Group C*-Algebras and K-theory

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Preface

These notes are about the formulation of the Baum-Connes conjecture in operator algebra theory and the proofs of some cases of it. They are aimed at readers who have some prior familiarity with K-theory for C^*-algebras (up to and including the Bott Periodicity theorem). I hope the notes will be suitable for a second course in operator K-theory.

The lectures begin by reviewing K-theory and the Bott periodicity theorem. Much of the Baum-Connes theory has to do with broadening the periodicity theorem in one way or another, and for this reason quite some time is spent formulating and proving the theorem in a way which is suited to later extensions. Following that, the lectures turn to the machinery of bivariant K-theory and the formulation of the Baum-Connes conjecture. The main objective of the notes is reached in Lecture 4, where the conjecture is proved for groups which act properly and isometrically on affine Euclidean spaces. The remaining lectures deal with partial results which are important in applications and with counterxamples to various overly optimistic strengthenings of the conjecture.

Despite their length the notes are not complete in every detail, and the reader will have to turn to the references, or his own inner resources, to fill some gaps. In addition the lectures contain no discussion of applications or connections to geometry, topology and harmonic analysis, nor do they cover the remarkable work of Vincent Lafforgue. For the former see [7]; for the latter see [62, 44].

The notes are based on joint work carried out over a period of many years now with many people: Paul Baum, Alain Connes, Erik Guentner, Gennadi Kasparov, Vincent Lafforgue, John Roe, Georges Skandalis and Jody Trout. It is a pleasure to thank them all. I am especially grateful to Erik Guentner for writing the first draft of these notes and for his valuable assistance throughout their creation. Both authors were partially supported by NSF grants during the preparation of this paper.

Nigel Higson

1 K-Theory

In the first three lectures we shall be developing machinery needed to formulate the Baum-Connes conjecture and prove some cases of it. We shall presume some prior familiarity with C^*-algebra K-theory, but we shall also develop a 'spectral' picture of K-theory from scratch. In Lecture 1 we shall prove the Bott periodicity theorem in C^*-algebra K-theory in a way which will be suited to generalization in subsequent lectures.

1.1 Review of K-theory

We begin by briefly reviewing the rudiments of C^*-algebra K-theory, up to and including the Bott periodicity theorem. As the reader knows, C^*-algebra K-theory is a development of the topological K-theory of Atiyah and Hirzebruch [4]. But the basic definition is completely algebraic in nature:

Definition 1.1. *Let A be a ring with a multiplicative unit. The group $K_0(A)$ is the abelian group generated by the set of isomorphism classes of finitely generated and projective (unital, right) A-modules, subject to the relations $[E] + [F] = [E \oplus F]$.*

Remark 1.1. Functional analysts usually prefer to formulate the basic definition in terms of equivalence classes idempotents in the matrix rings $M_n(A)$. This is because in several contexts idempotents arise more naturally than modules. We shall use both definitions below, bearing in mind that they are related by associating to an idempotent $P \in M_n(A)$ the projective module $E = PA^n$.

The group $K_0(A)$ is functorial in A since associated to a ring homomorphism $A \to B$ there is an induction operation on modules, $E \mapsto E \otimes_A B$.

Most of the elementary algebraic theory of the functor $K_0(A)$ is a consequence of a structure theorem involving pull-back diagrams like this one:

$$\begin{tikzcd} A \arrow[r, "q_1"] \arrow[d, "q_2"'] & A_1 \arrow[d, "p_1"] \\ A_2 \arrow[r, "p_2"'] & B \end{tikzcd} \qquad A = \{(a_1, a_2) \in A_1 \oplus A_2 \mid p_1(a_1) = p_2(a_2)\}.$$

Theorem 1.1. *Assume that in the above diagram at least one of the two homomorphisms into B is surjective. If E_1 and E_2 are finitely generated and projective modules over A_1 and A_2, and if $F\colon p_{1*}E_1 \to p_{2*}E_2$ is an isomorphism of B modules, then the A-module*

$$E = \{(e_1, e_2) \in E_1 \times E_2 \mid F(e_1 \otimes 1) = e_2 \otimes 1\}$$

is finitely generated and projective. Moreover, up to isomorphism, every finitely generated and projective module over A has this form. □

This is proved in the first few pages of Milnor's algebraic K-theory book [49]. The theorem describes projective modules over A in terms of projective modules over A_1, projective modules over A_2, and invertible maps between projective modules over B. It leads very naturally to the definition of a group $K_1(B)$ in terms of invertible matrices, but at this point the purely algebraic and the C^*-algebraic theories diverge, as a result of an important homotopy invariance principle.

Definition 1.2. *Let A be a C^*-algebra. Denote by $A[0,1]$ the C^*-algebra of continuous functions from the unit interval $[0,1]$ into A.*

We shall similarly denote by $A(X)$ the C^*-algebra of continuous functions from a compact space X into a C^*-algebra A.

Theorem 1.2. *Let A be a C^*-algebra with unit. If E is a finitely generated and projective module over $A[0,1]$ then the induced modules over A obtained by evaluation at $0 \in [0,1]$ and $1 \in [0,1]$ are isomorphic to one another.* □

As a result, K-theory is a *homotopy functor* in the sense of the following definition:

Definition 1.3. *A* homotopy *of $*$-homomorphisms between C^*-algebras is a family of homomorphisms $\varphi_t \colon A \to B$ ($t \in [0,1]$), for which the maps $t \mapsto \varphi_t(a)$ are continuous, for all $a \in A$. A functor F on the category of C^*-algebras is a* homotopy functor *if all the homomorphisms φ_t in any homotopy induce one and the same map $F(\varphi_t) \colon F(A) \to F(B)$.*

We shall now define the K-theory group $K_1(A)$.

Definition 1.4. *Let A be a C^*-algebra with unit. Denote by $M_n(A)$ the C^*-algebra of $n \times n$ matrices with entries in A and denote by $GL_n(A)$ the group of invertible elements in $M_n(A)$. View $GL_n(A)$ as a subgroup of each $GL_{n+k}(A)$ via the embeddings*

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

Denote by $K_1(A)$ the direct limit of the component groups $\pi_0(GL_n(A))$:

$$K_1(A) = \varinjlim \pi_0(GL_n(A)).$$

Remark 1.2. This is a group, thanks to the group structure in $GL_n(A)$, and in fact an abelian group since $\left(\begin{smallmatrix} XY & 0 \\ 0 & I \end{smallmatrix}\right)$ is homotopic to $\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$, and hence to $\left(\begin{smallmatrix} YX & 0 \\ 0 & I \end{smallmatrix}\right)$.

Returning to our pullback diagram and Theorem 1.1, it is now straightforward to derive all but the dotted part of the following six-term 'Mayer-Vietoris' exact sequence of K-theory groups:

$$\begin{tikzcd} K_0(A) \arrow[r] & K_0(A_1) \oplus K_0(A_2) \arrow[r] & K_0(B) \arrow[d, dotted] \\ K_1(B) \arrow[u] & K_1(A_1) \oplus K_1(A_2) \arrow[l] & K_1(A). \arrow[l] \end{tikzcd} \qquad (1)$$

The diagram is completed (along the dotted arrow) as follows. Consider first the pullback diagram

$$\begin{tikzcd} A(S^1) \arrow[r, "q_1"] \arrow[d, "q_2"'] & A_1(S^1) \arrow[d, "p_1"] \\ A_2(S^1) \arrow[r, "p_2"'] & B(S^1) \end{tikzcd}$$

involving algebras of functions on the circle S^1. The Mayer-Vietoris sequence associated to it,

$$\begin{tikzcd} K_0(A(S^1)) \arrow[r] & K_0(A_1(S^1)) \oplus K_0(A_2(S^1)) \arrow[r] & K_0(B(S^1)) \\ K_1(B(S^1)) \arrow[u] & K_1(A_1(S^1)) \oplus K_1(A_2(S^1)) \arrow[l] & K_1(A(S^1)), \arrow[l] \end{tikzcd} \tag{2}$$

maps to the Mayer-Vietoris sequence (1) via the operation ε of evaluation at $1 \in S^1$, and in fact this map is the projection onto a direct summand since ε has a one-sided inverse consisting of the inclusion of the constant functions into the various algebras of functions on S^1. The complementary summands are computed using the following two results:

Theorem 1.3. *Let A be a C^*-algebra. The kernel of the evaluation homomorphism*

$$\varepsilon\colon K_0(A(S^1)) \to K_0(A)$$

is naturally isomorphic to $K_1(A)$. □

This is a simple application of the partial Mayer-Vietoris sequence (think of $A(S^1)$ as assembled by a pullback operation from two copies of $A[0,1]$).

Theorem 1.4. *Let A be a C^*-algebra. The kernel of the evaluation homomorphism*

$$\varepsilon\colon K_1(A(S^1)) \to K_1(A)$$

is naturally isomorphic to $K_0(A)$.

This is much harder; it is one formulation of the Bott periodicity theorem. But granting ourselves the result for a moment, we can complete the diagram (1) by the simple device of viewing its horizontal reflection (with the K_1-groups on the top) as a direct summand of the diagram (2). The required connecting map $\partial\colon K_0(B) \to K_1(A)$ appears as a direct summand of the connecting map $\partial\colon K_1(B(S^1)) \to K_0(A(S^1))$.

The full Mayer-Vietoris sequence is a powerful computational tool, especially for commutative algebras. For example it implies that the functors $X \mapsto K_j(A(X))$ constitute a cohomology theory on compact spaces (as in algebraic topology). A simple consequence is the formula

$$K_0(A(S^2)) \cong K_0(A) \oplus K_0(A)$$

which is a perhaps more familiar formulation of Bott periodicity.

Let us conclude our review of K-theory with a quick look at the proof of Theorem 1.4. The launching point is the definition of a map

$$\beta : K_0(A) \to K_1(A(S^1))$$

by associating to the class of an idempotent $P \in M_n(A)$ the element

$$u_P(z) = zP + (1 - P) \tag{3}$$

in $GL_n(A(S^1))$. The following argument (due to Atiyah and Bott [6]) then shows that this *Bott homomorphism* is an isomorphism onto the kernel of the evaluation map $\varepsilon\colon K_1(A(S^1)) \to K_1(A)$. The key step is to show β is surjective; the proof in injectivity is a minor elaboration of the surjectivity argument[3] and we shall not comment on it further.

By an approximation argument involving trigonometric polynomials the proof of surjectivity quickly reduces to showing that a polynomial loop of invertible matrices

$$u(z) = b_0 + zb_1 + \cdots + z^m b_m, \quad b_j \in M_n(A),$$

which defines a element of the kernel of the evaluation map must lie in the image of β. By elementary row operations, the loop $u(z)$ is equivalent to the 'linear' loop

$$v(z) = \begin{pmatrix} b_0 & b_1 & \dots & b_{m-1} & b_m \\ -z & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -z & 1 \end{pmatrix} = Az + B,$$

for suitable matrices A and B. Evaluating at $z = 1$ and bearing in mind that v is in the kernel of the evaluation map we see that $A + B$ is path connected to I (in some suitable $GL_N(A)$) and so v is equivalent to

$$w(z) = (A + B)^{-1}(Az + B) = Cz + (I - C).$$

The final step of the argument is for our purposes the most interesting, since in involves in a crucial way the spectral theory of elements in C^*-algebras. Since $w(z)$ is invertible for all $z \in S^1$ the spectrum of C contains no element on the line $\mathrm{Re}(z) = \frac{1}{2}$ in $\mathbb{C}$. If P denotes the idempotent associated to the part of the spectrum of C to the right of this line (obtained from the Riesz functional calculus) then $w(z)$ is homotopic to the path

$$u_P(z) = Pz + (1 - P),$$

the K-theory class of which is of course in the image of β. This concludes the proof.

In the following sections we shall recast the definition of K-theory and the proof of Bott periodicity in a way which brings spectral theory very much to prominence. As we shall eventually see, this is an important first step toward our principal goal of computing K-theory for group C^*-algebras.

[3] As Shmuel Weinberger puts it, uniqueness is a relative form of existence.

1.2 Graded C*-Algebras

To proceed further with K-theory we shall find it convenient to work with *graded* C^*-algebras, which are defined as follows.

Definition 1.5. *Let A be a C^*-algebra. A grading on A is a $*$-automorphism α of A satisfying $\alpha^2 = 1$. Equivalently, a grading is a decomposition of A as a direct sum of two $*$-linear subspaces, $A = A_0 \oplus A_1$, with the property that $A_i A_j \subset A_{i+j}$, where $i, j \in \mathbb{Z}/2$. Elements of A_0 (for which $\alpha(a) = a$) are said to be of* even grading-degree *while elements of A_1 (for which $\alpha(a) = -a$) are of* odd grading-degree*.*

Example 1.1. The *trivial grading* on A is defined by the $*$-automorphism $\alpha = \mathrm{id}$, or equivalently by setting $A_0 = A$ and $A_1 = 0$.

In fact, we shall require only a very small collection of non-trivially graded C^*-algebras, among which the following two are the most important.

Example 1.2. Let $\mathcal{H}$ be a graded Hilbert space; that is, a Hilbert space equipped with an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. The C^*-algebras $\mathcal{K}(\mathcal{H})$ of compact operators and $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ are graded. To describe the grading, think of an operator T on $\mathcal{H}$ as a 2×2 matrix of operators. We declare the diagonal matrices to be even and the off-diagonal ones to be odd.

Example 1.3. Let $\mathcal{S} = C_0(\mathbb{R})$, the C^*-algebra of continuous, complex-valued functions on $\mathbb{R}$ which vanish at infinity, and define a grading on $\mathcal{S}$ by the decomposition

$$\mathcal{S} = C_0(\mathbb{R}) = \{ \text{ even functions } \} \oplus \{ \text{ odd functions } \}.$$

The grading operator is the automorphism $f(x) \mapsto f(-x)$.

Warning: In K-theory it is customary to introduce the C^*-algebra $C_0(\mathbb{R})$ in connection with the operation of 'suspension'. But in what follows the algebra $\mathcal{S}$ will play a quite different role.

Definition 1.6. *A graded C^*-algebra A is inner-graded if there exists a self-adjoint unitary ε in the multiplier algebra of A which implements the grading automorphism α on A:*

$$\alpha(a) = \varepsilon\, a\, \varepsilon, \quad \textit{for all } a \in A.$$

Examples 1.5 *The trivial grading on a C^*-algebra A is inner: take $\varepsilon = 1$. In addition the gradings on $\mathcal{K}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ are inner: take ε to be the operator which is $+I$ on H_0 and $-I$ on H_1. However the grading on $\mathcal{S}$ is not inner.*

All the fundamental constructions on C^*-algebras have graded counterparts, and we shall require below some familiarity with the notion of tensor product for graded C^*-algebras. As is the case with ungraded C^*-algebras, tensor products of graded C^*-algebras are defined as completions of the algebraic graded tensor product. And as is the case in the ungraded world, there is not usually a unique such completion.

Let us introduce the symbol ∂a defined by

$$\partial a = \begin{cases} 0, & \text{if } a \in A_0 \\ 1, & \text{if } a \in A_1. \end{cases}$$

An element $a \in A$ is *homogeneous* if $a \in A_0$ or $a \in A_1$. Keep in mind that ∂a is defined only when a is a homogeneous element.

Definition 1.7. *Let A and B be graded C^*-algebras. Let $A \widehat{\odot} B$ be the algebraic tensor product of the linear spaces underlying A and B. Define a multiplication, involution and grading on $A \widehat{\odot} B$ by means of the following formulas involving elementary tensors:*

$$\begin{aligned} (a_1 \widehat{\odot} b_1)(a_2 \widehat{\odot} b_2) &= (-1)^{\partial b_1 \partial a_2} a_1 a_2 \widehat{\odot} b_1 b_2, \\ (a \widehat{\odot} b)^* &= (-1)^{\partial a \partial b} a^* \widehat{\odot} b^* \\ \partial(a \widehat{\odot} b) &= \partial a + \partial b, \quad (mod\, 2), \end{aligned}$$

for all homogeneous elements $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. (The multiplication and involution are extended by linearity to all of $A \widehat{\odot} B$.)

The construction of $A \widehat{\odot} B$ satisfies the usual associativity and commutativity rules but with occasional twists. For example, an isomorphism $A \widehat{\odot} B \to B \widehat{\odot} A$ is defined by

$$a \widehat{\odot} b \longmapsto (-1)^{\partial a \partial b} b \widehat{\odot} a. \tag{4}$$

Definition 1.8. *The graded commutator of elements in a graded C^*-algebra is given by the formula*

$$[a, b] = ab - (-1)^{\partial a \partial b} ba,$$

on homogeneous elements (this is extended by linearity to all elements).

Lemma 1.1. *If C is a graded C^*-algebra and if $\varphi\colon A \to C$ and $\psi\colon B \to C$ are graded $*$-homomorphisms*[4] *whose images graded-commute (meaning that all graded commutators $[\varphi(a), \psi(b)]$ are zero) then there is a unique graded $*$-homomorphism from $A \widehat{\odot} B$ into C which maps $a \widehat{\odot} b$ to $\varphi(a)\psi(b)$.* □

Example 1.4. Let $\mathcal{H}$ be a graded Hilbert space and denote by $\mathcal{H} \widehat{\otimes} \mathcal{H}$ the ordinary Hilbert space tensor product, but considered as a graded Hilbert space. The construction of the lemma produces a graded $*$-homomorphism from the tensor product algebra $\mathcal{B}(\mathcal{H}) \widehat{\odot} \mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ which takes the homogeneous elementary tensor $S \widehat{\odot} T$ to the operator

$$v \otimes w \mapsto Sv \otimes (-1)^{\partial v \partial T} Tw.$$

[4] A $*$-homomorphism is *graded*, or *grading-preserving*, if it maps homogeneous elements to homogeneous elements of the same grading-degree.

Definition 1.9. *Let A and B be graded C^*-algebras and let $A\widehat{\odot}B$ be their algebraic tensor product. The maximal graded tensor product, which we will denote by $A\widehat{\otimes}B$, or occasionally by $A\widehat{\otimes}_{max}B$, is the completion of $A\widehat{\odot}B$ in the norm*

$$\|\sum a_i\widehat{\odot}b_i\| = \sup \|\sum_i \varphi(a_1)\psi(b_i)\|,$$

where the supremum is taken over graded-commuting pairs of graded $*$-homomorphisms, mapping A and B into a common third graded C^*-algebra C.

Warning: Our use of the undecorated symbol $\widehat{\otimes}$ to denote the *maximal* tensor product (as opposed to the *minimal* one, which we shall define in a moment) runs counter to ordinary C^*-algebra usage. In situations where the choice of tensor product really is crucial we shall try to write $\widehat{\otimes}_{max}$.

Remark 1.3. It is clear from the definition that the tensor product $\widehat{\otimes}$ is functorial: if $\varphi : A \to C$ and $\psi : B \to D$ are graded $*$-homomorphisms then there is a unique graded $*$-homomorphism $\varphi\widehat{\otimes}\psi : A\widehat{\otimes}B \to C\widehat{\otimes}D$ mapping $a\widehat{\otimes}b$ to $\varphi(a)\widehat{\otimes}\psi(b)$, for all $a \in A$ and $b \in B$.

Example 1.5. If one of A or B is inner-graded then the ungraded C^*-algebra underlying the graded tensor product $A\widehat{\otimes}B$ is isomorphic to the usual tensor product of the ungraded C^*-algebras underlying A and B. If say A is inner-graded then the isomorphism $A\widehat{\otimes}B \to A \otimes B$ is defined by

$$a\widehat{\otimes}b \longmapsto a\varepsilon^{\partial b} \otimes b.$$

We also note that the graded tensor product of two inner-graded C^*-algebras is itself inner-graded. Indeed

$$\varepsilon_A\widehat{\otimes}\varepsilon_B \in \mathcal{M}(A\widehat{\otimes}B) = \mathcal{M}(A \otimes B).$$

For the most part we shall use the maximal tensor product of graded C^*-algebras, but occasionally we shall work with the following 'minimal' product:

Definition 1.10. *Let A and B be graded C^*-algebras and let $A\widehat{\odot}B$ be their algebraic tensor product. The minimal graded tensor product of A and B is the completion of $A\widehat{\odot}B$ in the representation obtained by first faithfully representing A and B as graded subalgebras of $\mathcal{B}(\mathcal{H})$, and then mapping $B(\mathcal{H})\widehat{\odot}\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H}\widehat{\otimes}\mathcal{H})$ as above.*

The minimal tensor product is also functorial, but from our point of view it has some serious shortcomings. These will be explained in the next lecture.

