(f_1) + \alpha_2 \mu_t(f_2) \\ \mu_t(f_1 f_2) - \mu_t(f_1)\mu_t(f_2) \\ \mu_t(f^*) - \mu_t(f)^* \end{array} \right\} = 0.$$

As a result, the family of maps $\mu_t \colon C^*_\lambda(G_0) \to C^*_\lambda(G)$ is an asymptotic morphism in the sense of [CH90].[6] As explained in [CH90] and reviewed below, asymptotic morphisms induce homomorphisms on K-theory groups, and as was noted in [BCH94, p. 263], the following is equivalent to the Connes-Kasparov conjecture:

7.1. **Conjecture.** *Let* G *be a connected Lie group and let* K *be a maximal compact subgroup of* G*. The* K*-theory map*

$$\mu \colon K_*(C^*_\lambda(G_0)) \longrightarrow K_*(C^*_\lambda(G))$$

associated to the asymptotic morphism $\{\mu_t\}$ *obtained from the deformation* $\nu(K,G)$ *is an isomorphism.*

For a proof of the equivalence, see [Con94, Prop. 9, p. 141]. Of course, since we noted earlier that the Connes-Kasparov conjecture has now been proved for all G, the same goes for Conjecture 7.1.

[5] This idea is already strongly suggested by the form of the Penington-Plymen proof described in Section 4.

[6] Although here we have parametrized the functions μ_t by $t \in (0,1]$ rather than $t \in [1,\infty)$, as is done in [CH90].

The K-theory map in Conjecture 7.1 can be constructed from the diagram

$$\begin{tikzcd} & K_*(\mathcal{C}) \arrow[dl] \arrow[dr] & \\ K_*(C^*_\lambda(G_0)) \arrow[rr, dashed, "\mu"'] & & K_*(C^*_\lambda(G)) \end{tikzcd}$$

in which the downward maps are induced from the evaluation maps at $t = 0$ and $t = 1$. The left downward map is an isomorphism because, as a result of triviality of the field $\{C^*_\lambda(G_t)\}$ away from zero, the kernel of evaluation at $t = 0$ is isomorphic to the algebra of continuous functions from $[0, 1]$ into $C^*_\lambda(G)$ that vanish at $t = 0$, and this C^*-algebra has zero K-theory. The dashed arrow can therefore be filled in so as to make the diagram commute.

To prove Conjecture 7.1 we shall show that the right downward arrow is an isomorphism, and to this it suffices to show that, for every n, evaluation at $t = 1$ gives an isomorphism

$$K_*(\mathcal{J}_1 + \cdots + \mathcal{J}_n) \xrightarrow[\cong]{} K_*(J_1 + \cdots + J_n).$$

This is because $\cup_n(\mathcal{J}_1 + \cdots + \mathcal{J}_n)$ is dense in $\mathcal{C}$, so that by the continuity of K-theory,

$$K_*(\mathcal{C}) = K_*\big(\varinjlim(\mathcal{J}_1 + \cdots + \mathcal{J}_n)\big) = \varinjlim K_*\big(\mathcal{J}_1 + \cdots + \mathcal{J}_n\big),$$

and the same for $C^*_\lambda(G)$. By the five lemma, excision in K-theory and induction, the isomorphism statement that we now wish to prove will follow from the fact that the evaluation map on subquotients

$$K_*(\mathcal{C}_n) \longrightarrow K_*(C_n)$$

is an isomorphism for all n. Finally, in the diagram

$$\begin{tikzcd} K_*(\mathcal{C}_n) \arrow[r] & K_*(C_n) \\ K_*(p_n\mathcal{C}_n p_n) \arrow[r] \arrow[u] & K_*(p_n C_n p_n), \arrow[u] \end{tikzcd}$$

the vertical maps, induced from inclusions, are isomorphisms. Indeed we noted earlier that if p is a projection in the multiplier algebra of any C^*-algebra C, then the map $K_*(pCp) \to K_*(CpC)$ is an isomorphism. But our calculations of $p_n\mathcal{C}_n p_n$ and $p_n C_n p_n$ show that there is a diagram

$$\begin{tikzcd} K_*(p_n\mathcal{C}_n p_n) \arrow[r] & K_*(p_n C_n p_n) \\ K_*\big(C_0(\hat{\mathfrak{a}}/W_n \times [0,1])\big) \arrow[r] \arrow[u, "\cong"] & K_*\big(C_0(\hat{\mathfrak{a}}/W_n)\big), \arrow[u, "\cong"'] \end{tikzcd}$$

and by homotopy invariance of K-theory the bottom map, also induced from evaluation at $t = 1$, is evidently an isomorphism.

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