Exercise 1.6 *Show that the minimal and maximal completions of $A\widehat{\odot}\mathcal{K}(\mathcal{H})$ and $\mathcal{S}\widehat{\odot}A$ are the same.*

Exercise 1.1. Describe the tensor product C^*-algebra $\mathcal{S}\widehat{\otimes}\mathcal{S}$ (note that although $\mathcal{S}$ itself is a commutative C^*-algebra, the tensor product $\mathcal{S}\widehat{\otimes}\mathcal{S}$ is not).

Exercise 1.7 *Show that $\mathcal{K}(\mathcal{H})\widehat{\otimes}\mathcal{K}(\mathcal{H}') \cong \mathcal{K}(\mathcal{H}\widehat{\otimes}\mathcal{H}')$.*

1.3 Amplification

The graded C^*-algebra $\mathcal{S} = C_0(\mathbb{R})$ will play a special role for us. Using it we shall enrich, or 'amplify', the category of graded C^*-algebras and $*$-homomorphisms.

To do so we introduce two $*$-homomorphisms, as follows:

$$\eta\colon \mathcal{S} \to \mathbb{C} \quad \text{and} \quad \Delta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}.$$

The first is defined by $\eta(f) = f(0)$. In the world of ungraded C^*-algebras and K-theory η is not so interesting since it is homotopic to the zero $*$-homomorphism. But as $*$-homomorphism of graded C^*-algebras η is definitely non-trivial, even at the level of K-theory (which we will come to in the next section). The defining formula for Δ,

$$\Delta\colon f(X) \mapsto f(X\widehat{\otimes}1 + 1\widehat{\otimes}X),$$

is explained as follows. Denote by $\mathcal{S}_R$ the quotient of $\mathcal{S}$ consisting of functions on the interval $[-R, R]$ (the quotient map is the operation of restriction of functions) and denote by $X_R \in \mathcal{S}_R$ the function $x \mapsto x$. If $f \in \mathcal{S}$ then we can apply the functional calculus to the self-adjoint element $X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R \in \mathcal{S}_R\widehat{\otimes}\mathcal{S}_R$ to obtain an element $f(X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R) \in \mathcal{S}_R\widehat{\otimes}\mathcal{S}_R$.

Lemma 1.2. *There is a unique graded $*$-homomorphism $\Delta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}$ whose composition with the quotient map $\mathcal{S}\widehat{\otimes}\mathcal{S} \to \mathcal{S}_R\widehat{\otimes}\mathcal{S}_R$ is the $*$-homomorphism*

$$\Delta\colon f \mapsto f(X_R\widehat{\otimes}1 + 1\widehat{\otimes}X_R),$$

for every $R > 0$. $\square$

Exercise 1.8 *Show that the intersection of the kernels of the maps $\mathcal{S}\widehat{\otimes}\mathcal{S} \to \mathcal{S}_R\widehat{\otimes}\mathcal{S}_R$ is zero. This proves the uniqueness part of the Lemma.*

Remark 1.4. If the self-adjoint homogeneous elements u and v in $\mathcal{S}$ are defined by

$$u(x) = e^{-x^2}, \quad \text{and} \quad v(x) = xe^{-x^2}.$$

then

$$\Delta(u) = u\widehat{\otimes}u \quad \text{and} \quad \Delta(v) = u\widehat{\otimes}v + v\widehat{\otimes}u.$$

Since u and v generate the C^*-algebra $\mathcal{S}$, formulas involving Δ and η can often be verified by checking them on u and v.

Remark 1.5. Another approach to the definition of Δ is to use the theory of unbounded multipliers. See the short appendix to this lecture.

The $*$-homomorphisms η and Δ provide $\mathcal{S}$ with a sort of coalgebra structure: the diagrams

$$
\begin{tikzcd}
\mathcal{S} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{S}\widehat{\otimes}\mathcal{S} \arrow[d, "1\widehat{\otimes}\Delta"] \\
\mathcal{S}\widehat{\otimes}\mathcal{S} \arrow[r, "\Delta\widehat{\otimes}1"'] & \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}\mathcal{S}
\end{tikzcd},
\qquad
\begin{tikzcd}
 & \mathcal{S} \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\Delta"] & \\
\mathcal{S} & & \mathcal{S} \\
 & \mathcal{S}\widehat{\otimes}\mathcal{S} \arrow[ul, "\eta\widehat{\otimes}1"] \arrow[ur, "1\widehat{\otimes}\eta"'] &
\end{tikzcd}
\tag{5}
$$

commute, as is easily verified by considering the elements u and $v \in \mathcal{S}$.

Definition 1.11. *Let A be a graded C^*-algebra. The amplification of A is the graded tensor product $\mathcal{S}A = \mathcal{S}\widehat{\otimes}A$.*

Definition 1.12. *The* amplified category of graded C^*-algebras *is the category whose objects are the graded C^*-algebras and for which the morphisms from A to B are the graded $*$-homomorphisms from $\mathcal{S}A$ to B. Composition of morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$ in the amplified category is given by the following composition of $*$-homomorphisms:*

$$\mathcal{S}A \xrightarrow{\Delta\widehat{\otimes}1} \mathcal{S}^2 A \xrightarrow{\mathcal{S}(\varphi)} \mathcal{S}B \xrightarrow{\psi} C.$$

Exercise 1.2. Using (5) verify that the composition law is associative and that the $*$-homomorphisms $\mathcal{S}A \to A$ obtained by taking the tensor product of the augmentation $\eta\colon \mathcal{S} \to \mathbb{C}$ with the identity map on A serve as identity morphisms for this composition law.

Remark 1.6. Most features of the category of graded C^*-algebras pass to the amplified category. One example is the tensor product operation: given amplified morphisms from $\varphi_1\colon A_1 \to B_1$ and $\varphi_2\colon A_2 \to B_2$ there is a tensor product morphism from $A_1\widehat{\otimes}A_2$ to $B_1\widehat{\otimes}B_2$ (in other words a $*$-homomorphism from $\mathcal{S}(A_1\widehat{\otimes}A_2)$ into $B_1\widehat{\otimes}B_2$) defined by the composition of $*$-homomorphisms

$$\mathcal{S}(A_1\widehat{\otimes}A_2) = \mathcal{S}\widehat{\otimes}A_1\widehat{\otimes}A_2 \xrightarrow{\Delta\widehat{\otimes}1\widehat{\otimes}1} \mathcal{S}^2\widehat{\otimes}A_1\widehat{\otimes}A_2 \cong \mathcal{S}A_1\widehat{\otimes}\mathcal{S}A_2 \xrightarrow{\varphi_1\widehat{\otimes}\varphi_2} B_1\widehat{\otimes}B_2$$

(the formula incorporates the transposition isomorphism (4)).

Exercise 1.3. Show that the tensor product is functorial (compatible with composition) and associative.

1.4 Stabilization

A second means of enriching the notion of $*$-homomorphism is the process of *stabilization*. This is of course very familiar in K-theory: stabilization means replacing a C^*-algebra A with $A\widehat{\otimes}\mathcal{K}(\mathcal{H})$, its tensor product with the C^*-algebra of compact operators.

If A is a trivially graded C^*-algebra with unit then each projection p in $A \otimes \mathcal{K}(\mathcal{H})$ determines a projective module over A (namely $p(A \otimes \mathcal{K}(\mathcal{H}))$ with the obvious right action of A) and in fact the set of isomorphism classes of finitely generated A-modules is identified in this way with the set of homotopy classes of projections in $A \otimes \mathcal{K}(\mathcal{H})$. For this reason stabilization is a central idea in K-theory.

Let us now return to the graded situation. There are $*$-homomorphisms

$$\mathbb{C} \to \mathcal{K}(\mathcal{H}) \quad \text{and} \quad \mathcal{K}(\mathcal{H}) \widehat{\otimes} \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$$

defined by mapping $\lambda \in \mathbb{C}$ to λe, where e is the projection onto a one-dimensional, grading-degree zero subspace of $\mathcal{H}$, and by identifying $\mathcal{H} \widehat{\otimes} \mathcal{H}$ with $\mathcal{H}$ by a grading-degree zero unitary isomorphism. These play a role similar to the maps η and Δ introduced in the previous section. There is no canonical choice of the projection e or the isomorphism $\mathcal{H} \widehat{\otimes} \mathcal{H} \cong \mathcal{H}$, and for this reason we cannot 'stabilize' the category of C^*-algebras in quite the way we amplified it in the previous section. But at the level of homotopy the situation is better:

Lemma 1.3. *Let $\mathcal{H}$ and $\mathcal{H}'$ be graded Hilbert spaces. Any two grading-preserving isometries from $\mathcal{H}$ into $\mathcal{H}'$ induce graded $*$-homomorphisms from $\mathcal{K}(\mathcal{H})$ to $\mathcal{K}(\mathcal{H}')$ which are homotopic through graded $*$-homomorphisms.* $\square$

As as result there are canonical, up to homotopy, maps $\mathbb{C} \to \mathcal{K}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H}) \widehat{\otimes} \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$. We could therefore create a stabilized homotopy category, in which the morphisms from A to B are the homotopy classes of graded $*$-homomorphisms from A to $B \widehat{\otimes} \mathcal{K}(\mathcal{H})$. We could even stabilized and amplify simultaneously, and create the category in which the morphisms between C^*-algebras A and B are the homotopy classes of graded $*$-homomorphisms from $\mathcal{S}A$ to $B \widehat{\otimes} \mathcal{K}(\mathcal{H})$. We won't exactly do this, but the reader will notice echoes of this construction in the following sections.

1.5 A Spectral Picture of K-Theory

We are going provide a 'spectral' description of K-theory which is well adapted to Fredholm index theory and to an eventual bivariant generalization. Actually our definition is a back formation from the bivariant theory described in [13, 14, 27] (it is also closely related to various other approaches to K-theory).

For the rest of this section we shall fix a graded Hilbert space $\mathcal{H}$ whose even and odd grading-degree parts are both countably infinite-dimensional. Unless explicitly noted otherwise we shall be working with graded C^*-algebras and grading-preserving $*$-homomorphisms between them.

Definition 1.13. *We shall denote by $[A, B]$ the set of homotopy classes of grading-preserving $*$-homomorphisms between the graded C^*-algebras A and B.*

With this notation in hand, our description of K-theory is quite simple:

Definition 1.14. *If A is a graded C^*-algebra then we define*

$$K(A) = [\mathcal{S}, A \widehat{\otimes} \mathcal{K}(\mathcal{H})].$$

For the moment $K(A)$ is just a set, although we will soon give it the structure of an abelian group. But first let us give two examples of classes in $K(A)$ to help justify the definition.

Example 1.6. Take $A = \mathbb{C}$. Let D be an unbounded self-adjoint operator on the graded Hilbert space $\mathcal{H}$ of the form

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

(in other words D is a grading-degree one operator) and assume that D has compact resolvent. (For example, D might be a Dirac-type operator on a compact manifold.) The functional calculus

$$\psi_D \colon f \mapsto f(D)$$

defines a graded $*$-homomorphism $\psi_D \colon \mathcal{S} \to \mathcal{K}(\mathcal{H})$ and hence a class in $K(\mathbb{C})$.

Example 1.7. Suppose that A is unital and trivially graded, so that the K-theory group $K_0(A)$ of Section 1.1 can be described in terms of equivalence classes of projections in $A \widehat{\otimes} \mathcal{K}(\mathcal{H})$. If p_0, p_1 are two such projections, acting on the even and odd parts of the graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, then the formula

$$\psi_p \colon f \longmapsto \begin{pmatrix} f(0)p_0 & 0 \\ 0 & f(0)p_1 \end{pmatrix}$$

defines a grading preserving $*$-homomorphism from $\mathcal{S}$ to $A \widehat{\otimes} \mathcal{K}(\mathcal{H})$.

The second example is related to the first as follows: if D is a self-adjoint, grading-degree one, compact resolvent operator on $\mathcal{H}$ then the family

$$\psi_s \colon f \mapsto f(s^{-1}D), \qquad s \in [0,1]$$

is a homotopy from the $*$-homomorphism ψ_D at $s = 1$ to the $*$-homomorphism ψ_p at $s = 0$, where $p = p_0 \oplus p_1$ is the projection onto the kernel of D.

Before reading any further the reader may enjoy solving the following problem.

Exercise 1.9 *Prove that $K(\mathbb{C}) \cong \mathbb{Z}$ in such a way that to the class of the $*$-homomorphism ψ_D of Example 1.6 is associated the Fredholm index of D_+.*

Let us turn now to the operation of addition on $K(A)$. This is given by the direct sum operation which associates to a pair of $*$-homomorphisms ψ_1 and ψ_2 the $*$-homomorphism

$$\psi_1 \oplus \psi_2 \colon \mathcal{S} \to A \widehat{\otimes} \mathcal{K}(\mathcal{H} \oplus \mathcal{H}).$$

(One identifies $\mathcal{H} \oplus \mathcal{H}$ with $\mathcal{H}$ by some degree zero unitary isomorphism to complete the definition; at the level of homotopy any two such identifications are equivalent.)

The zero element is the class of the zero homomorphism. To prove the existence of additive inverses it is convenient to make the following preliminary observation which will be important for other purposes as well. The proof is a simple exercise with the functional calculus.

Lemma 1.4. *Let D be any graded C^*-algebra and let $\psi\colon \mathcal{S} \to D$ be a grading-preserving $*$-homomorphism. Adjoin units to $\mathcal{S}$ and D, extend ψ, and form the unitary element*

$$U_\psi = \psi\left(\frac{x-i}{x+i}\right)$$

in the unitalization of D. The correspondence $\varphi \leftrightarrow U_\psi$ is a bijection between the set of $*$-homomorphisms $\psi\colon \mathcal{S} \to D$ and the set of unitary elements U in the unitalization of D which are equal to 1 modulo D and which are mapped to their adjoints by the grading automorphism: $\alpha(U) = U^*$. $\square$

Definition 1.15. *If D is a graded C^*-algebra then by a* Cayley transform *for D we shall mean a unitary in the unitalization of D which is equal to the identity, modulo D, and which is switched to its adjoint by the grading automorphism.*

Returning to the question of additive inverses in $K(A)$, if U is the Cayley transform of ψ then it is tempting to say that the additive inverse to ψ should be represented by the Cayley transform U^*. But this is not quite right; we must also view U^* as a Cayley transform for $A\widehat{\otimes}\mathcal{K}(\mathcal{H}^{\text{opp}})$, where $\mathcal{H}^{\text{opp}}$ is the Hilbert space $\mathcal{H}$ but with the grading reversed. The rotation homotopy

$$\begin{pmatrix} \cos(t)U & \sin(t)I \\ -\sin(t)I & \cos(t)U^* \end{pmatrix}$$

is then a path of Cayley transforms for $A\widehat{\otimes}\mathcal{K}(\mathcal{H} \oplus \mathcal{H}^{\text{opp}})$ connecting $\left(\begin{smallmatrix} U & 0 \\ 0 & U^* \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, which is in turn connected to the identity.

Remark 1.7. In terms of $*$-homomorphisms rather than Cayley transforms, the additive inverse of ψ is represented by the $*$-homomorphism

$$\psi^{\text{opp}} = \psi \circ \alpha\colon \mathcal{S} \to A\widehat{\otimes}\mathcal{K}(\mathcal{H}^{\text{opp}})$$

obtained by composing ψ with the grading automorphism on $\mathcal{S}$ and also reversing the grading on the Hilbert space $\mathcal{H}$.

Remark 1.8. In the next lecture we shall give an account of additive inverses using the comultiplication map Δ we introduced in the previous section.

Proposition 1.10 *On the category of trivially graded and unital C^*-algebras the functor $K(A)$ defined in this section is naturally isomorphic to the K-theory functor $K_0(A)$ introduced at the beginning of this lecture.*

Proof. We have already seen that $K(A)$ is the group of path components of the space of Cayley transforms for $A \otimes \mathcal{K}(\mathcal{H})$ (we can dispense with the graded tensor product here since A is trivially graded). If $\varepsilon = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$ is the grading operator and if U is a Cayley transform then εU is a self-adjoint unitary whose $+1$ spectral projection,

$$P = \tfrac{1}{2}(\varepsilon U + 1),$$

is equal to the $+1$ spectral projection $P_\varepsilon = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ of ε, modulo $A \otimes \mathcal{K}(\mathcal{H})$. Conversely if P is a projection which is equal to P_ε modulo $A \otimes \mathcal{K}(\mathcal{H})$ then the formula

$$U = \varepsilon(2P - I)$$

defines a Cayley transform for $A \otimes \mathcal{K}(\mathcal{H})$. We therefore have a new description of the new $K(A)$, as the group of path components of the projections which are equal to P_ε, modulo $A \otimes \mathcal{K}(\mathcal{H})$. We leave it to the reader to determine that the formula $[P] \mapsto [P] - [P_\varepsilon]$ is an isomorphism from this new component space to the usual $K_0(A)$ (the argument involves the familiar stability property of K-theory).

Exercise 1.11 *Denote by $\mathcal{C}_1$ the C^*-algebra $\mathbb{C} \oplus \mathbb{C}$ with grading operator $\lambda_1 \oplus \lambda_2 \mapsto \lambda_2 \oplus \lambda_1$ (this is an example of a* Clifford algebra *— see Section 1.11). Show that if A is trivially graded and unital then $K(A \widehat{\otimes} \mathcal{C}_1) \cong K_1(A)$.*

Exercise 1.4. Show that if a graded C^*-algebra B is the closure of the union of a direct system of graded C^*-subalgebras B_α then the natural map

$$\varinjlim K(B_\alpha) \longrightarrow K(B)$$

is an isomorphism. (*Hint:* Show that every Cayley transform for $B \widehat{\otimes} \mathcal{K}(\mathcal{H})$ is a limit of Cayley transforms for the subalgebras $B_\alpha \widehat{\otimes} \mathcal{K}(\mathcal{H})$.)

1.6 Long Exact Sequences

Although it is not absolutely necessary we shall invoke some ideas of elementary homotopy theory to construct the K-theory long exact sequences. For this purpose let us introduce the following space:

Definition 1.16. *Let A be a graded C^*-algebra. Denote by $\mathbb{K}(A)$ the space of all graded $*$-homomorphisms from $\mathcal{S}$ into $A \widehat{\otimes} \mathcal{K}(\mathcal{H})$, equipped with the topology of pointwise convergence (so that $\psi_\alpha \to \psi$ iff $\psi_\alpha(f) \to \psi(f)$ in the norm topology, for every $f \in \mathcal{S}$). Thus:*

$$\mathbb{K}(A) = \mathrm{Map}(\mathcal{S}, A \widehat{\otimes} \mathcal{K}(\mathcal{H})).$$

Remark 1.9. As it happens, the space $\mathbb{K}(A)$ is a spectrum in the sense of homotopy theory—see for example [1]—but we shall not need the homotopy-theoretic notion of spectrum in these lectures.

The space $\mathbb{K}(A)$ has a natural base-point, namely the zero homomorphism from $\mathcal{S}$ into $A\widehat{\otimes}\mathcal{K}(\mathcal{H})$. It also has a more or less natural 'direct sum' operation

$$\mathbb{K}(A) \times \mathbb{K}(A) \to \mathbb{K}(A)$$

which associates to a pair of $*$-homomorphisms ψ_1 and ψ_2 the $*$-homomorphism $\varphi_1 \oplus \varphi_2$ into $A\widehat{\otimes}\mathcal{K}(\mathcal{H}\oplus\mathcal{H})$. (One identifies $\mathcal{H}\oplus\mathcal{H}$ with $\mathcal{H}$ by some degree zero unitary isomorphism to complete the definition; at the level of homotopy any two such identifications are equivalent.) It is of course this operation which gives the addition operation on the groups $K(A) = \pi_0(\mathbb{K}(A))$. By a general principle in homotopy theory the direct sum operation agrees with the group operations on the higher homotopy groups $\pi_n(\mathbb{K}(A))$, for $n \geq 1$.

As for the higher groups $\pi_n(\mathbb{K}(A))$, they may be identified as follows. There is an obvious homeomorphism of spaces

$$\mathbb{K}(C_0(\mathbb{R}^n) \otimes A) \cong \Omega^n \mathbb{K}(A).$$

Indeed by evaluation at points of $\mathbb{R}^n$ we obtain from an element of $\mathbb{K}(C_0(\mathbb{R}^n) \otimes A)$ a map from $\mathbb{R}^n$ to $\mathbb{K}(A)$ which converges to the zero homomorphism at infinity, or in other words a pointed map from the one-point compactification S^n of $\mathbb{R}^n$ into $\mathbb{K}(A)$, which is to say an element of $\Omega^n\mathbb{K}(A)$. It follows that

$$\pi_n(\mathbb{K}(A)) = \pi_0(\Omega^n\mathbb{K}(A)) \cong K(C_0(\mathbb{R}^n) \otimes A).$$

Definition 1.17. *Let A be a graded C^*-algebra. The higher K-theory groups of A are the homotopy groups of the space $\mathbb{K}(A)$:*

$$K_n(A) = \pi_n(\mathbb{K}(A)), \qquad n \geq 0.$$

The space $\mathbb{K}(A)$, and therefore also the groups $K_n(A)$, are clearly functorial in A. They are well adapted to the construction of long exact sequences, as the following computation shows:

Lemma 1.5. *If $A \to B$ is a surjective homomorphism of graded C^*-algebras then the induced map from $\mathbb{K}(A)$ to $\mathbb{K}(B)$ is a fibration.*

Recall that a map $X \to Y$ is a (Serre) fibration if for every map from a cube (of any finite dimension) into Y, and for every lifting to X of the restriction of f to a face of the cube, there is an extension to a lifting defined on the whole cube.

Proof. Think of $\mathbb{K}(A)$ as the space of Cayley transforms for $A\widehat{\otimes}\mathcal{K}(\mathcal{H})$, and thus as a space of unitary elements. The proof that the map $\mathbb{K}(A) \to \mathbb{K}(B)$ is a fibration is then only a small modification of the usual proof that the map of unitary groups corresponding to a surjection of C^*-algebras is a fibration.

The fiber of the map $\mathbb{K}(A) \to \mathbb{K}(B)$ (meaning the inverse image of the base-point) is of course $\mathbb{K}(J)$ where the ideal J is the kernel of the surjection. So elementary homotopy theory now provides us with long exact sequences

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow K_{n+1}(B) \longrightarrow K_n(J) \longrightarrow K_n(B) \longrightarrow \cdots$$

(ending at $K(B)$) as well as Mayer-Vietoris sequences

$$\ldots \longrightarrow K_{n+1}(B) \longrightarrow K_n(A) \longrightarrow K_n(A_1) \oplus K_n(A_2) \longrightarrow K_n(B) \longrightarrow \ldots$$

associated to pullback squares of the sort we considered in the first part of this lecture.

1.7 Products

A key feature of our spectral picture of K-theory is that it is very well adapted to *products*. Recall that in the realm of ungraded C^*-algebras there is a product operation

$$K_0(A) \otimes K_0(B) \to K_0(A \otimes B)$$

defined for unital C^*-algebras by the prescription $[p] \otimes [q] = [p \otimes q]$. This is the first in a sequence of more and more complicated, and more and more powerful, product operations, which culminates with the famous Kasparov product in bivariant K-theory.

In our spectral picture the product is defined using the 'comultiplication' map Δ that we introduced during our discussion of graded C^*-algebras. Using Δ we obtain a map of spaces

$$\mathbb{K}(A) \times \mathbb{K}(B) \to \mathbb{K}(A \widehat{\otimes} B)$$

by associating to a pair (ψ_A, ψ_B) the composition

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S} \widehat{\otimes} \mathcal{S} \xrightarrow{\psi_A \widehat{\otimes} \psi_B} \big(A \widehat{\otimes} \mathcal{K}(\mathcal{H})\big) \widehat{\otimes} \big(B \widehat{\otimes} \mathcal{K}(\mathcal{H})\big) \cong A \widehat{\otimes} B \widehat{\otimes} \mathcal{K}(\mathcal{H})$$

(in the last step we employ a transposition isomorphism and we also pick an isomorphism $\mathcal{H} \widehat{\otimes} \mathcal{H} \cong \mathcal{H}$). Taking homotopy groups we obtain pairings

$$K_i(A) \otimes K_j(B) \to K_{i+j}(A \widehat{\otimes} B),$$

as required.

Example 1.8. Suppose that $A = B = \mathbb{C}$ and that ψ_1 and ψ_2 are the functional calculus homomorphisms associated to self-adjoint operators D_1 and D_2, as in Example 1.6. Then the product of ψ_1 and ψ_2 is the functional calculus homomorphism for the self-adjoint operator[5] $D_1 \widehat{\otimes} I + I \widehat{\otimes} D_2$. This type of formula is familiar from index theory; in fact it is the standard construction of an operator whose Fredholm index is the *product* of the indices of D_1 and D_2. It is this example which dictates our use of the comultiplication Δ.

The various features of the product are summarized in the following two results.

[5] To be accurate, the formula defines an essentially self-adjoint operator defined on the algebraic tensor product of the domains of D_1 and D_2.

Proposition 1.1. *The K-theory product has the following properties:*

(a) *It is associative.*
(b) *It is commutative, in the sense that if $x \in K(A)$ and $y \in K(B)$, and if $\tau\colon A\widehat{\otimes}B \to B\widehat{\otimes}A$ is the transposition isomorphism, then $\tau_*(x \times y) = y \times x$.*
(c) *It is functorial, in the sense that if $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ are graded $*$-homomorphisms then $(\varphi\widehat{\otimes}\psi)_*(x \times y) = \varphi_*(x) \times \psi_*(y)$.* $\square$

Remark 1.10. In item (b), if we take $x \in K_i(A)$ and $y \in K_j(B)$ then the appropriate formula is $\tau_*(x \times y) = (-1)^{ij} y \times x$.

Proposition 1.2. *Denote by $1 \in K(\mathbb{C})$ the class of the homomorphism which maps the element $f \in \mathcal{S}$ to the element $f(0)P \in \mathcal{K}(\mathcal{H})$, where P is the orthogonal projection onto a one-dimensional, grading-degree zero subspace of $\mathcal{H}$. If A is any graded C^*-algebra and if $x \in K(B)$ then under the isomorphism $\mathbb{C}\widehat{\otimes} B \cong B$ the class $1 \times x$ corresponds to x.* $\square$

1.8 Asymptotic Morphisms

We are now going to introduce a concept which can be used as a tool to compute K-theory for C^*-algebras. Other tools are available (for example Kasparov's theory or the theory of C^*-algebra extensions) but we shall work almost exclusively with asymptotic morphisms in these lectures.

Definition 1.18. *Let A and B be graded C^*-algebras. An asymptotic morphism from A to B is a family of functions $\varphi_t : A \to B$, $t \in [1,\infty)$ satisfying the continuity condition that for all $a \in A$*

$$t \longmapsto \varphi_t(a) : [1,\infty) \to B \text{ is bounded and continuous}$$

and the asymptotic conditions that for all $a, a_1, a_2 \in A$ and $\lambda \in \mathbb{C}$

$$\left.\begin{array}{l} \varphi_t(a_1a_2) - \varphi_t(a_1)\varphi_t(a_2) \\ \varphi_t(a_1 + a_2) - \varphi_t(a_1) - \varphi_t(a_2) \\ \varphi_t(\lambda a) - \lambda\varphi_t(a) \\ \varphi_t(a^*) - \varphi_t(a)^* \end{array}\right\} \to 0, \quad \text{as } t \to \infty.$$

If A and B are graded we shall require that in addition

$$\alpha(\varphi_t(a)) - \varphi_t(\alpha(a)) \to 0 \quad \text{as } t \to \infty,$$

where α denotes the grading automorphism. We shall denote an asymptotic morphism with a dashed arrow, thus: $\varphi : A \dashrightarrow B$.

In short, an asymptotic morphism is a one-parameter family of maps from A to B which are asymptotically $*$-homomorphisms.

We shall postpone for a little while the presentation of nontrivial examples of asymptotic morphisms (the main ones are given in Sections 1.12 and 2.6). As for trivial examples, observe that each $*$-homomorphism from A to B can be viewed as a (constant) asymptotic morphism from A to B.

It is usually convenient to work with equivalence classes of asymptotic morphisms, as follows:

Definition 1.19. *Two asymptotic morphisms φ^1, $\varphi^2 : A \dashrightarrow B$ are (asymptotically) equivalent if for all $a \in A$*

$$\lim_{t\to\infty} \left\|\varphi^1_t(a) - \varphi^2_t(a)\right\| = 0.$$

Up to equivalence, an asymptotic morphism $\varphi\colon A \dashrightarrow B$ is exactly the same thing as a $*$-homomorphism from A into the following *asymptotic algebra* associated to B.

Definition 1.20. *Let B be a graded C^*-algebra. Denote by $\mathrm{i}(B)$ the C^*-algebra of bounded, continuous functions from $[1,\infty)$ into B, and denote by $\mathrm{i}_0(B)$ the ideal comprised of functions which vanish at infinity. The asymptotic C^*-algebra of B is the quotient C^*-algebra*

$$\mathfrak{A}(B) = \mathrm{i}(B)/\mathrm{i}_0(B).$$

If $\varphi\colon A \to \mathfrak{A}(B)$ is a $*$-homomorphism then by composing φ with a set-theoretic section of the quotient mapping from $\mathrm{i}(B)$ to $\mathfrak{A}(B)$ we obtain an asymptotic morphism from A to B; its equivalence class is independent of the choice of section. Conversely an asymptotic morphism can be viewed as a function from A into $\mathrm{i}(B)$, and by composing with the quotient map into $\mathfrak{A}(B)$ we obtain a $*$-homomorphism from A to $\mathfrak{A}(B)$ which depends only on the asymptotic equivalence class of the asymptotic morphism.

Suppose now that we are given an asymptotic morphism

$$\varphi\colon A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \dashrightarrow B\widehat{\otimes}\mathcal{K}(\mathcal{H}).$$

If $\psi\colon \mathcal{S} \to A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ is a graded $*$-homomorphism then the composition

$$\mathcal{S} \xrightarrow{\psi} A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \overset{\varphi}{\dashrightarrow} B\widehat{\otimes}\mathcal{K}(\mathcal{H}) \tag{6}$$

is an asymptotic morphism from $\mathcal{S}$ into $B\widehat{\otimes}\mathcal{K}(\mathcal{H})$.

Lemma 1.6. *Every asymptotic morphism from $\mathcal{S}$ into a graded C^*-algebra D is asymptotic to a family of graded $*$-homomorphisms from $\mathcal{S}$ to D.*

Proof. We saw previously that a $*$-homomorphism from $\mathcal{S}$ to D is the same thing as Cayley transform for D — a unitary in the unitalization of D (equal to 1 modulo D) which is switched to its adjoint by the grading automorphism. In the same way, by making use of the asymptotic algbra $\mathfrak{A}(D)$ we see that an asymptotic morphism from $\mathcal{S}$ to D is the same thing, up to equivalence, as a norm continuous family of elements X_t in the unitalization, equal to 1 modulo D, which are asymptotically unitary and asymptotically switched to their adjoints by the grading automorphism. But such an 'asymptotic Cayley transform' family can, for large t, be altered to produce a family of *actual* Cayley transforms: first replace X_t by

$$Y_t = \frac{1}{2}(X_t + \alpha(X_t^*))$$

(this ensures that the grading automorphism switches the element and its adjoint) and then unitarize by forming

$$U_t = Y_t(Y_t^* Y_t)^{-\frac{1}{2}}$$

(note that Y_t is invertible for large t). Since X_t and U_t are asymptotic we have shown that every asymptotic morphism from $\mathcal{S}$ into a C^*-algebra is asymptotic to a family of $*$-homomorphisms (corresponding to U_t), as required.

Definition 1.21. *Two asymptotic morphisms φ^0 and φ^1 from A to B are* homotopic *if there is an asymptotic morphism φ from A to $B[0,1]$ from which φ^0 and φ^1 can be recovered by evaluation at $0, 1 \in [0,1]$. Homotopy is an equivalence relation and we shall use the notation*

$$[\![A, B]\!] = \{ \text{ homotopy classes of asymptotic morphisms from } A \text{ to } B \ \}.$$

There is a natural map from $[A, B]$ into $[\![A, B]\!]$ since each $*$-homomorphism can be regarded as a constant asymptotic morphism. It follows easily from the previous lemma that:

Proposition 1.3. *If D is any graded C^*-algebra then the natural map*

$$[\mathcal{S}, D] \longrightarrow [\![\mathcal{S}, D]\!]$$

is an isomorphism. $\square$

Returning to the composition (6), it gives rise to the following diagram:

$$\begin{tikzcd}
{[\mathcal{S}, A\widehat{\otimes}\mathcal{K}(\mathcal{H})]} \arrow[r, "\text{composition with } \varphi"] & {[\![\mathcal{S}, B\widehat{\otimes}\mathcal{K}(\mathcal{H})]\!]} \\
& {[\mathcal{S}, B\widehat{\otimes}\mathcal{K}(\mathcal{H})]} \arrow[u, "\cong"']
\end{tikzcd}$$

We arrive at the following conclusion: composition with $\varphi\colon A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \dashrightarrow B\widehat{\otimes}\mathcal{K}(\mathcal{H})$ induces a homomorphism $\varphi_*\colon K(A) \to K(B)$.

1.9 Asymptotic Morphisms and Tensor Products

The construction of maps $\varphi_*\colon K(A) \to K(B)$ from asymptotic morphisms has several elaborations which are quite important. They rely on the following observation:

Lemma 1.7. *Let D be a C^*-algebra and let $\varphi\colon A \dashrightarrow B$ be an asymptotic morphism between C^*-algebras. There is an asymptotic morphism $\varphi\widehat{\otimes}1\colon A\widehat{\otimes}D \dashrightarrow B\widehat{\otimes}D$ such that, on elementary tensors,*

$$(\varphi\widehat{\otimes}1)_t\colon a\widehat{\otimes}d \mapsto \varphi_t(a)\widehat{\otimes}d.$$

Moreover this formula determines $\varphi\widehat{\otimes}1$ uniquely, up to asymptotic equivalence.

Proof. Assume for simplicity that B and D are unital (the general case, which can be attacked by adjoining units, is left to the reader). There are graded $*$-homomorphisms from A and D into the asymptotic algebra $\mathfrak{A}(B\widehat{\otimes}D)$, determined by the formulas $a \mapsto \varphi_t(a)\widehat{\otimes}1$ and $d \mapsto 1\widehat{\otimes}d$. They graded commute and so determine a homomorphism $\varphi\widehat{\otimes}1\colon A\widehat{\otimes}D \to \mathfrak{A}(B\widehat{\otimes}D)$. This in turn determines an asymptotic morphism $\varphi\widehat{\otimes}1\colon A\widehat{\otimes}D \dashrightarrow B\widehat{\otimes}D$, as required. Two asymptotic morphisms which are asymptotic on the elementary tensors $a\widehat{\otimes}d$ determine $*$-homomorphisms into $\mathfrak{A}(B\widehat{\otimes}D)$ which are equal on elementary tensors, and hence equal everywhere. From this it follows that the two asymptotic morphisms are equivalent.

Remark 1.11. It is clear from the argument that it is crucial here to use the *maximal* tensor product.

Here then are the promised elaborations:

(a) An asymptotic morphism $\varphi\colon A \dashrightarrow B$ determines an asymptotic morphism from $A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ to $B\widehat{\otimes}\mathcal{K}(\mathcal{H})$ by tensor product, and hence a K-theory map $\varphi_*\colon K(A) \to K(B)$.

(b) An asymptotic morphism $\varphi\colon A \dashrightarrow B\widehat{\otimes}\mathcal{K}(\mathcal{H})$ determines an asymptotic morphism from $A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ to $B\widehat{\otimes}\mathcal{K}(\mathcal{H})\widehat{\otimes}\mathcal{K}(H)$ by tensor product. After identifying $\mathcal{K}(\mathcal{H})\widehat{\otimes}\mathcal{K}(\mathcal{H})$ with $\mathcal{K}(\mathcal{H})$ we can apply the construction of the previous section to obtain a map $\varphi_*\colon K(A) \to K(B)$.

(c) An asymptotic morphism $\varphi\colon \mathcal{S}\widehat{\otimes}A \dashrightarrow B$ determines an asymptotic morphism from $\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ to $B\widehat{\otimes}\mathcal{K}(\mathcal{H})\widehat{\otimes}\mathcal{K}(H)$ by tensor product. If $\psi\colon \mathcal{S} \to A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ represents a class in $K(A)$ then by forming the composition

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S}\widehat{\otimes}\mathcal{S} \xrightarrow{1\widehat{\otimes}\psi} \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \overset{\varphi\widehat{\otimes}1}{\dashrightarrow} B\widehat{\otimes}\mathcal{K}(\mathcal{H})$$

we obtain a class in $K(B)$, and we obtain a K-theory map $\varphi_*\colon K(A) \to K(B)$.

(d) Combining (b) and (c), an asymptotic morphism $\varphi\colon \mathcal{S}\widehat{\otimes}A \to B\widehat{\otimes}\mathcal{K}(\mathcal{H})$ determines a K-theory map $\varphi_*\colon K(A) \to K(B)$.

1.10 Bott Periodicity in the Spectral Picture

We are going to formulate and prove the Bott periodicity theorem using the spectral picture of K-theory, products, and a line of argument which is due to Atiyah [5]. In the course of doing so we shall introduce many of the ideas which will feature in our later discussion of the Baum-Connes conjecture.

In this section present an abstract outline of the argument; in the next three sections we shall fill in the details using the theory of Clifford algebras to construct suitable K-theory classes and asymptotic morphisms.

Definition 1.22. *Let us say that a graded C^*-algebra B has the* rotation property *if the automorphism $b_1 \widehat{\otimes} b_2 \mapsto (-1)^{\partial b_1 \partial b_2} b_2 \widehat{\otimes} b_1$ which interchanges the two factors in the tensor product $B \widehat{\otimes} B$ is homotopic to a tensor product $*$-homomorphism $1 \otimes \iota\colon B \widehat{\otimes} B \to B \widehat{\otimes} B$.*

Example 1.9. The trivially graded C^*-algebra $B = C_0(\mathbb{R}^{2n})$ has this property (with $\iota = 1$).[6]

Theorem 1.12. *Let B be a graded C^*-algebra with the rotation property. Suppose there exists a class $b \in K(B)$ and an asymptotic morphism*

$$\alpha\colon \mathcal{S} \widehat{\otimes} B \to \mathcal{K}(\mathcal{H})$$

with the property that the induced K-theory homomorphism $\alpha_*\colon K(B) \to K(\mathbb{C})$ maps b to 1. Then for every C^*-algebra A the maps

$$\alpha_*\colon K(A \widehat{\otimes} B) \to K(A) \quad \textit{and} \quad \beta_*\colon K(A) \to K(A \widehat{\otimes} B)$$

induced by α and by multiplication by the K-theory class b are inverse to one another.

Proof. From our definitions it is clear that the diagram

$$\begin{array}{ccc} K(C) \otimes K(A \widehat{\otimes} B) & \xrightarrow{K\text{-theory product}} & K(C \widehat{\otimes} A \widehat{\otimes} B) \\ \Big\downarrow{\scriptstyle 1 \otimes \alpha_*} & & \Big\downarrow{\scriptstyle \alpha_*} \\ K(C) \otimes K(A) & \xrightarrow[K\text{-theory product}]{} & K(C \widehat{\otimes} A) \end{array}$$

commutes. Let us express this by saying that the maps $\alpha_*\colon K(A \widehat{\otimes} B) \to K(A)$ are *multiplicative*. It follows directly from the multiplicative property that α_* is left-inverse to the map $\beta_*\colon K(A) \to K(A \widehat{\otimes} B)$:

$$\alpha_*(\beta_*(x)) = \alpha_*(x \times b) = x \times \alpha_*(b) = x \times 1 = x.$$

To prove that α_* is also left-inverse to β_* we introduce the isomorphisms

$$\sigma\colon A \widehat{\otimes} B \to B \widehat{\otimes} A$$

and

$$\tau\colon B \widehat{\otimes} A \widehat{\otimes} B \to B \widehat{\otimes} A \widehat{\otimes} B$$

which interchange the first and last factors in the tensor products. Note that

[6] So does $B = C_0(\mathbb{R}^{2n+1})$, but Theorem 1.12 does not apply in the odd-dimensional case.

$$\sigma_*(y) \times z = \tau_*(z \times y), \qquad \forall y \in K(A \widehat{\otimes} B), z \in K(B).$$

Since B has the rotation property, τ is homotopic to the tensor product $\iota \widehat{\otimes} 1 \widehat{\otimes} 1$, where ι is as in Definition 1.22. Therefore, setting $z = b$ above, we get

$$\sigma_*(y) \times b = \tau_*(b \times y) = \iota_*(b) \times y.$$

Applying α_* we deduce that

$$\sigma_*(y) = \alpha_*(\sigma_*(y) \times b) = \alpha_*(\iota_*(b) \times y) = \iota_*(b) \times \alpha_*(y)$$

(the first and last inequalities follow from the multiplicative property of α_*). Applying another flip isomorphism we conclude that $y = \alpha_*(y) \times \iota_*(b)$. This shows that multiplication by $\iota_*(b)$ is left-inverse to α_*. Therefore α_*, being both left and right invertible, is invertible. Moreover the left inverse β_* is necessarily a two-sided inverse.

Remark 1.12. It follows that $\iota_*(b) = b$. This fact can be checked in the example presented in the next section.

1.11 Clifford Algebras

We begin by venturing a bit further into the realm of graded C^*-algebras. We are going to introduce the (complex) *Clifford algebras*, which are a familiar presence in K-theory and index theory.

Definition 1.23. *Let V be a finite-dimensional Euclidean vector space (that is, a real vector space equipped with a positive-definite inner product). The complex Clifford algebra of V is the graded complex C^*-algebra generated by a linear copy of V, whose elements are self-adjoint and of grading-degree one, subject to the relations $v^2 = \|v\|^2 \cdot 1$ for every $v \in V$.*

Remark 1.13. The Clifford algebra can be concretely constructed from the complexified tensor algebra $T(V)$ be dividing $T(V)$ the ideal generated by the elements $v \otimes v - \|v\|^2 \cdot 1$.

It follows immediately from the definition that if $e_1, \ldots, e_n$ is an orthonormal basis for V then regarded as members of $\mathrm{Cliff}(V)$ these elements satisfy the relations

$$e_j^2 = 1 \quad \text{and} \quad e_i e_j + e_j e_i = 0 \quad \text{if } i \neq j.$$

The monomials $e_{i_1} \cdot \cdots \cdot e_{i_p}$, where $1 \leq i_1 < \cdots < i_p \leq n$ span $\mathrm{Cliff}(V)$ as a complex linear space. In fact these monomials constitute a basis for $\mathrm{Cliff}(V)$. The monomial $e_{i_1} \cdot \cdots \cdot e_{i_p}$ has grading-degree p (mod 2).

Example 1.10. The C^*-algebra $\mathrm{Cliff}(\mathbb{R})$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$, with e_1 corresponding to $(1, -1)$. The grading automorphism transposes the two copies of $\mathbb{C}$.

Example 1.11. The C^*-algebra $\mathrm{Cliff}(\mathbb{R}^2)$ is isomorphic to $M_2(\mathbb{C})$ in such a way that

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The (inner) grading is given by the grading operator $\varepsilon = ie_1e_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Remark 1.14. More generally, each even Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{2k})$ is a matrix algebra $M_{2^k}(\mathbb{C})$, graded by $\varepsilon = i^k e_1 \dots e_{2k} = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$; each odd Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{2k+1})$ is a direct sum $M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$, graded by the automorphism which switches the summands.

Definition 1.24. *Let V by a finite-dimensional Euclidean vector space. Denote by $\mathcal{C}(V)$ the graded C^*-algebra of continuous functions, vanishing at infinity, from V into* $\mathrm{Cliff}(V)$. *(The grading on $\mathcal{C}(V)$ comes from* $\mathrm{Cliff}(V)$ *alone—thus for example an even function is a function which takes values in the even part of* $\mathrm{Cliff}(V)$.*)*

Example 1.12. Thus $\mathcal{C}(\mathbb{R}^1)$ is isomorphic to $C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$ (and the grading automorphism switches the summands) while the C^*-algebra $\mathcal{C}(\mathbb{R}^2)$ is isomorphic to $M_2(C_0(\mathbb{R}^2))$, graded by $\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$.

Suppose now that V and W are finite-dimensional Euclidean vector spaces. Each of V and W is of course a subspace of $V \oplus W$, and there are corresponding inclusions of $\mathrm{Cliff}(V)$ and $\mathrm{Cliff}(W)$ into $\mathrm{Cliff}(V \oplus W)$. They determine a $*$-isomorphism

$$\mathrm{Cliff}(V) \widehat{\otimes} \mathrm{Cliff}(W) \cong \mathrm{Cliff}(V \oplus W)$$

(this can be checked either by computing with the standard linear bases for the Clifford algebras, or by checking that that the tensor product $\mathrm{Cliff}(V) \widehat{\otimes} \mathrm{Cliff}(W)$ has the defining property of the Clifford algebra $\mathrm{Cliff}(V \oplus W)$).

Proposition 1.13 *Let V and W be finite-dimensional Euclidean spaces. The map $f_1 \widehat{\otimes} f_2 \mapsto f$, where $f(v+w) = f_1(v)f_2(w)$ determines an isomorphism of graded C^*-algebras*

$$\mathcal{C}(V \oplus W) \cong \mathcal{C}(V) \widehat{\otimes} \mathcal{C}(W)$$

Proof. This follows easily by combining the isomorphism $\mathrm{Cliff}(V) \widehat{\otimes} \mathrm{Cliff}(W) \cong \mathrm{Cliff}(V \widehat{\otimes} W)$ above with the isomorphism $C_0(V) \otimes C_0(W) \cong C_0(V \oplus W)$.

Proposition 1.4. *Let V be a finite-dimensional Euclidean vector space. The C^*-algebra $\mathcal{C}(V)$ has the rotation property.*

Proof. Let $g \colon W_1 \to W_2$ be an isometric isomorphism of finite-dimensional Euclidean vector spaces. There is a corresponding $*$-isomorphism $g_* \colon \mathrm{Cliff}(W_1) \to \mathrm{Cliff}(W_2)$ and also a $*$-isomorphism

$$g_{**} \colon \mathcal{C}(W_1) \to \mathcal{C}(W_2)$$

defined by $(g_{**}f)(w_2) = g_*(f(g^{-1}w_2))$. Under the isomorphism

$$\mathcal{C}(V)\widehat{\otimes}\mathcal{C}(V) \cong \mathcal{C}(V \oplus V)$$

of Proposition 1.13 the flip isomorphism on the tensor product corresponds to the $*$-automorphism τ_{**} of $\mathcal{C}(V \oplus V)$ associated to the map τ which exchanges the two copies of V in the direct sum $V \oplus V$. But τ is homotopic, through isometric isomorphisms of $V \oplus V$, to the map $(v_1, v_2) \mapsto (v_1, -v_2)$, and so τ_{**} is homotopic to $1\widehat{\otimes}\iota_{**}$, where $\iota\colon V \to V$ is multiplication by -1.

Of course, as we noted earlier, the algebra $C_0(V)$ has the rotation property too. The virtue of dealing with $\mathcal{C}(V)$ rather than the plainer object $C_0(V)$ is that with Clifford algebras to hand we can present in a very concise fashion the following important element of the group $K(\mathcal{C}(V))$.

Definition 1.25. *Denote by $C\colon V \to \mathrm{Cliff}(V)$ the function $C(v) = v$ which includes V as a real linear subspace of self-adjoint elements in* $\mathrm{Cliff}(V)$.

This is a continuous function on V into $\mathrm{Cliff}(V)$, but

$$C(v)^2 = \|v\|^2 \cdot 1$$

so C does not vanish at infinity (far from it) and it is therefore not an element of $\mathcal{C}(V)$. However if $f \in \mathcal{S}$ then the function $f(C)$ defined by

$$v \mapsto f(C(v)), \qquad v \in V,$$

where f is applied to the element $C(v) \in \mathrm{Cliff}(V)$ in the sense of the functional calculus, *does* belong to $\mathcal{C}(V)$ and the assignment $\beta\colon f \mapsto f(C)$ is a $*$-homomorphism from $\mathcal{S}$ to $\mathcal{C}(V)$.

Definition 1.26. *The* Bott element $b \in K(\mathcal{C}(V))$ *is the K-theory class of the $*$-homomorphism $\beta\colon \mathcal{S} \to \mathcal{C}(V)$ defined by $\beta\colon f \mapsto f(C)$.*

Remark 1.15. The function C is an example of an *unbounded multiplier* of the C^*-algebra $\mathcal{C}(V)$. See the appendix.

Example 1.13. Bearing in mind the isomorphisms of Examples 1.10 and 1.11, we have

$$C(x) = (x, -x), \qquad x \in \mathbb{R}^1$$

and

$$C(z) = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \qquad z \in \mathbb{C} \cong \mathbb{R}^2.$$

We can now formulate the Bott periodicity theorem.

Theorem 1.14. *For every graded C^*-algebra A and every finite-dimensional Euclidean space V the* Bott map

$$\beta\colon K(A) \to K(A\widehat{\otimes}\mathcal{C}(V)),$$

defined by $\beta(x) = x \times b$, is an isomorphism of abelian groups.

We shall prove the theorem in the next two sections by constructing a suitable asymptotic morphism α and proving that $\alpha_*(b) = 1$.

Remark 1.16. To relate the above theorem to more familiar formulations of Bott periodicity we note, as we did earlier, that if $n = 2k$ is even then the Clifford algebra $\mathcal{C}_n$ is isomorphic to $M_{2^k}(C_0(\mathbb{R}^n))$, from which it follows that if A is trivially graded then

$$K(A\widehat{\otimes}\mathcal{C}(\mathbb{R}^{2k})) \cong K(A \otimes C_0(\mathbb{R}^{2k})).$$

The 'graded' theorem above therefore implies the more familiar isomorphism

$$K(A \otimes C_0(\mathbb{R}^{2k})) \cong K(A).$$

1.12 The Dirac Operator

We are going to construct an asymptotic morphism as in the following result. (The actual proof of the theorem will be carried out in the next section.)

Theorem 1.15. *There exists an asymptotic morphism*

$$\alpha\colon \mathcal{S}\widehat{\otimes}\mathcal{C}(V) \dashrightarrow \mathcal{K}(\mathcal{H})$$

for which the induced homomorphism $\alpha\colon K(\mathcal{C}(V)) \to K(\mathbb{C})$ *maps the Bott element* $b \in K(\mathcal{C}(V))$ *to* $1 \in K(\mathbb{C})$.

Definition 1.27. *Let V be a finite-dimensional Euclidean vector space. Let us provide the finite-dimensional linear space underlying the algebra* $\mathrm{Cliff}(V)$ *with the Hilbert space structure for which the monomials $e_{i_1}\cdots e_{i_p}$ (associated to an orthonormal basis of V) are orthonormal. The Hilbert space structure so obtained is independent of the choice of $e_1,\dots,e_n$. Denote by $\mathcal{H}(V)$ the infinite-dimensional complex Hilbert space of square-integrable* $\mathrm{Cliff}(V)$*-valued functions on V, Thus:*

$$\mathcal{H}(V) = L^2(V, \mathrm{Cliff}(V)).$$

The Hilbert space $\mathcal{H}(V)$ is a graded Hilbert space, with grading inherited from $\mathrm{Cliff}(V)$.

Definition 1.28. *Let V be a finite-dimensional Euclidean vector space and let $e, f \in V$. Define linear operators on the finite-dimensional graded Hilbert space underlying* $\mathrm{Cliff}(V)$ *by the formulas*

$$\begin{aligned} e(x) &= e \cdot x \\ \widehat{f}(x) &= (-1)^{\partial x} x \cdot f. \end{aligned}$$

Observe that the operator $e\colon \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$ is self-adjoint while the operator $\widehat{f}\colon \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$ is skew-adjoint.

Exercise 1.5. Let $e_1, \dots, e_n$ be an orthonormal basis for V. Show that if $i_1 < \cdots < i_p$ then the 'number' operator

$$N = \sum_{i=1}^{n} \hat{e}_i e_i$$

maps the monomial $e_{i_1} \cdots e_{i_p}$ in $\mathrm{Cliff}(V)$ to $(2p - n)e_{i_1} \cdots e_{i_p}$.

Definition 1.29. *Let V be a finite-dimensional Euclidean vector space. Denote by $\mathfrak{s}(V)$ the dense subspace of $\mathcal{H}(V)$ comprised of Schwartz-class $\mathrm{Cliff}(V)$-valued functions:*

$$\mathfrak{s}(V) = \textit{Schwartz-class } \mathrm{Cliff}(V)\textit{-valued functions.}$$

The Dirac operator of V is the unbounded operator D on $\mathcal{H}(V)$, with domain $\mathfrak{s}(V)$, defined by

$$(Df)(v) = \sum_{1}^{n} \widehat{e}_i\Big(\frac{\partial f}{\partial x_i}(v)\Big),$$

where $e_1, \dots, e_n$ is an orthonormal basis of V and $x_1, \dots, x_n$ are the corresponding coordinates on V.

Since the individual $\widehat{e}_i$ are skew-adjoint and since they commute with the partial derivatives we see that D is formally self-adjoint on $\mathfrak{s}(V)$.

Lemma 1.8. *Let V be a finite-dimensional Euclidean vector space. The Dirac operator on V is essentially self-adjoint. If $f \in \mathcal{S}$, if $h \in \mathcal{C}(V)$ and if M_h is the operator of pointwise multiplication by h on the Hilbert space $\mathcal{H}(V)$, then the product $f(D)M_h$ is a compact operator on $\mathcal{H}(V)$.*

Proof. The operator D is a constant coefficient operator acting on a Schwartz space of vector valued functions on $V \cong \mathbb{R}^n$. It has the form $D = \sum_{i=1}^{n} E_i \frac{\partial}{\partial x_i}$, where the matrices E_i are skew adjoint. Under the Fourier transform (a unitary isomorphism) D corresponds to the multiplication operator $\hat{D} = \sqrt{-1}\sum_{i=1}^{n} E_i \xi_i$, and from this we see that $\hat{D}$, and hence D, is essentially self-adjoint. Moreover from the formula

$$\hat{D}^2 = \big(\sqrt{-1}\sum_{i=1}^{n} E_i \xi_i\big)^2 = \|\xi\|^2,$$

for all $\xi \in \mathbb{R}^n$, it follows that if say $f(x) = e^{-ax^2}$ then $f(\hat{D})$ is pointwise multiplication by $e^{-\|\xi\|^2}$, and therefore the inverse Fourier transform $f(D)$ is convolution by $e^{-\frac{1}{4}\|x\|^2}$ (give or take a constant). It follows that $h \in \mathcal{C}(V)$ is compactly supported then $f(D)M_h$ is a Hilbert-Schmidt operator, and is therefore compact. The lemma follows from this since the set of $f \in \mathcal{S}$ for which $f(D)M_h$ is compact, for all h, is an ideal in $\mathcal{S}$, while the function e^{-x^2} generates $\mathcal{S}$ as an ideal.

We are almost ready to define our asymptotic morphism α.

Definition 1.30. *Let V be a finite-dimensional Euclidean space. If $h \in \mathcal{C}(V)$ and if $t \in [1, \infty)$ then denote by $h_t \in \mathcal{C}(V)$ the function $h_t(v) = h(t^{-1}v)$.*

Lemma 1.9. *Let V be a finite-dimensional Euclidean space with Dirac operator D. For every $f \in \mathcal{S}$ and $h \in \mathcal{C}(V)$ we have*

$$\lim_{t\to\infty} \big\| \big[f(t^{-1}D), M_{h_t} \big] \big\| = 0,$$

where $M_{h_t} \in \mathcal{B}(\mathcal{H}(V))$ is the operator of pointwise multiplication by h_t and $f(t^{-1}D)$ is defined using the functional calculus of unbounded operators.

Remark 1.17. The commutator $[\,,\,]$ here is the graded commutator of Definition 1.8.

Proof. By an approximation argument involving the Stone-Weierstrass theorem it suffices to consider the cases where $f(x) = (x \pm i)^{-1}$ and where h is smooth and compactly supported. We compute

$$\big[(t^{-1}D \pm iI)^{-1}, M_{h_t} \big] = t^{-1}(t^{-1}D \pm iI)^{-1} \, [M_{h_t}, D] \, (t^{-1}D \pm iI)^{-1},$$

which has norm bounded by $t^{-1} \| \, [M_{h_t}, D] \, \|$. But the commutator of M_{h_t} with D is the operator of pointwise multiplication by (minus) the function

$$v \mapsto t^{-1} \sum_{i=1}^{n} \hat{e}_i \big(\frac{\partial h}{\partial x_i}(t^{-1}v) \big).$$

So its norm is $\mathcal{O}(t^{-1})$, and the proof is complete.

Proposition 1.5. *There is, up to equivalence, a unique asymptotic morphism*

$$\alpha_t \colon \mathcal{S} \widehat{\otimes} \mathcal{C}(V) \to \mathcal{K}(\mathcal{H}(V))$$

for which, on elementary tensors,

$$\alpha_t(f \widehat{\otimes} h) = f(t^{-1}D) M_{h_t}.$$

Proof. For $t \in [1, \infty)$ define a linear map $\alpha_t : \mathcal{S} \widehat{\odot} \mathcal{C}(V) \to \mathcal{B}(\mathcal{H}(V))$ by the formula

$$\alpha_t(f \widehat{\otimes} h) = f(t^{-1}D) M_{h_t}.$$

Lemma 1.9 shows that the maps α_t define a homomorphism from $\mathcal{S} \widehat{\odot} \mathcal{C}(V)$ into $\mathfrak{A}(\mathcal{B}(\mathcal{H}(V)))$. By the universal property of the tensor product $\widehat{\otimes}$ this extends to a $*$-homomorphism defined on $\mathcal{S} \widehat{\otimes} \mathcal{C}(V)$. Now, although neither of the operators $f(t^{-1}D)$ or M_{h_t} are compact it follows from elementary elliptic operator theory that their product is compact. So our $*$-homomorphism actually maps $\mathcal{S} \widehat{\otimes} \mathcal{C}(V)$ into the subalgebra $\mathfrak{A}(\mathcal{K}(\mathcal{H}(V))) \subseteq \mathfrak{A}(\mathcal{B}(\mathcal{H}(V)))$. Therefore we obtain an asymptotic morphism as required.

Remark 1.18. The presence of h_t, instead of the plainer h, in the definition of α is not at this stage very important. The 't' could be removed without any problem. But later on it will turn out to have been convenient to have used h_t.

Exercise 1.6. Show that if J is an ideal in a C^*-algebra A then there is a short exact sequence of asymptotic algebras

$$0 \longrightarrow \mathfrak{A}(J) \longrightarrow \mathfrak{A}(A) \longrightarrow \mathfrak{A}(A/J) \longrightarrow 0.$$

1.13 The Harmonic Oscillator

In this section we shall verify that $\alpha_*(b) = 1$, which will complete the proof of the Bott periodicity theorem. Actually we shall make a more refined computation which will be required later on.

We begin by taking a second look at the basic construction of Section 1.11.

Definition 1.31. *Let V be a finite-dimensional Euclidean vector space. The Clifford operator is the unbounded operator on $\mathcal{H}(V)$, with domain the Schwartz space $\mathfrak{s}(V)$, which is given by the formula*

$$(Cf)(v) = \sum_{i=1}^{n} x_i e_i(f(v)),$$

where x_i are the coordinates on V dual to the orthonormal basis e_i of V (the definition of C is independent of the choice of basis).

The Clifford operator is essentially self-adjoint on the domain $\mathfrak{s}(V)$. So if $f \in \mathcal{S}$ we may form the bounded operator $f(C) \in \mathcal{B}(\mathcal{H}(V))$ by the functional calculus.

Lemma 1.10. *Let V be a finite-dimensional Euclidean vector space and let $\beta\colon \mathcal{S} \to \mathcal{C}(V)$ be the homomorphism of Definition 1.26. If $\mathcal{C}(V)$ is represented on the Hilbert space $\mathcal{H}(V)$ by pointwise multiplication operators then the composition*

$$\mathcal{S} \xrightarrow{\beta} \mathcal{C}(V) \xrightarrow{M} \mathcal{B}(\mathcal{H}(V))$$

maps $f \in \mathcal{S}$ to $f(C) \in \mathcal{B}(\mathcal{H}(V))$. $\square$

We shall compute the compostion $\alpha_*(b)$ by analyzing the following operator:

Definition 1.32. *Let V be a finite-dimensional Euclidean vector space. Define an unbounded operator B on $\mathcal{H}(V)$, with domain $\mathfrak{s}(V)$, by the formula*

$$(Bf)(v) = \sum_{1}^{n} x_i e_i(f(v)) + \sum_{1}^{n} \widehat{e}_i\Big(\frac{\partial f}{\partial x_i}(v)\Big).$$

Thus $B = C + D$, where C is the Clifford operator and D is the Dirac operator.

Example 1.14. Suppose $V = \mathbb{R}$. Then

$$B = \begin{pmatrix} 0 & x - d/dx \\ x + d/dx & 0 \end{pmatrix},$$

if we identify $\mathcal{H}(V)$ with $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ in the way suggested by Example 1.10.

Observe that the operator B maps the Schwartz space $\mathfrak{s}(V)$ into itself. So the operator $H = B^2$ is defined on $\mathfrak{s}(V)$.

Proposition 1.16 *Let V be a finite-dimensional euclidean vector space of dimension n, let $B = C + D$ as above. There exists within $\mathfrak{s}(V)$ an orthonormal basis for $\mathcal{H}(V)$ consisting of eigenvectors for B^2 such that*

(a) *the eigenvalues are nonnegative integers, and each eigenvalue occurs with finite multiplicity, and*
(b) *the eigenvalue 0 occurs precisely once and the corresponding eigenfunction is* $\exp(-\frac{1}{2}\|v\|^2)$

Proof. Let us consider the case $V = \mathbb{R}$ first. Here,

$$B^2 = \begin{pmatrix} x^2 - \frac{d^2}{dx^2} - 1 & 0 \\ 0 & x^2 - \frac{d^2}{dx^2} + 1 \end{pmatrix},$$

and so it suffices to prove that within the Schwartz subspace of $L^2(\mathbb{R})$ there is an orthonormal basis of eigenfunctions for the operator

$$H = x^2 - \frac{d^2}{dx^2},$$

for which the eigenvalues are positive integers (with finite multiplicities) and for which the eigenvalue 1 appears with multiplicity one. This is a well-known computation, and is done as follows. Define $K = x + \frac{d}{dx}$ and $L = x - \frac{d}{dx}$, and let $f_1(x) = e^{-\frac{1}{2}x^2}$. Observe that

$$H = KL - I = LK + I$$

and that $Kf_1 = 0$, so that $Hf_1 = f_1$. It follows that $HL = LH + 2L$ and $HL^n = L^nH + 2nL^n$. So if we define $f_{n+1} = L^n f_1$ then $Hf_{n+1} = (2n+1)f_{n+1}$. The functions f_{n+1} are orthogonal (being eigenfunctions of the symmetric operator H with distinct eigenvalues), nonzero, and they span $L^2(\mathbb{R})$ (since, by induction, f_{n+1} is a polynomial of degree n times f_1). So after L^2-normalization we obtain the required basis.

The general case follows from the (purely algebraic) calculation

$$B^2 = C^2 + D^2 + N = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} -\frac{\partial^2}{\partial x_i^2} + (2p - n) \quad \text{on } \mathcal{H}_p(V),$$

where N is the number operator introduced in Exercise 1.5 and $\mathcal{H}_p(V)$ denotes the subspace of $\mathcal{H}(V)$ comprised of functions $V \to \mathrm{Cliff}(V)$ whose values are combinations of the degree p monomials $e_{i_1} \cdots e_{i_p}$. From this an eigenbasis for B^2 may be found by separation of variables.

We shall use the following consequences of this computation:

Corollary 1.1. *Let V be a finite-dimensional Euclidean vector space. Let $B = B_V$ be the Bott-Dirac operator of V, considered as an unbounded operator on $\mathcal{H}(V)$ with domain $\mathfrak{s}(V)$. Then*

(a) *B is essentially self-adjoint*
(b) *B has compact resolvent.*
(c) *The kernel of B is one-dimensional and is generated by the function* $\exp(-\|v\|^2)$.

Theorem 1.17. *Let V be a finite-dimensional Euclidean vector space. The composition*

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S}\widehat{\otimes}\mathcal{S} \xrightarrow{1\widehat{\otimes}\beta} \mathcal{S}\widehat{\otimes}\mathcal{C}(V) \overset{\alpha}{\dashrightarrow} \mathcal{K}(\mathcal{H}(V))$$

is asymptotically equivalent to the asymptotic morphism $\gamma\colon \mathcal{S} \dashrightarrow \mathcal{K}(\mathcal{H})$ defined by

$$\gamma_t(f) = f(t^{-1}B) \qquad (t \geq 1).$$

The idea of the proof is to check the equivalence of the asymptotic morphisms $\alpha \circ \beta$ and γ on the generators

$$u(x) = e^{-x^2}, \quad \text{and} \quad v(x) = xe^{-x^2}.$$

of the C^*-algebra $\mathcal{S}$. Since for example

$$\gamma_t(u) = e^{-tH} \quad \text{and} \quad \alpha_t(\beta(u)) = e^{-tD^2}e^{-tC^2}$$

(the latter thanks to Lemma 1.10) we shall need to know that e^{-tH} is asymptotic to $e^{-tD^2}e^{-tC^2}$. For this purpose we invoke Mehler's formula:

Proposition 1.6 (Mehler's Formula). *Let V be a finite-dimensional Euclidean space and let C and D be the Clifford and Dirac operators for V. The operators D^2, C^2 and $C^2 + D^2$ are essentially self-adjoint on the Schwartz space $\mathfrak{s}(V)$, and if $s > 0$ then*

$$e^{-s(C^2+D^2)} = e^{-\frac{1}{2}s_1C^2}e^{-s_2D^2}e^{-\frac{1}{2}s_1C^2},$$

where $s_1 = (\cosh(2s) - 1)/\sinh(2s)$ and $s_2 = \sinh(2s)/2$. In addition,

$$e^{-s(C^2+D^2)} = e^{-\frac{1}{2}s_1D^2}e^{-s_2C^2}e^{-\frac{1}{2}s_1D^2},$$

for the same s_1 and s_2. $\square$

See for example [16]. Note that the second identity follows from the first upon taking the Fourier transform on $L^2(\mathbb{R})$, which interchanges the operators D^2 and C^2.

Lemma 1.11. *If X is any unbounded self-adjoint operator then there are asymptotic equivalences*

$$e^{-\frac{1}{2}\tau_1X^2} \sim e^{-\frac{1}{2}t^{-2}X^2}, \qquad e^{-\tau_2X^2} \sim e^{-t^{-2}X^2}$$

and

$$t^{-1}Xe^{-\frac{1}{2}\tau_1X^2} \sim t^{-1}Xe^{-\frac{1}{2}t^{-2}X^2}, \qquad t^{-1}Xe^{-\tau_2X^2} \sim t^{-1}Xe^{-t^{-2}X^2}$$

where $\tau_1 = (\cosh(2t^{-2}) - 1)/\sinh(2t^{-2})$ and $\tau_2 = \sinh(2t^{-2})/2$.

Remark 1.19. By 'asymptotic equivalence' we mean here that the differences between the left and right hand sides in the above relations all converge to zero, in the operator norm, as t tends to infinity.

Proof (Proof of the Lemma). By the spectral theorem it suffices to consider the same problem with the self-adjoint operator X replaced by a real variable x and the operator norm replaced by the supremum norm on $C_0(\mathbb{R})$. The lemma is then a simple calculus exercise, based on the Taylor series $\tau_1, \tau_2 = t^{-2} + o(t^{-2})$.

Lemma 1.12. *If* $f, g \in \mathcal{S} = C_0(\mathbb{R})$ *then*

$$\lim_{t\to\infty} \left\| \left[f(t^{-1}C), g(t^{-1}D) \right] \right\| = 0.$$

Proof. For any fixed $f \in \mathcal{S}$, the set of $g \in \mathcal{S}$ for which the lemma holds is a C^*-subalgebra of $C_0(\mathbb{R})$. So by the Stone-Weierstrass theorem it suffices to prove the lemma when g is one of the resolvent functions $(x \pm i)^{-1}$. It furthermore suffices to consider the case where f is a smooth and compactly supported function. In this case we have

$$\left\| \left[f(t^{-1}C), (t^{-1} \pm i)^{-1} \right] \right\| \leq \left\| \left[f(t^{-1}C), t^{-1}D \right] \right\|$$

by the commutator identity for resolvents. But then

$$\left\| \left[f(t^{-1}C), t^{-1}D \right] \right\| \leq t^{-2} \cdot \text{constant} \cdot \| \operatorname{grad}(f(C)) \|.$$

This proves the lemma.

Proof (Proof of Theorem 1.17). Denote by $N \colon \mathcal{H}(V) \to \mathcal{H}(V)$ the 'number operator' which multiplies the degree p component of $\mathcal{H}(V)$ by $2n - p$. We observed in the proof of Proposition 1.16 that

$$B^2 = C^2 + D^2 + N,$$

and let us observe now that the operator N commutes with C^2 and D^2. As a result,

$$e^{t^{-2}B^2} = e^{t^{-2}(C^2+D^2)} e^{t^{-2}N}$$

and therefore, by Mehler's formula,

$$e^{-t^{-2}B^2} = e^{-\frac{1}{2}\tau_1 C^2} e^{-\tau_2 D^2} e^{-\frac{1}{2}\tau_1 C^2} e^{-t^{-2}N},$$

It follows from Lemma 1.11 that

$$e^{-t^{-2}B^2} \sim e^{\frac{1}{2}t^{-2}C^2} e^{-t^{-2}D^2} e^{-\frac{1}{2}t^{-2}C^2} e^{-t^{-2}N},$$

and hence from Lemma 1.12 that

$$e^{-t^{-2}B^2} \sim e^{-t^{-2}C^2} e^{-t^{-2}D^2}$$

(since the operator N is bounded the operators $e^{-t^{-2}N}$ converge in norm to the identity operator). Now the homomorphism $\beta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{C}(V)$ maps $u(x) = e^{-x^2}$ to $u\widehat{\otimes}u(C)$, and applying α_t we obtain

$$\alpha_t(\beta(u)) = u(t^{-1}C)u(t^{-1}D) = e^{-t^{-2}C^2}e^{t^{-2}D^2},$$

as we noted earlier. But $\gamma_t(u) = e^{-t^{-2}B^2}$, and so we have shown that $\alpha_t(\beta(u))$ and $\gamma_t(u)$ are asymptotic to one another. A similar computation shows that if $v(x) = xe^{-x^2}$ then $\alpha_t(\beta(v))$ and $\gamma_t(v)$ are asymptotic to one another. Since u and v generate $\mathcal{S}$, this completes the proof.

Corollary 1.2. *The homomorphism* $\alpha_*\colon K(\mathcal{C}(V)) \to K(\mathbb{C})$ *maps the element* $b \in K(\mathcal{C}(V))$ *to the element* $1 \in K(\mathbb{C})$.

Proof. The class $\alpha_*(b)$ is represented by the composition of the $*$-homomorphism β with the asymptotic morphism α. By Theorem 1.17, this composition is asymptotic to the asymptotic morphism

$$\gamma_t(f) = f(t^{-1}B).$$

But each map γ_t is actually a $*$-homomorphism, and so the asymptotic morphism γ is homotopic to the single $*$-homomorphism $f \mapsto f(B)$. Now denote by p the projection onto the kernel of B. The formula

$$f \longmapsto \begin{cases} f(s^{-1}D), & \text{if } s \in (0,1] \\ \begin{pmatrix} f(0)p & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } s = 0, \end{cases}$$

defines a homotopy proving that $\alpha_*(b) = 1$.

Appendix: Unbounded Multipliers

Any C^*-algebra A may be regarded as a right Hilbert module over itself (see the book [45] for an introduction to Hilbert modules). An *unbounded (essentially self-adjoint) multiplier* of A is then an essentially self-adjoint operator on the Hilbert module A, in the sense of the following definition:

Definition 1.33. *(Compare [45, Chapter 9].) Let A be a C^*-algebra and let $\mathcal{E}$ be a Hilbert A-module. An* essentially self-adjoint *operator on $\mathcal{E}$ is an A-linear map T from a dense A-submodule $\mathcal{E}_T \subseteq \mathcal{E}$ into $\mathcal{E}$ with the following properties:*

(a) $\langle Tv, w\rangle = \langle v, Tw\rangle$, *for all* $v, w \in \mathcal{E}_T$.
(b) *The operator $I + T^2$ is densely defined and has dense range.*

If T is essentially self-adjoint then the closure of T (the graph of which is the closure of the graph of T) is self-adjoint and regular, which means that the operators

$(\overline{T} \pm iI)$ are bijections from the domain of $\overline{T}$ to $\mathcal{E}$, and that the inverses $(\overline{T} \pm iI)^{-1}$ are adjoints of one another. See [45, Chapter 9] again.

If T is essentially self-adjoint then there is a functional calculus $*$-homomorphism from $\mathcal{S} = C_0(\mathbb{R})$ into the bounded, adjoinable operators on $\mathcal{E}$. It maps $(x \pm i)^{-1}$ to $(\overline{T} \pm iI)^{-1}$.

In the case where $\mathcal{E} = A$, if the densely defined operators $(T \pm iI)^{-1}$ are given by right multiplication with elements of A, then the functional calculus homomorphism maps $\mathcal{S}$ into A (acting on A as right multiplication operators). If A is graded, if the domain A_T of T is graded, and if T has odd grading-degree (as a map from the graded space A_T into the graded space A) then the functional calculus homomorphism is a graded $*$-homomorphism.

Example 1.15. If $A = \mathcal{S}$ then the operator $X\colon f(x) \mapsto xf(x)$, defined on say the compactly supported functions, is essentially self-adjoint.

Lemma 1.13. *If X_1 is an essentially self-adjoint multiplier of A_1 and if X_2 is essentially self-adjoint multiplier of A_2, then $X_1\widehat{\otimes}1 + 1\widehat{\otimes}X_2$, with domain $A_{X_1}\widehat{\odot}A_{X_2}$, is an essentially self-adjoint multiplier of $A_1\widehat{\otimes}A_2$.* □

Example 1.16. Using the lemma we can define $\Delta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}$ by $\Delta(f) = f(X\widehat{\otimes}1 + 1\widehat{\otimes}X)$.

2 Bivariant K-Theory

We saw in the last section that asymptotic morphisms between C^*-algebras determine maps between K-theory groups. In this lecture we shall organize homotopy classes of asymptotic morphisms into a *bivariant* version of K-theory, whose purpose is to streamline the computation of K-theory groups via asymptotic morphisms. In doing so we shall be following the lead of Kasparov (see [39, 37, 38]), although the theory we obtain, called E-theory [13, 14, 27], will in fact be a minor modification of Kasparov's KK-theory.

2.1 The E-Theory Groups

Definition 2.1. *Let A and B be separable, graded C^*-algebras. We shall denote by $E(A,B)$ the set of homotopy classes of asymptotic morphisms from $\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ to $B\widehat{\otimes}\mathcal{K}(\mathcal{H})$. Thus:*

$$E(A,B) = [\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathcal{K}(\mathcal{H})]\!].$$

Example 2.1. Each $*$-homomorphism φ from A to B, or more generally from $\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ to $B\widehat{\otimes}\mathcal{K}(\mathcal{H})$, determines an element of $E(A,B)$. This element depends only on the homotopy class of φ, and will be denoted $[\varphi] \in E(A,B)$.

The sets $E(A,B)$ come equipped with an operation of addition, given by direct sum of asymptotic morphisms, and the zero asymptotic morphism provides a zero element for this addition.

Lemma 2.1. *The abelian monoids $E(A,B)$ are in fact abelian groups.*

Proof. Let $\varphi\colon \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \dashrightarrow B\widehat{\otimes}\mathcal{K}(\mathcal{H})$ be an asymptotic morphism. Define an asymptotic morphism

$$\varphi^{\mathrm{opp}}\colon \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \dashrightarrow B\widehat{\otimes}\mathcal{K}(\mathcal{H}^{\mathrm{opp}})$$

by the formula $\varphi_t^{\mathrm{opp}}(x) = \varphi_t(\alpha(x))$, where α is the grading automorphism. We shall show that φ^{opp} defines an additive inverse to φ in $E(A,B)$.

For a fixed scalar $s \geq 0$ the formula

$$\Phi_t^s\colon f\widehat{\otimes}x \mapsto f\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}\begin{pmatrix} \varphi_t(x) & 0 \\ 0 & \varphi_t^{\mathrm{opp}}(x) \end{pmatrix} \qquad f \in \mathcal{S}, \quad x \in \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H})$$

defines an asymptotic morphism Φ^s from $\mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ into $B\widehat{\otimes}\mathcal{K}(\mathcal{H}\oplus\mathcal{H}^{\mathrm{opp}})$. By composing Φ^s with the comultiplication $\Delta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}$ we obtain asymptotic morphisms

$$\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \xrightarrow{\Delta\widehat{\otimes}1} \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}) \overset{\Phi^s}{\dashrightarrow} B\widehat{\otimes}\mathcal{K}(\mathcal{H}\oplus\mathcal{H}^{\mathrm{opp}})$$

which constitute a homotopy (parametrized by $s \in [0,\infty]$) connecting $\varphi\oplus\varphi^{\mathrm{opp}}$ to 0.

Remark 2.1. The above argument provides another proof that the K-theory groups described in the last lecture are in fact groups.

If e is a rank-one projection in $\mathcal{K}(\mathcal{H})$ then by composing asymptotic morphisms with the $*$-homomorphism which maps the element $f\widehat{\otimes}a \in \mathcal{S}\widehat{\otimes}A$ to the element $f\widehat{\otimes}a\widehat{\otimes}e \in \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ we obtain a map (of sets, or in fact abelian groups)

$$[\![\mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{K}(\mathcal{H}), B\widehat{\otimes}\mathcal{K}(\mathcal{H})]\!] \longrightarrow [\![\mathcal{S}\widehat{\otimes}A, B\widehat{\otimes}\mathcal{K}(\mathcal{H})]\!].$$

Lemma 2.2. *The above map is a bijection.*

Proof. The inverse is given by tensor product with the identity on $\mathcal{K}(\mathcal{H})$. Details are left to the reader as an exercise.

The groups $E(A,B)$ are contravariantly functorial in A and covariantly functorial in B on the category of graded C^*-algebras.

Proposition 2.1. *The functor $E(\mathbb{C},B)$ on the category of graded C^*-algebras is naturally isomorphic to $K(B)$.*

Proof. This follows from Proposition 1.3 and Lemma 2.2.

2.2 Composition of Asymptotic Morphisms

The main feature of E-theory is the existence of a bilinear 'composition law'

$$E(A,B) \otimes E(B,C) \to E(A,C)$$

which is associative in the sense that the two possible iterated pairings

$$E(A,B) \otimes E(B,C) \otimes E(C,D) \to E(A,D)$$

are equal, and which gathers the E-theory groups together into an additive category (the objects are separable graded C^*-algebras, the morphisms from A to B are the elements of the abelian group $E(A,B)$, and the above pairing is the composition law).

The E-theory category plays an important role in the computation of C^*-algebra K-theory groups, as follows. To compute the K-theory of a C^*-algebra A one can, on occasion, find a C^*-algebra B and elements of $E(A,B)$ and $E(B,A)$ whose compositions are the identity morphisms in $E(A,A)$ and $E(B,B)$. Composition with these two elements of $E(A,B)$ and $E(B,A)$ now gives a pair of mutually inverse maps between $E(\mathbb{C},A)$ and $E(\mathbb{C},B)$. But as we noted in the last section $E(\mathbb{C},A)$ and $E(\mathbb{C},B)$ are the K-theory groups $K(A)$ and $K(B)$. It therefore follows that $K(A) \cong K(B)$. Therefore, assuming that $K(B)$ can be computed, so can $K(A)$. This is the main strategy for computing the K-theory of group C^*-algebras.

In this section and the next we shall lay the groundwork for the construction of the composition pairing. The following sequence of definitions and lemmas presents a reasonably conceptual approach to the problem. The proofs are all very simple, and by and large they are omitted. Details can be found in the monograph [27].

We begin by repeating a definition from the last lecture.

Definition 2.2. *Let B be a graded C^*-algebra. Denote by $\mathfrak{i}(B)$ the C^*-algebra of bounded, continuous functions from $[1,\infty)$ into B, and denote by $\mathfrak{i}_0(B)$ the ideal comprised of functions which vanish at infinity. The asymptotic C^*-algebra of B is the quotient C^*-algebra*

$$\mathfrak{A}(B) = \mathfrak{i}(B)/\mathfrak{i}_0(B).$$

Observe (as we did in the last section) that an asymptotic morphism $\varphi\colon A \dashrightarrow B$ defines a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}(B)$ in the obvious manner and that two asymptotic morphism from A to B define the same $*$-homomorphism from A to $\mathfrak{A}(B)$ precisely when they are asymptotically equivalent.

The asymptotic algebra construction $B \mapsto \mathfrak{A}(B)$ is a functor, since a $*$-homomorphism from B to C induces a $*$-homomorphism from $\mathfrak{A}(B)$ to $\mathfrak{A}(C)$ by composition.

Definition 2.3. *The asymptotic functors $\mathfrak{A}^0, \mathfrak{A}^1, \ldots$ are defined by $\mathfrak{A}^0(B) = B$ and*

$$\mathfrak{A}^n(B) = \mathfrak{A}(\mathfrak{A}^{n-1}(B)).$$

Two $*$-homomorphisms $\varphi^0, \varphi^1 \colon A \to \mathfrak{A}^n(B)$ are n-homotopic if there exists an $*$-homomorphism $\Phi : A \to \mathfrak{A}^n(B[0,1])$ from which the $*$-homomorphisms φ^0 and φ^1 are recovered as the compositions

$$A \longrightarrow \mathfrak{A}^n(B[0,1]) \xrightarrow{\text{evaluate at } 0,\,1} \mathfrak{A}^n(B).$$

Lemma 2.3. [27, Proposition 2.3] *The relation of n-homotopy is an equivalence relation on the set of $*$-homomorphisms from A to $\mathfrak{A}^n(B)$.* $\square$

Definition 2.4. *Let A and B be graded C^*-algebras. Denote by $[\![A,B]\!]_n$ the set of n-homotopy classes of $*$-homomorphisms from A to $\mathfrak{A}^n(B)$:*

$$[\![A,B]\!]_n = \{ \text{ n-Homotopy classes of $*$-homomorphisms from A to $\mathfrak{A}^n(B)$ } \}.$$

Example 2.2. Observe that $[\![A,B]\!]_0$ is the set of homotopy classes of $*$ - homomorphisms and $[\![A,B]\!]_1$ is the set of homotopy classes of asymptotic morphisms.

Remark 2.2. The relation of n-homotopy is *not* the same thing as homotopy: homotopic $*$-homomorphisms into $\mathfrak{A}^n(B)$ are n-homotopic, but not *vice-versa*, in general.

There is a natural transformation of functors, from $\mathfrak{A}^n(B)$ to $\mathfrak{A}^{n+1}(B)$, defined by including $\mathfrak{A}^n(B)$ as constant functions in $\mathfrak{A}^{n+1}(B) = \mathfrak{A}(\mathfrak{A}^n(B))$. A second and different natural transformation from $\mathfrak{A}^n(B)$ to $\mathfrak{A}^{n+1}(B)$ may be defined by including B into $\mathfrak{A}(B)$ as constant functions, and then applying the functor $\mathfrak{A}^n$ to this inclusion. Both natural transformations are compatible with homotopy in the sense that they define maps

$$[\![A,B]\!]_n \longrightarrow [\![A,B]\!]_{n+1}.$$

Lemma 2.4. [27, Proposition 2.8] *The above natural transformations define the same map $[\![A,B]\!]_n \longrightarrow [\![A,B]\!]_{n+1}$* $\square$.

With the above maps the sets $[\![A,B]\!]_n$ are organized into a directed system

$$[\![A,B]\!]_1 \to [\![A,B]\!]_2 \to [\![A,B]\!]_3 \to \cdots$$

Definition 2.5. *Let A and B be graded C^*-algebras. Denote by $[\![A,B]\!]_\infty$ the direct limit of the above directed system.*

Proposition 2.2. [27, Proposition 2.12] *Let $\varphi \colon A \to \mathfrak{A}^n(B)$ and $\psi \colon B \to \mathfrak{A}^m(C)$ be $*$-homomorphisms. The class of the composite $*$-homomorphism*

$$A \xrightarrow{\varphi} \mathfrak{A}^n(B) \xrightarrow{\mathfrak{A}^n(\psi)} \mathfrak{A}^{n+m}(C).$$

in the set $[\![A,C]\!]_\infty$ depends only on the classes of φ and ψ in the sets $[\![A,B]\!]_\infty$ and $[\![B,C]\!]_\infty$. The composition law

$$[\![A,B]\!]_\infty \times [\![B,C]\!]_\infty \to [\![A,C]\!]_\infty$$

so defined is associative. $\square$

Exercise 2.1. Show that the identity $*$-homomorphism from A to A determines an element of $[\![A,A]\!]_\infty$ which serves as an identity morphism for the above composition law.

Thanks to Proposition 2.2 and the exercise we obtain a category:

Definition 2.6. *The asymptotic category is the category whose objects are the graded C^*-algebras, whose are elements of the sets $[\![A,B]\!]_\infty$, and whose composition law is the process described in Proposition 2.2.*

Observe that there is a functor from the category of graded C^*-algebras and $*$-homomorphisms into the asymptotic category (which is the identity on objects and which assigns to a $*$-homomorphism $\varphi\colon A \to B$ its class in $[\![A,B]\!]_\infty$).

Exercise 2.2. Show that K-theory, thought of as a functor from graded C^*-algebras to abelian groups, factors through the asymptotic category.

2.3 Operations

We want to define tensor products, amplifications and other operations on the asymptotic category. For this purpose we introduce the following definitions.

Definition 2.7. *Let F be a functor from the category of graded C^*-algebras to itself. If B is a graded C^*-algebra and if $f \in F(B[0,1])$ then define a function $\hat{f}$ from $[0,1]$ into $F(B)$ by assigning to $t \in [0,1]$ the image of f under the homomorphism $F(\varepsilon_t)\colon F(B[0,1]) \to F(B)$, where ε_t is evaluation at t. The functor F is* continuous *if for every B and every $f \in F(B[0,1])$ the function $\hat{f}$ is continuous.*

Example 2.3. The tensor product functors $A \mapsto A\widehat{\otimes}B$ (for both the minimal and maximal tensor product) are continuous.

Definition 2.8. *A functor F from the category of graded C^*-algebras to itself is* exact *if for every short exact sequence*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

the induced sequence

$$0 \longrightarrow F(J) \longrightarrow F(A) \longrightarrow F(A/J) \longrightarrow 0$$

is also exact.

Exercise 2.3. The *maximal* tensor product functor $A \mapsto A\widehat{\otimes}_{max}B$ is exact.

Remark 2.3. In contrast the *minimal* tensor product functor $A \mapsto A\widehat{\otimes}_{min}B$ is not exact for every B. See [66] for examples (and also Lecture 6).

If F is a continuous functor then the construction of $\hat{f}$ from f described in Definition 2.7 determines a natural transformation

$$F(B[0,1]) \to F(B)[0,1].$$

The same process also determines natural transformations

$$F(\mathrm{i}(B)) \to \mathrm{i}(F(B) \quad \text{and} \quad F(\mathrm{i}_0(B)) \to \mathrm{i}_0(F(B))$$

(recall that $\mathrm{i}(B)$ is the C^*-algebra of bounded and continuous functions from $[1,\infty)$ into B and $\mathrm{i}_0(B)$ is the ideal of functions vanishing at infinity). So if F is in addition an exact functor then we obtain an induced map from $F(\mathfrak{A}(B))$ into $\mathfrak{A}(F(B))$, as indicated in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & F(\mathrm{i}_0(B)) \arrow[r] \arrow[d] & F(\mathrm{i}(B)) \arrow[r] \arrow[d] & F(\mathfrak{A}(B)) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \mathrm{i}_0(F(B)) \arrow[r] & \mathrm{i}(F(B)) \arrow[r] & \mathfrak{A}(F(B)) \arrow[r] & 0. \end{tikzcd}$$

Proposition 2.3. [27, Theorem 3.5] *Let F be a continuous and exact functor on the category of graded C^*-algebras. The process which assigns to each $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n(B)$ the composition*

$$F(A) \xrightarrow{F(\varphi)} F(\mathfrak{A}^n(B)) \longrightarrow \mathfrak{A}^n(F(B))$$

defines a functor on the asymptotic category. □

Applying this to the (maximal) tensor product functors we obtain the following result.

Proposition 2.1 [27, Theorem 4.6] *There is a functorial tensor product $\widehat{\otimes}_{max}$ on the asymptotic category.* □

With a tensor product operation in hand we can construct an amplified asymptotic category in the same way we constructed the amplification of the category of C^*-algebras and $*$-homomorphisms in Definition 1.12.

Definition 2.9. *The* amplified asymptotic category *is the category whose objects are the graded C^*-algebras and for which the morphisms from A to B are the elements of $[\![\mathcal{S}\widehat{\otimes}A, B]\!]_\infty$. Composition of morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$ in the amplified asymptotic category is given by the following composition of morphisms in the asymptotic category:*

$$\mathcal{S}\widehat{\otimes}A \xrightarrow{\Delta\widehat{\otimes}1} \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}A \xrightarrow{1\widehat{\otimes}\varphi} \mathcal{S}\widehat{\otimes}B \xrightarrow{\psi} C.$$

2.4 The E-Theory Category

The main technical theorem in E-theory is the following:

Theorem 2.2. [27, Theorem 2.16] *Let A and B be graded C^*-algebras and assume that A is separable. The natural map of $[\![A, B]\!]_1$ into the direct limit $[\![A, B]\!]_\infty$ is a bijection. Thus every morphism from A to B in the asymptotic category is represented by a unique homotopy class of asymptotic morphisms from A to B.* □

Unlike the results of the previous two sections, this is a little delicate. We refer the reader to [27] for details.

It follows from Theorem 2.2 and Definition 2.1 that the group $E(A, B)$ (for A separable) may be identified with the set of morphisms in the amplified asymptotic category from $A\widehat{\otimes}\mathcal{K}(\mathcal{H})$ to $B\widehat{\otimes}\mathcal{K}(\mathcal{H})$. As a result we obtain a pairing

$$E(A, B) \otimes E(B, C) \longrightarrow E(A, C)$$

from the composition law in the asymptotic category. We have now reached the main objective of the lecture:

Theorem 2.3. *The E-theory groups $E(A, B)$ are the morphism groups in an additive category $\boldsymbol{E}$ whose objects are the separable graded C^*-algebras. There is a functor from the homotopy category of graded separable C^*-algebras and graded $*$-homomorphisms into $\boldsymbol{E}$ which is the identity on objects.* □

Remark 2.4. If $\varphi\colon A \to B$ is a $*$-homomorphism and if $\psi\colon B \dashrightarrow C$ is an asymptotic morphism then φ and ψ determine elements $[\varphi] \in E(A, B)$ and $[\psi] \in E(B, C)$. In addition the (naive) composition $\psi \circ \varphi$ is an asymptotic morphism from A to C, and so defines an element $[\psi \circ \varphi] \in E(A, C)$. We have that $[\psi \circ \varphi] = [\psi] \circ [\varphi]$. The same applies to compositions of $*$-homomorphisms and asymptotic morphisms the other way round, and also to compositions in the amplified category.

The tensor product functor on the asymptotic category extends to the amplified asymptotic category (compare Remark 1.6), and we obtain a tensor product in E-theory:

Theorem 2.4. *There is a functorial tensor product $\widehat{\otimes}_{max}$ on the E-theory category which is compatible with the tensor product on C^*-algebras via the functor from the category of graded separable C^*-algebras and graded $*$-homomorphisms into the E-theory category.* □

The minimal tensor product does not carry over to E-theory, but we have at least a partial result. First, here is some standard C^*-algebra terminology.

Definition 2.10. *A (graded) C^*-algebra B is* exact *if, for every short exact sequence of graded C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

the sequence of minimal tensor products

$$0 \longrightarrow J\widehat{\otimes}_{min}B \longrightarrow A\widehat{\otimes}_{min}B \longrightarrow A/J\widehat{\otimes}_{min}B \longrightarrow 0$$

is exact.

In other words, B is exact if and only if the functor $A \mapsto A\widehat{\otimes}_{min}B$ is exact.

Theorem 2.5. *Let B be a separable, graded and exact C^*-algebra. There is a functor $A \mapsto A\widehat{\otimes}_{min}B$ on the E-theory category. In particular, if A_1 and A_2 are isomorphic in the E-theory category then $A_1\widehat{\otimes}_{min}B$ and $A_2\widehat{\otimes}_{min}B$ are isomorphic there too.* □

We shall return to the topic of minimal tensor products in Lecture 6.

2.5 Bott Periodicity

Our proof of Bott periodicity in Lecture 1 may be recast as a computation in E-theory, as follows.

Definition 2.11. *Let V be a finite-dimensional Euclidean vector space. Denote by $\beta \in E(\mathbb{C}, \mathcal{C}(V))$ the E-theory class of the $*$-homomorphism $\beta\colon \mathcal{S} \to \mathcal{C}(V)$ introduced in Definition 1.26. Denote by $\alpha \in E(\mathcal{C}(V), \mathbb{C})$ the E-theory class of the asymptotic morphism $\alpha\colon \mathcal{S}\widehat{\otimes}\mathcal{C}(V) \dashrightarrow \mathcal{K}(\mathcal{H}(V))$ introduced in Proposition 1.5.*

Proposition 2.4. *The composition*

$$\mathbb{C} \xrightarrow{\beta} \mathcal{C}(V) \xrightarrow{\alpha} \mathbb{C}$$

in the E-theory category is the identity morphism $\mathbb{C} \to \mathbb{C}$.

Proof. This follows from Remark 2.4 and Theorem 1.17, as in the proof of Corollary 1.2.

A small variation on the rotation argument we discussed in Section 1.10 now proves the following basic result:

Theorem 2.6. *The morphisms $\alpha\colon \mathcal{C}(V) \to \mathbb{C}$ and $\beta\colon \mathbb{C} \to \mathcal{C}(V)$ in the E-theory category are mutual inverses.* □

2.6 Excision

The purpose of this section is to discuss the construction of 6-term exact sequences in E-theory. First, we need a simple definition.

Definition 2.12. *Let A be a C^*-algebra. The* suspension *of A is the C^*-algebra*

$$\Sigma A = \{\, f \in A[0,1] \,:\, f(0) = f(1) = 0 \,\}.$$

In other words ΣA is the tensor product of A with $\Sigma = C_0(0,1)$. If A is graded then so is ΣA (the algebra Σ itself is given the trivial grading).

Theorem 2.7. *The suspension map*

$$E(A,B) \longrightarrow E(\Sigma A, \Sigma B)$$

is an isomorphism. Moreover there are natural isomorphisms

$$E(A,B) \cong E(\Sigma^2 A, B) \quad \textit{and} \quad E(A,B) \cong E(A, \Sigma^2 B).$$

Proof. It follows from Bott periodicity that Σ^2 is isomorphic to $\mathbb{C}$ in the E-theory category, and this proves the second part of the theorem. With the periodicity isomorphisms available, we obtain an inverse to the suspension map by simply suspending a second time.

Here then are the main theorems in the section:

Theorem 2.8. *Let B be a graded C^*-algebra and let I be an ideal in a separable C^*-algebra A. There is a functorial six-term exact sequence*

$$\begin{tikzcd}
E(A/I,B) \arrow[r] & E(A,B) \arrow[r] & E(I,B) \arrow[d] \\
E(I,\Sigma B) \arrow[u] & E(A,\Sigma B) \arrow[l] & E(A/I,\Sigma B) \arrow[l]
\end{tikzcd}$$

Theorem 2.9. *Let A be a graded C^*-algebra and let J be an ideal in a separable C^*-algebra B. There is a functorial six-term exact sequence*

$$\begin{tikzcd}
E(A,J) \arrow[r] & E(A,B) \arrow[r] & E(A,B/J) \arrow[d] \\
E(A,\Sigma B/J) \arrow[u] & E(A,\Sigma B) \arrow[l] & E(A,\Sigma J) \arrow[l]
\end{tikzcd}$$

For simplicity we shall discuss only the second of these two theorems (the proofs of the two theorems are similar, although the second is a little easier in some respects). For a full account of both see [27, Chapters 5 and 6].

The proof of Theorem 2.9 has two parts. The first is a construction borrowed from elementary homotopy theory, involving the following notion:

Definition 2.13. *Let $\pi\colon B \to C$ be a $*$-homomorphism of (graded) C^*-algebras. The* mapping cone *of π is the C^*-algebra*

$$C_\pi = \left\{ b \oplus f \in B \oplus C[0,1] \;:\; \pi(b) = f(0) \quad \textit{and} \quad f(1) = 0 \right\}.$$

Proposition 2.5. *Let $\pi\colon B \to C$ be a $*$-homomorphism. For every C^*-algebra A there is a long exact sequence of pointed sets*

$$\cdots \longrightarrow [\![A, \Sigma B]\!] \longrightarrow [\![A, \Sigma C]\!] \longrightarrow [\![A, C_\pi]\!] \longrightarrow [\![A, B]\!] \longrightarrow [\![A, C]\!].$$

The proposition may be formulated for homotopy classes of asymptotic morphisms, as above, or for homotopy classes of ordinary $*$-homomorphisms (compare [59]). The proofs are the same in both cases. There are $*$-homomorphisms

$$\cdots \longrightarrow \Sigma B \longrightarrow \Sigma C \longrightarrow C_\pi \longrightarrow B \longrightarrow C,$$

which supply the maps in the proposition, and since the composition of any two successive $*$-homomorphisms in this sequence is null-homotopic, the composition of any two successive maps of the sequence in the proposition is trivial. Let us prove exactness at the $[\![A, B]\!]$ term. If the composition

$$A \overset{\varphi}{\dashrightarrow} B \xrightarrow{\pi} C$$

is null homotopic then a null homotopy gives an asymptotic morphism from $\Phi\colon A \dashrightarrow C[0,1)$. The pair comprised of φ and Φ now determines an asymptotic morphism from A into C_π, as required. For more details see [27, Chapter 5].

Corollary 2.1. *Let $\pi: B \to C$ be a $*$-homomorphism. For every C^*-algebra A there is a functorial six-term exact sequence*

$$\begin{array}{ccccc} E(A, C_\pi) & \longrightarrow & E(A,B) & \longrightarrow & E(A,C) \\ \uparrow & & & & \downarrow \\ E(A, \Sigma C) & \longleftarrow & E(A, \Sigma B) & \longleftarrow & E(A, \Sigma C_\pi) \end{array}$$

This follows from Proposition 2.5 and Theorem 2.7. To prove Theorem 2.9 it remains to replace C_π with J in the above corollary, in the case where $\pi\colon B \to C$ is a surjection with kernel J. To this end, observe that there is an inclusion $j \mapsto j \oplus 0$ of J into C_π. Using the following construction one can show that this inclusion is an isomorphism in the E-theory category.

Theorem 2.10. ([27, Chapter 5].) *Let J be an ideal in a separable graded C^*-algebra A. There is a norm-continuous family $\{u_t\}_{t\in[1,\infty)}$ of degree-zero elements in J such that*

(a) $0 \le u_t \le 1$ *for all t,*
(b) $\lim_{t\to\infty} \|u_t j - j\| = 0$, *for all $j \in J$, and*
(c) $\lim_{t\to\infty} \|u_t a - a u_t\| = 0$, *for all $a \in A$.*

If $s\colon A/J \to A$ is any set-theoretic section of the quotient mapping then the formula

$$\varphi_t(f \otimes x) = f(u_t)s(x)$$

defines an asymptotic morphism from $\Sigma A/J$ into J. □

Theorem 2.11. ([27, Proposition 5.14].) *Let J be an ideal in a separable, graded C^*-algebra A. The asymptotic morphism associated to the extension*

$$0 \longrightarrow \Sigma J \longrightarrow A[0,1) \longrightarrow C_\pi \longrightarrow 0$$

determines an element of $E(\Sigma C_\pi, \Sigma J)$ *which is inverse to the element of* $E(\Sigma J, \Sigma C_\pi)$ *which is determined by the inclusion of* J *into* C_π. □

In view of Theorem 2.7 it now follows that $J \cong \mathbb{C}_\pi$ in the E-theory category, and the proof of Theorem 2.9 is complete.

2.7 Equivariant Theory

We are now going to define an equivariant version of E-theory which will be particularly useful for computing the K-theory of group C^*-algebras. To keep matters as simple as possible we shall work here with countable and *discrete* groups, although it is possible to consider arbitrary second countable, locally compact groups.

The following definition provides the main idea behind the equivariant theory:

Definition 2.14. *Let* G *be a countable discrete group and let* A *and* B *be graded* G-C^**-algebras (that is, graded* C^**-algebras equipped with actions of* G *by grading-preserving* $*$*-automorphisms). An* equivariant asymptotic morphism *from* A *to* B *is an asymptotic morphism* $\varphi\colon A \dashrightarrow B$ *such that*

$$\varphi_t(g \cdot a) - g \cdot (\varphi_t(a)) \to 0, \quad \textit{as } t \to \infty,$$

for all $a \in A$ *and all* $g \in G$.

Homotopy is defined just as in the non-equivariant case, and we set

$$[\![A, B]\!]^G = \{ \text{Homotopy classes of asymptotic morphisms from } A \text{ to } B \}.$$

If B is a G-C^*-algebra then so is the asymptotic algebra $\mathfrak{A}(B)$, and an equivariant asymptotic morphism from A to B is the same thing, up to equivalence, as an equivariant $*$-homomorphism from A to $\mathfrak{A}(B)$.[7] Thanks to this observation it is a straightforward matter to define an equivariant version of the asymptotic category that we constructed in Section 2.2. The higher asymptotic algebras $\mathfrak{A}^n(B)$ are G-C^*-algebras; we define $[\![A, B]\!]^G_n$ to be the set of n-homotopy classes of equivariant $*$-homomorphism from A to $\mathfrak{A}^n(B)$; and we define

$$[\![A, B]\!]^G_\infty = \varinjlim [\![A, B]\!]^G_n.$$

These are the morphism sets of a category, using the composition law described in Proposition 2.2, and this category may be 'amplified', as in Section 1.3. Finally, if A is separable (and assuming, as we shall throughout, that G is countable) then the canonical map gives an isomorphism

$$[\![A, B]\!]^G \xrightarrow[\cong]{} [\![A, B]\!]^G_\infty.$$

[7] This is one place where our assumption that G is discrete is helpful: if G is not discrete then the action of G on $\mathfrak{A}(B)$ is not necessarily continuous.

See [27] for details.

To define the equivariant E-theory groups it remains to introduce a stabilization operation which is appropriate to the equivariant context.

Definition 2.15. *Let G be a countable discrete group. The standard G-Hilbert space $\mathcal{H}_G$ is the infinite Hilbert space direct sum*

$$\mathcal{H}_G = \oplus_{n=0}^{\infty} \ell^2(G),$$

equipped with the regular representation of G on each summand and graded so the even numbered summands are even and the odd numbered summands are odd.

The standard G-Hilbert space has the following universal property:

Lemma 2.5. *If $\mathcal{H}$ is any separable graded G-Hilbert space[8] then the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}_G$ is unitarily equivalent to $\mathcal{H}_G$ via a grading-preserving, G-equivariant unitary isomorphism of Hilbert spaces.*

Proof. Denote by $\mathcal{H}_0$ the Hilbert space $\mathcal{H}$ equipped with the trivial G-action. The formula $v \otimes [g] \mapsto g^{-1} \cdot v \otimes [g]$ defines a unitary isomorphism from $\mathcal{H} \otimes \ell^2(G)$ to $\mathcal{H}_0 \otimes \ell^2(G)$, and from it we obtain a unitary isomorphism

$$\mathcal{H} \widehat{\otimes} \mathcal{H}_G \xrightarrow[\cong]{} \mathcal{H}_0 \widehat{\otimes} \mathcal{H}_G .$$

Since $\mathcal{H}_0 \widehat{\otimes} \mathcal{H}_G$ is just a direct sum of copies of $\mathcal{H}_G$ it is clear that $\mathcal{H}_0 \widehat{\otimes} \mathcal{H}_G \cong \mathcal{H}_G$. Hence $\mathcal{H} \widehat{\otimes} \mathcal{H}_G \cong \mathcal{H}_G$, as required.

Definition 2.16. *Let G be a countable discrete group and let A and B be graded, separable G-C^*-algebras. Denote by $E_G(A, B)$ the set of homotopy classes of equivariant asymptotic morphisms from $\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}_G)$ to $B \widehat{\otimes} \mathcal{K}(\mathcal{H}_G)$,*

$$E_G(A, B) = [\![\mathcal{S} \widehat{\otimes} A \widehat{\otimes} \mathcal{K}(\mathcal{H}_G), B \widehat{\otimes} \mathcal{K}(\mathcal{H}_G)]\!]^G .$$

Remark 2.5. The virtue of working with the Hilbert space $\mathcal{H}_G$, as in the above definition, is that if $\mathcal{H}$ is *any* separable graded G-Hilbert space and if $\varphi \colon \mathcal{S} \widehat{\otimes} A \dashrightarrow B \widehat{\otimes} \mathcal{K}(\mathcal{H})$ is an equivariant asymptotic morphism then φ determines an element of $E_G(A, B)$. To see this, simply tensor φ by $\mathcal{K}(\mathcal{H}_G)$ and apply Lemma 2.5.

Remark 2.6. The construction described in the previous remark has a generalization which will be important in Lecture 4. Suppose that $\mathcal{H}$ is a separable, graded Hilbert space which is equipped with a *continuous family* of unitary G-actions, parametrized by $t \in [1, \infty)$. The continuity requirement here is pointwise strong continuity, so that if $g \in G$ and $k \in \mathcal{K}(\mathcal{H})$ then $g \cdot_t k$ is norm-continuous in t. Suppose now that A and B are G-C^*-algebras and that

[8] A *graded G-Hilbert space* is a graded Hilbert space equipped with unitary representations of G on its even and odd grading-degree summands.

$$\varphi\colon \mathcal{S}\widehat{\otimes}A \dashrightarrow B\widehat{\otimes}\mathcal{K}(\mathcal{H})$$

is an asymptotic morphism which is equivariant with respect to the given family of G-actions, in the sense that

$$\lim_{t\to\infty} \|\varphi_t(g\cdot x) - g\cdot_t(\varphi_t(x))\| = 0,$$

for all $g \in G$ and $x \in \mathcal{S}\widehat{\otimes}A$. Then φ too determines an element of $E_G(A,B)$. Indeed, after we tensor with $\mathcal{K}(\mathcal{H}_G)$ and apply the procedure in the proof of Lemma 2.5 we obtain an asymptotic morphism into $B\widehat{\otimes}\mathcal{K}(\mathcal{H}_0\widehat{\otimes}\mathcal{H}_G)$ which is equivariant in the usual sense for the single, fixed representation of G on $\mathcal{H}_0\widehat{\otimes}\mathcal{H}_G$.

Remark 2.7. One final comment: it is essential that in Definition 2.16 we include a factor of $\mathcal{K}(\mathcal{H}_G)$ in both arguments. If we were to leave one out then we would obtain a quite different (and not very useful) object.

By comparing the definition of $E_G(A,B)$ to the construction of the equivariant, amplified asymptotic category we immediately obtain the following result:

Theorem 2.12. *The E_G-theory groups $E_G(A,B)$ are the morphism sets of an additive category whose objects are the separable graded G-C^*-algebras. There is a functor from the homotopy category of graded G-C^*-algebras and graded G-equivariant $*$-homomorphisms into the equivariant E-theory category which is the identity on objects.* □

The equivariant E theory category has a tensor product $\widehat{\otimes}_{max}$. Moreover there are six-term exact sequences of E-theory groups associated to short exact sequences of G-C^*-algebras. The precise statements and proofs are only minor modifications of what we saw in the non-equivariant case, and we shall omit them here. See [27].

2.8 Crossed Products and Descent

In order to apply equivariant E-theory to the problem of computing C^*-algebra K-theory one must first apply a descent operation which transfers computations in equivariant E-theory to computations in the nonequivariant theory. This involves the notion of crossed product C^*-algebra, and we begin with a rapid review of the basic definitions (see [53]) for more details).

Definition 2.17. *Let G be a discrete group and let A be a G-C^*-algebra. A* covariant representation *of A in a C^*-algebra B is a pair (φ,π) consisting of a $*$-homomorphism φ from A into a C^*-algebra B and a group homomorphism π from G into the unitary group of the multiplier algebra of B which are related by the formulas*

$$\pi(g)\varphi(a)\pi(g^{-1}) = \varphi(g\cdot a), \quad \textit{for all } a\in A,\ g\in G.$$

Definition 2.18. *Let G be a discrete group and let A be a G-C^*-algebra. The linear space $C_c(G,A)$ of finitely-supported, A-valued functions on G is an involutive algebra with respect to the convolution multiplication and involution defined by*

$$f_1 \star f_2(g) = \sum_{h\in G} f_1(h)\ (h \cdot (f_2(h^{-1}g)))$$
$$f^*(g) = g \cdot (f(g^{-1})^*)$$

Observe that a covariant representation of A in a C^*-algebra B determines a $*$-homomorphism $\varphi \times \pi$ from $C_c(G,A)$ into B by the formula

$$(\varphi \times \pi) f = \sum_{g\in G} \varphi(f(g))\pi(g) \quad \text{for all } f \in C_c(G,A).$$

Definition 2.19. *The* full crossed product C^*-algebra $C^*(G,A)$ *is the completion of the $*$-algebra $C_c(G,A)$ in the smallest C^*-algebra norm which makes all the $*$-homomorphisms $\varphi \times \pi$ continuous.*

Example 2.4. Setting $A = \mathbb{C}$ we obtain the *full group C^*-algebra $C^*(G)$*.

If A is graded, and if G acts by grading - preserving automorphisms, then $C^*(G,A)$ has a natural grading too (the grading automorphism acts pointwise on functions in $C_c(G,A)$).

Remark 2.8. The C^*-algebra $C^*(G,A)$ contains a copy of A and the multiplier algebra of $C^*(G,A)$ contains a copy of G within its unitary group. Elements of $C_c(G,A)$ can be written as finite sums $\sum_{g\in G} a_g \cdot g$, where $a_g \in A$ and $a_g = 0$ for almost all g. It will usually be convenient to use this means of representing elements. For example the grading automorphism is

$$\sum_{g\in G} a_g \cdot g \mapsto \sum_{g\in G} \alpha(a_g) \cdot g.$$

The full crossed product is a functor from G-C^*-algebras to C^*-algebras which is (extending the terminology of Section 2.3 in the obvious way) both continuous and exact. As a result, there is a *descent functor* from the equivariant asymptotic category to the asymptotic category,

$$[\![A,B]\!]^G_\infty \longrightarrow [\![C^*(G,A), C^*(G,B)]\!]_\infty.$$

In order to obtain a corresponding functor in E-theory we need the following computation:

Lemma 2.6. *Let G be a discrete group, let B be a G-C^*-algebra and let $\mathcal{H}$ be a G-Hilbert space on which the group element $g \in G$ acts as the unitary operator $U_g\colon \mathcal{H} \to \mathcal{H}$. The formula*

$$\sum_{g\in G} (b_g \widehat{\otimes} k_g) \cdot g \mapsto \sum_{g\in G} (b_g \cdot g) \widehat{\otimes} k_g U_g$$

determines an isomorphism of C^*-algebras

$$C^*(G, B\widehat{\otimes}\mathcal{K}(\mathcal{H})) \xrightarrow[\cong]{} C^*(G, B)\widehat{\otimes}\mathcal{K}(\mathcal{H}).$$

Proof. The formula defines an algebraic $*$-isomorphism from $C_c(G, B\widehat{\odot}\mathcal{K}(\mathcal{H}))$ to $C_c(G, B)\widehat{\odot}\mathcal{K}(\mathcal{H})$. Examining the definitions of the norms for the max tensor product and full crossed product we see that the $*$-isomorphism extends to a $*$-isomorphism of C^*-algebras.

Combining the lemma with the descent functor between asymptotic categories we obtain the following result:

Theorem 2.13. *There is a* descent functor *from the equivariant E-theory category to the E-theory category which maps the G-C^*-algebra A to the full crossed product C^*-algebra $C^*(G, A)$, and which maps the E-theory class of a G-equivariant $*$-homomorphism $\varphi\colon A \to B$ to the E-theory class of the induced $*$-homomorphism from $C^*(G, A)$ to $C^*(G, B)$.* □

Corollary 2.2. *Let G be a countable discrete group. Suppose that A and B are separable G-C^*-algebras and that A and B are isomorphic objects in the equivariant E-theory category. Then $K(C^*(G, A))$ is isomorphic to $K(C^*(G, B))$.* □

2.9 Reduced Crossed Products

We also wish to apply equivariant E-theory to the computation of K-theory for *reduced* crossed products. Here the operation of descent works smoothly for a large class of groups, as the following discussion shows, but not so well for all groups, as we shall see in Lecture 6.[9]

In the following definition we shall use, in a very modest way, the notion of Hilbert module. See [45] for a treatment of this subject.

Definition 2.20. *Let A be a G-C^*-algebra and denote by $\ell^2(G, A)$ the Hilbert A-module comprised of functions $\xi\colon G \to A$ for which the series $\sum_g \xi(g)^*\xi(g)$ is norm-convergent in A. The regular representation of A is the covariant representation (φ, π) into the bounded, adjoinable operators on $\ell^2(G, A)$ given by the formulas*

$$(\varphi(a)\xi)(h) = (h^{-1} \cdot a)\xi(h), \qquad \xi \in \ell^2(G, A),$$

and

$$(\pi(g)\xi)(h) = \xi(g^{-1}h), \qquad \xi \in \ell^2(G, A).$$

The regular representation determines a $*$-homomorphism from the crossed product algebra $C^*(G, A)$ into the C^*-algebra of bounded, adjoinable operators on $\ell^2(G, A)$.

[9] It should be pointed out here that Kasparov's KK-theory has no such limitation in this respect. However it has other shortcomings. Indeed as we shall see in Lecture 6 there is no ideal bivariant K-theory for C^*-algebras.

Definition 2.21. *Let A be a G-C^*-algebra. The* reduced crossed product algebra *$C^*_\lambda(G,A)$ is the image of $C^*(G,A)$ in the regular representation.*

Example 2.5. Setting $A=\mathbb{C}$ we obtain the *reduced group C^*-algebra $C^*_\lambda(G)$*.

Like the full crossed product, the reduced crossed product is a functor from (graded) G-C^*-algebras to (graded) C^*-algebras. However unlike the full crossed product the reduced crossed product is not exact for every G (although inexact examples are hard to come by — see Lecture 6). This prompts us to make the following definition:

Definition 2.22. *A discrete group G is* exact *if the functor $A \mapsto C^*_\lambda(G,A)$ is exact in the sense of Definition 2.8.*

There is a very simple and beautiful characterization of exact groups, due to Kirchberg and Wassermann [43].

Proposition 2.6. *A discrete group G is exact if and only if its reduced group C^*-algebra $C^*_\lambda(G)$ is exact.*

Proof (Proof (sketch)). Exactness of $C^*_\lambda(G)$ is implied by exactness of G since in the case of trivial G-actions the reduced crossed product $C^*_\lambda(G,A)$ is the same thing as $A\otimes_{min} C^*_\lambda(G)$ (note that $C^*_\lambda(G)$ is trivially graded, so $\otimes_{min}=\widehat{\otimes}_{min}$ here). The reverse implication is argued as follows. If $C^*_\lambda(G)$ is exact then the sequence

$$0 \longrightarrow C^*(G,J)\otimes_{min} C^*_\lambda(G) \longrightarrow C^*(G,A)\otimes_{min} C^*_\lambda(G)$$
$$\longrightarrow C^*(G,A/J)\otimes_{min} C^*_\lambda(G) \longrightarrow 0$$

is exact. But for any G-C^*-algebra D there is a functorial embedding

$$C^*_\lambda(G,D) \longrightarrow C^*(G,D)\otimes_{min} C^*_\lambda(G)$$

defined by the formulas $g \mapsto g\otimes g$ and $d\mapsto d\otimes 1$, and moreover a functorial, continuous and linear left-inverse defined by $d\otimes 1\mapsto d$, $g\otimes g\mapsto g$ and $g\otimes h\mapsto 0$ if $g\neq h$. It follows that the sequence

$$0 \longrightarrow C^*_\lambda(G,J) \longrightarrow C^*_\lambda(G,A) \longrightarrow C^*_\lambda(G,A/J) \longrightarrow 0$$

is a direct summand of the minimal tensor product exact sequence above, and is therefore exact itself. For more details see Section 5 of [43]

Exercise 2.4. If B is an exact C^*-algebra and if B_1 is a C^*-subalgebra of B then B then B_1 is also exact.

Thanks to the exercise and to Proposition 2.6 it is possible to show that many classes of groups are exact. For example all discrete subgroups of connected Lie groups are exact and all hyperbolic groups (these will be discussed in Lecture 5) are exact too. Every amenable group is exact since in this case the reduced and full crossed product functors are one and the same. For more information on exactness see for example [67]. We shall also return to the subject in Section 4.5.

By retracing the steps we took in the previous section we arrive at the following result:

Theorem 2.14. *Let G be an exact, countable, discrete group. There is a* descent functor *from the equivariant E-theory category to the E-theory category which maps a G-C^*-algebra A to the reduced crossed product C^*-algebra $C^*_\lambda(G, A)$, and which maps the class of a G-equivariant $*$-homomorphism $\varphi\colon A \to B$ to the class of the induced $*$-homomorphism from $C^*_\lambda(G, A)$ to $C^*_\lambda(G, B)$.* □

Corollary 2.3. *Let G be an exact, countable, discrete group. Suppose that A and B are separable G-C^*-algebras and that A and B are isomorphic objects in the equivariant E-theory category. Then $K(C^*_\lambda(G, A))$ is isomorphic to $K(C^*_\lambda(G, B))$.* □

2.10 The Baum-Connes Conjecture

In this lecture we shall formulate the Baum-Connes conjecture and prove it in some simple cases, for example for finite groups and free abelian groups. We shall also sketch the proof of the conjecture for so-called ‘proper’ coefficient C^*-algebras. This result will play an important role in the next chapter. The proof for proper algebras is not difficult, but it is a little long-winded, and we shall refer the reader to the monograph [27] for the details.

We shall continue to work exclusively with *discrete* groups. Our formulation of the conjecture, which uses E-theory, is equivalent to the formulation in [7] which uses KK-theory. Indeed there is a natural transformation from KK to E which determines an isomorphism from the KK-theoretic ‘left-hand side’ of the Baum-Connes conjecture to its E-theoretic counterpart. The isomorphism can be proved either by a Mayer-Vietoris type of argument (see for example Lecture 5) or by directly constructing an inverse. See also the discussion in Section 4.6 which in many cases reduces the conjecture to a statement in K-theory, independent of both E-theory and KK-theory.[10] Our treatment using E-theory is quite well suited to the theorems we shall formulate and prove in Lecture 4. However a major drawback of E-theory is that it is not well suited to dealing with inexact groups. In any case, the E-theoretic and KK-theoretic developments of the Baum-Connes theory are very similar, and having studied by himself the basics of KK-theory the reader could develop the Baum-Connes conjecture in KK-theory simply by replacing E with KK throughout this lecture.

[10] In fact the argument of Section 4.6 can be made to apply to any discrete group, but we shall not go into this here.

2.11 Proper G-Spaces

Let G be a countable discrete group. Throughout this lecture we shall be dealing with Hausdorff and paracompact topological spaces X equipped with actions of G by homeomorphisms.

Definition 2.23. *A G-space X is* proper *if for every $x \in X$ there is a G-invariant open subset $U \subseteq X$ containing x, a finite subgroup H of G, and a G-equivariant map from U to G/H.*

The definition says that locally the orbits of G in X look like G/H.

Example 2.6. If H is a finite subgroup of G then the discrete homogeneous space G/H is proper. Moreover if Y is any (Hausdorff and paracompact) space with an H-action then the *induced space* $X = G \times_H Y$ (the quotient of $G \times Y$ by the diagonal action of H, with H acting on G by right multiplication) is proper.

In fact every proper G-space is locally induced from a finite group action:

Lemma 2.7. *A G-space X is proper if and only if for every $x \in X$ there is a G-invariant open subset $U \subseteq X$ containing X, a finite subgroup H of G, an H-space Y, and a G-equivariant homeomorphism from U to $G \times_H Y$.* □

Many proofs involving proper spaces proceed by reducing the case of a general proper G-space to the case of the local models $G \times_H W$, and hence to the case of finite group actions, using the lemma.

Lemma 2.8. *A locally compact G-space X is proper if and only if the map from $G \times X$ to $X \times X$ which takes (g, x) to (gx, x) is a proper map of locally compact spaces (meaning that the inverse image of every compact set is compact).* □

Example 2.7. If G is a discrete subgroup of a Lie group L, and if K is a compact subgroup of L, then the quotient space L/K is a proper G-space.

2.12 Universal Proper G-Spaces

Definition 2.24. *A proper G-space X is universal if for every proper G-space Y there exists a G-equivariant continuous map $Y \to X$, and if moreover this map is unique up to G-equivariant homotopy.*

It is clear from the definition that any two universal proper G-spaces are G-equivariantly homotopy equivalent. For this reason let us introduce the notation eG for a universal proper G-space (with the understanding that different models for eG will agree up to equivariant homotopy).

Proposition 2.15 *Let G be a countable discrete group. There exists a universal proper G-space.* □

Here is one simple construction (due to Kasparov and Skandalis [36]). Let X_1 be the space of (countably additive) measures on G with total mass 1 or less. This is a compact space in the topology of pointwise convergence. Let $X_{\frac{1}{2}}$ be the closed subspace of X_1 consisting of measures of total mass $\frac{1}{2}$ or less. The set-theoretic difference $X = X_1 \setminus X_{\frac{1}{2}}$ is a locally compact proper G-space which is universal.

In examples one can usually provide a much more concrete model. See [7] for examples (and see also Lectures 4 and 5 below). The following result, which we shall not prove, gives the general flavor of these constructions.

Proposition 2.7. *Let M be a complete and simply connected Riemannian manifold of nonpositive sectional curvature. If a discrete group G acts properly and isometrically on M then M is a universal G-space.* □

Remark 2.9. The manifold here could be infinite-dimensional.

2.13 G-Compact Spaces

Definition 2.25. *A proper G-space X is G*-compact *if there is a compact subset $K \subseteq X$ whose translates under the G-action cover X.*

If X is a G-compact proper G-space then X is locally compact and the quotient X/G is compact.

Definition 2.26. *Let X be a G-compact proper G-space. A cutoff function for X is a continuous function $\theta\colon X \to [0,1]$ such that*

(a) $\operatorname{supp}(\theta)$ *is compact, and*
(b) $\sum_{g\in G} \theta^2(g\cdot x) = 1$*, for all $x \in X$.*

Observe that the sum in (b) is locally finite. Every G-compact proper G-space admits a cutoff function. Moreover any two cutoff functions are, in a sense, homotopic: if θ_0 and θ_1 are cutoff functions then the functions

$$\theta_t = \sqrt{t\theta_1^2 + (1-t)\theta_0^2}, \quad t \in [0,1]$$

are all cutoff functions.

Lemma 2.9. *Let θ be a cutoff function for the G-compact proper G-space X. The formula*

$$p(g)(x) = \theta(g^{-1}x)\,\theta(x).$$

defines a projection in $C_c(G, C_c(X))$, and hence in $C^*(G,X)$. The K-theory class of this projection is independent of the choice of cutoff function.

Remark 2.10. We are using here the streamlined notation $C^*(G,X)$ in place of $C^*(G,C_0(X))$.

Proof (Proof of the Lemma). A computation shows that p is a projection (note that the sum involved in the definition of p is in fact finite). If θ_0 and θ_1 are cutoff functions then associated to the homotopy of cutoff functions θ_t defined above there is a homotopy of projections p_t, and therefore θ_0 and θ_1 give rise to the same K-theory class, as required.

Definition 2.27. *We will call the unique K-theory class of projections associated to cutoff functions the unit class:*

$$[p] \in K(C^*(G,X)) \cong E(\mathbb{C}, C^*(G,X)).$$

Exercise 2.16 *(See [54, Thm 6.1].) Let X be a proper G-space. Show that the full and reduced crossed products $C^*(G,X)$ and $C^*_\lambda(G,X)$ are isomorphic.*

Hint: *One approach is to show that if $f \in C_c(G, C_c(X))$, and if $1+f$ is invertible in $C^*_\lambda(G,X)$, then the inverse actually lies in $1 + C_c(G, C_c(X))$. It follows that $1+f$ is invertible in $C^*(G,X)$ too, and therefore, the map $C^*(G,X) \to C_\lambda(G,X)$ is spectrum-preserving, and hence isometric.*

Remark 2.11. As a result of the exercise, we can obviously define a unit class in $K(C^*_\lambda(G,X))$ too.

2.14 The Assembly Map

In this section we shall further streamline our notation and write $E_G(X,D)$ in place of $E_G(C_0(X),D)$. Observe that $E_G(X,D)$ is covariantly functorial on the category of G-compact proper G-spaces X.

Definition 2.28. *Let G be a countable discrete group and let D be a separable G-C^*-algebra. The assembly map*

$$\mu\colon E_G(X,D) \to K(C^*(G,D))$$

is the composition

$$E_G(X,D) \xrightarrow{\text{descent}} E(C^*(G,X), C^*(G,D)) \xrightarrow{[p]} E(\mathbb{C}, C^*(G,D))$$

where the first map is the descent homomorphism of Section 2.8 and the second is composition with the unit class $[p] \in E(\mathbb{C}, C^*(G,X))$.

Definition 2.29. *Let G be a countable group and let D be a G-C^*-algebra. The topological K-theory of G with coefficients in a G-C^*-algebra D is defined by*

$$K^{top}(G,D) = \varinjlim_{\substack{X \subseteq \mathrm{e}G \\ G\text{-inv, } G\text{-cpt}}} E_G(X,D),$$

where the limit is taken over the collection of G-invariant and G-compact subspaces $X \subseteq \mathrm{e}G$, directed by inclusion.

To explain the limit, note that if $X \subseteq Y \subseteq \mathrm{e}G$ are G-compact proper G-spaces then X is a closed subset of Y and restriction of functions defines a G-equivariant $*$-homomorphism from $C_0(Y)$ to $C_0(X)$. This induces a homomorphism from $E_G(X, D)$ to $E_G(Y, D)$.

If $X \subseteq Y \subseteq \mathrm{e}G$ are G-compact proper G-spaces then under the restriction map from $E(\mathbb{C}, C^*(G, Y))$ to $E(\mathbb{C}, C^*(G, X))$ the unit class for Y maps to the unit class for X; consequently the assembly maps for the various G-compact subsets of $\mathrm{e}G$ are compatible and pass to the direct limit:

Definition 2.30. *The* (full) Baum-Connes assembly map *with coefficients in a separable G-C^*-algebra D is the map*

$$\mu : K^{top}(G, D) \to K(C^*(G, D))$$

which is obtained as the limit of the assembly maps of Definition 2.28 for G-compact subspaces $X \subset \mathrm{e}G$.

Definition 2.31. *The* reduced Baum-Connes assembly map *with coefficients in a separable G-C^*-algebra D is the map*

$$\mu_\lambda : K^{top}(G, D) \to K(C^*_\lambda(G, D))$$

obtained by composing the full Baum-Connes assembly map μ with the map from $K(C^*(G, D))$ to $K(C^*_\lambda(G, D))$ induced from the quotient mapping from $C^*(G, D)$ onto $C^*_\lambda(G, D)$.

Remark 2.12. If G is exact and if X is a G-compact proper G-space then there is a reduced assembly map

$$\mu\colon E_G(C_0(X), D) \to K(C^*_\lambda(G, D)),$$

defined by means of a composition

$$E_G(C_0(X), D) \xrightarrow{\text{descent}} E(C^*_\lambda(G, X), C^*_\lambda(G, D)) \xrightarrow{[p]} E(\mathbb{C}, C^*_\lambda(G, D))$$

involving the reduced descent functor of Section 2.9. The Baum-Connes assembly map μ_λ may then be equivalently defined as a direct limit of such maps.

2.15 Baum-Connes Conjecture

The following is known as the *Baum-Connes Conjecture with coefficients* (the 'coefficients' being of course the auxiliary C^*-algebra D).

Conjecture 2.1. Let G be a countable discrete group. The Baum-Connes assembly map

$$\mu_\lambda : K^{top}(G, D) \to K(C^*_\lambda(G, D)).$$

is an isomorphism for every separable G-C^*-algebra D.

Not a great deal is known about this conjecture. We shall prove one of the main results (which covers, for example, amenable groups) in the next section. Unfortunately, thanks to some recent constructions of Gromov, the Baum-Connes conjecture with coefficients appears to be false, in general. See Lecture 6.

In the next conjecture, which is the official *Baum-Connes conjecture* for discrete groups, the coefficient algebra B is specialized to $D = \mathbb{C}$ and $D = C_0(0,1)$. We shall use the notations $K_*^{top}(G)$ and $K_*(C^*_\lambda(G))$ to denote topological and C^*-algebra K-theory in these two cases (this of course is customary usage in K-theory).

Conjecture 2.2. Let G be a countable discrete group. The Baum-Connes assembly map

$$\mu_\lambda : K_*^{top}(G) \to K_*(C^*_\lambda(G)).$$

is an isomorphism.

Somewhat more is known about this conjecture, thanks largely to the remarkable work of Lafforgue [44, 62]. For example, the conjecture is proved for all hyperbolic groups (we shall define these in Lecture 5). What is especially interesting is that, going beyond discrete groups, the Baum-Connes conjecture has now been proved for all reductive Lie and p-adic groups (this is part of what Lafforgue accomplished using his Banach algebra version of bivariant K-theory, although by invoking a good deal of representation theory many cases here had been confirmed prior to Lafforgue's work). Unfortunately we shall not have the time to discuss either Lafforgue's work or the topic of K-theory for non-discrete groups.

At the present time, the major open question seems to be whether or not the Baum-Connes conjecture (with or without coefficients, according to one's degree of optimism) is true for discrete subgroups of connected Lie groups. Even the case of uniform lattices in semisimple groups remains open.

Considerably more is known about the *injectivity* of the Baum-Connes assembly map, and fortunately this is all that is required in some of the key applications of the conjecture to geometry and topology. We shall say more about injectivity in Lecture 5.

Remark 2.13. We shall discuss in Lecture 6 the reason for working with $C^*(G, D)$ in place of $C^*_\lambda(G, D)$.

2.16 The Conjecture for Finite Groups

The reader can check for himself that the Baum-Connes conjecture is true (in fact it is a tautology) for the trivial, one-element group. Next come the finite groups. Here the conjecture is a theorem, and it is basically equivalent to a well-known result of Green and Julg which identifies equivariant K-theory and the K-theory of crossed product algebras in the case of finite groups. See [23, 35]. What follows is a brief account of this.

Theorem 2.17 (Green-Julg). *Let G be a finite group and let D be a G-C^*-algebra. The Baum-Connes assembly map*

$$\mu : K^{top}(G, D) \to K(C^*(G, D))$$

is an isomorphism for every G-C^*-algebra D.

Remark 2.14. If G is finite then $C^*(G, D) = C^*_\lambda(G, D)$ for every D.

If G is a finite group then eG can be taken to be the one point space. So the theorem provides an isomorphism

$$E_G(\mathbb{C}, D) \xrightarrow[\cong]{\mu} E(\mathbb{C}, C^*(G, D)).$$

The unit projection $p \in C^*(G)$ which is described in Lemma 2.9 is the function $p(g) = 1/|G|$, which is the central projection in $C^*(G)$ corresponding to the trivial representation of G (it acts as the orthogonal projection onto the G-fixed vectors in any unitary representation of G).

Theorem 2.17 is proved by defining an inverse to the assembly map μ. For this purpose we note that $C^*(G, D)$ may be identified with a fixed point algebra,

$$C^*(G, D) \xrightarrow[\cong]{} [D \otimes \mathrm{End}(\ell^2(G))]^G,$$

by mapping d to $\sum_{g \in G} g \cdot d \otimes p_g$ (where p_g is the projection onto the functions supported on $\{g\}$) and by mapping g to $1 \otimes \rho(g)$, where ρ is the right regular representation (the fixed point algebra is computed using the left regular representation). The displayed $*$-homomorphism can be thought of as an equivariant $*$-homomorphism from $C^*(G, D)$, equipped with the trivial action of G, into $D \otimes \mathrm{End}(\ell^2(G))$. It induces a homomorphism

$$E(\mathbb{C}, C^*(G, D)) \longrightarrow E_G(\mathbb{C}, D \otimes \mathrm{End}(\ell^2(G))).$$

But the left hand side here is $K(C^*(G, D))$ and the right hand side is $K^{top}(G, D)$, and it is not difficult to check that the above map inverts the assembly map μ, as required. For details see [27, Thm. 11.1].

2.17 Proper Algebras

Theorem 2.17 has an important extension to the realm of infinite groups, involving the following notion:

Definition 2.32. *A G-C^*-algebra B is* proper *if there exists a locally compact proper G-space X and an equivariant $*$-homomorphism φ from $C_0(X)$ into the grading-degree zero part of the center of the multiplier algebra of B such that $\varphi[C_0(X)] \cdot B$ is norm-dense in B.*

Remark 2.15. We shall say that B, as in the definition, is *proper over* Z. Throughout the lecture we shall deal with proper algebras which are *separable*.

The notion of proper algebra is due essentially to Kasparov [38], in whose work proper algebras appear in connection with RKK-theory, a useful elaboration of KK-theory. We shall not develop RKK here, or even its E-theoretic counterpart. While this limits the amount of machinery we must introduce, it will also make some of the arguments in this and later lectures a little clumsier than they need be.

Examples 2.18 *If G is finite every G-C^*-algebra is proper over the one point space. If Z is a proper G-space then $C_0(Z)$ is a proper G-C^*-algebra. If B is proper over Z then, for every G-C^*-algebra D, the tensor product $B\widehat{\otimes}D$ is also proper.*

Exercise 2.5. Prove that if B is proper then $C^*(G,B) = C^*_\lambda(G,B)$.

A guiding principle is that the action of a group on a proper algebra is more or less the same thing as the action of a *finite* group on a C^*-algebra. With this in mind the following theorem should not be surprising.

Theorem 2.19. [27, Theorem 13.1] *Let G be a countable discrete group and let B be a proper G-C^*-algebra. The Baum-Connes assembly map*

$$\mu : K^{top}(G,B) \to K(C^*(G,B))$$

is an isomorphism.

Remark 2.16. Thanks to Exercise 2.5, the assembly map μ_λ into $K(C^*_\lambda(G,B))$ is an isomorphism as well.

The proof of Theorem 2.19 is not difficult, but with the tools we have to hand it is rather long. So we shall just give a quick outline. The following computation is key not just to the proof of Theorem 2.19 but also to a number of results in Lecture 5.

Proposition 2.8. [27, Lemma 12.11] *Let H be a finite subgroup of a countable group G and let W be a locally compact space equipped with an action of H by homeomorphisms. If D is any G-C^*-algebra there is a natural isomorphism*

$$E_H(C_0(W),D) \cong E_G(C_0(G\times_H W),D),$$

where on the left hand side D is viewed as an H-C^*-algebra by restriction of the G-action.

Proof. The space W is included into $G\times_H W$ as the open set $\{e\}\times W$, and as a result there is an H-equivariant map from $C_0(W)$ into $C_0(G\times_H W)$. Composition with this map defines a 'restriction' homomorphism

$$E_G(C_0(G\times_H W),D) \xrightarrow{\mathrm{Res}} E_H(C_0(W),D)$$

To construct an inverse, the important observation to make is that every H-equivariant asymptotic morphism from $C_0(W)$ into D extends uniquely to a G-equivariant asymptotic morphism from $C_0(G\times_H W)$ into $D\otimes\mathcal{K}(\ell^2(G/H))$. Decorating this construction with copies of $\mathcal{S}$ and $\mathcal{K}(\mathcal{H})$ we obtain an inverse map

$$E_H(C_0(W), D) \longrightarrow E_G(C_0(G\times_H W), D)$$

as required.

Proposition 2.8 has the following immediate application:

Lemma 2.10. *Let G be a countable group. If the assembly map*

$$\mu\colon K^{top}(G, B) \to K(C^*(G, B))$$

is an isomorphism for every G-C^*-algebra B which is proper over a G-compact space Z, then it is an isomorphism for every G-C^*-algebra.

Proof. Every proper algebra is a direct limit of G-C^*-algebras which are proper over G-compact spaces. Since K-theory commutes with direct limits (see Exercise 1.4), as does the crossed product functor, to prove the lemma it suffices to prove that the same is true for the functor $D \mapsto K^{top}(G, D)$. In view of the definition of $K^{top}(G, D)$ it suffices to prove that if Z is a G-compact proper G-simplicial complex then the functor $E_G(C_0(Z), D)$ commutes with direct limits. By a Mayer-Vietoris argument the proof of this reduces to the case where Z is a proper homogeneous space G/H. But here we have a sequence of isomorphisms

$$E_G(C_0(G/H), D) \cong E_H(\mathbb{C}, D) \cong K(C^*(G, D)),$$

the first by Proposition 2.8 and the second by Theorem 2.17. Since K-theory commutes with direct limits the lemma is proved.

Lemma 2.11. *Let G be a countable group. If the assembly map*

$$\mu\colon K^{top}(G, B) \to K(C^*(G, B))$$

is an isomorphism for every G-C^*-algebra B which is proper over a proper homogeneous space $Z = G/H$ then it is an isomorphism for every G-C^*-algebra which is proper over a G-compact proper G-space.

Proof. This is another Mayer-Vietoris argument, this time in the B-variable. Observe that if B is proper over Z then to each G-invariant open set U in Z there corresponds an ideal $J = C_0(U)\cdot B$ of B. Using this, together with the long exact sequences in E-theory and the five lemma, an induction argument can be constructed on the number of G-invariant open sets needed to cover Z, each of which admits a G-map to a proper homogeneous space.

The proof of Theorem 2.19 therefore reduces to the case where B is proper over some proper homogeneous space G/H. Observe now that if B is proper over G/H then B is a direct sum of ideals corresponding to the points of G/H, and the ideal B_e corresponding to eH is an H-C^*-algebra . The proof is completed by developing a variant of the isomorphism in Proposition 2.8, and producing a commuting diagram

$$\begin{array}{ccc} K^{top}(G,B) & \xrightarrow{\mu} & K(C^*(G,B)) \\ \cong\big\downarrow & & \big\downarrow\cong \\ K^{top}(H,B_e) & \underset{\cong}{\xrightarrow{\mu}} & K(C^*(H_e,B)). \end{array}$$

See [27, Chapter 12] for details.

2.18 Proper Algebras and the General Conjecture

The following simple theorem provides a strategy for attacking the Baum-Connes conjecture for general coefficient algebras. The theorem, or its extensions and relatives, is invoked in nearly all approaches to the Baum-Connes conjecture. As we shall see in Lecture 5 the theorem is particularly useful as a tool to prove results about the injectivity of the Baum-Connes map.

Theorem 2.20. *Let G be a countable discrete group. Suppose there exists a proper G-C^*-algebra B and morphisms $\beta \in E_G(\mathbb{C},B)$ and $\alpha \in E_G(B,\mathbb{C})$ in the equivariant E-theory category such that*

$$\alpha \circ \beta = 1 \in E_G(\mathbb{C},\mathbb{C}).$$

Then the Baum-Connes assembly map $\mu : K^{top}(G,D) \to K(C^*(G,D))$ is an isomorphism for every separable G-C^*-algebra D. If in addition G is exact then the reduced Baum-Connes assembly map

$$\mu_\lambda : K^{top}(G,D) \to K(C^*_\lambda(G,D))$$

is an isomorphism.

Proof. Let G be a countable discrete group, let D be a separable G-C^*-algebra and let α and β be as in the statement of the theorem. Consider the following diagram:

$$\begin{array}{ccc} K^{top}(G,\mathbb{C}\widehat{\otimes}D) & \xrightarrow{\mu} & K(C^*(G,\mathbb{C}\widehat{\otimes}D)) \\ \beta_*\big\downarrow & & \big\downarrow\beta_* \\ K^{top}(G,B\widehat{\otimes}D) & \underset{\cong}{\xrightarrow{\mu}} & K(C^*(G,B\widehat{\otimes}D)) \\ \alpha_*\big\downarrow & & \big\downarrow\alpha_* \\ K^{top}(G,\mathbb{C}\widehat{\otimes}D) & \underset{\mu}{\longrightarrow} & K(C^*(G,\mathbb{C}\widehat{\otimes}D)). \end{array}$$

The horizontal maps are the assembly maps; the vertical maps are induced from E-theory classes $\beta \otimes 1 \in E_G(\mathbb{C}\widehat{\otimes}D, B\widehat{\otimes}D)$ and $\alpha \otimes 1 \in E_G(B\widehat{\otimes}D, \mathbb{C}\widehat{\otimes}D)$. The diagram is commutative. Since the C^*-algebra B is proper, so is the tensor product $B\widehat{\otimes}D$ and therefore by the Theorem 2.19 the middle horizontal map is an isomorphism. By assumption, the compositions of the vertical maps on the left, and hence also on the right hand side are the identity. It follows that the top horizontal map is an isomorphism too. The statement concerning reduced crossed products is proved in exactly the same way.

2.19 Crossed Products by the Integers

In this section we shall apply the approach outlined in the previous section to just about the simplest example possible beyond finite groups: the free abelian group $G = \mathbb{Z}^n$. What follows will serve as a model for the more elaborate constructions in the next lecture. For this reason it might be worth the reader's while to study the present case quite carefully.

Let G act by translations on $\mathbb{R}^n$ in the usual way and then let G act on the graded C^*-algebra $\mathcal{C}(\mathbb{R}^n)$ that we introduced in Lecture 1 by $(g \cdot f)(v) = f(g \cdot v)$.

Exercise 2.6. With this action of the free abelian group $\mathbb{Z}^n$, the C^*-algebra $\mathcal{C}(\mathbb{R}^n)$ is proper.

We are going to produce a factorization

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\beta"'] \arrow[rr, bend left, "1"] & \mathcal{C}(\mathbb{R}^n) \arrow[r, "\alpha"'] & \mathbb{C} \end{tikzcd}$$

in $\mathbb{Z}^n$-equivariant E-theory. The elements α and β are very small modifications of the objects we defined in Lecture 1 while studying Bott Periodicity.

Definition 2.33. *Denote by $\beta\colon \mathcal{S} \to \mathcal{C}(\mathbb{R}^n)$ the $*$-homomorphism that was introduced in Definition 1.26, and for $t \geq 1$ denote by $\beta_t\colon \mathcal{S} \to \mathcal{C}(\mathbb{R}^n)$ the $*$-homomorphism $\beta_t(f) = \beta(f_t)$, where $f_t(x) = f(t^{-1}x)$.*

Thus $\beta_t(f) = f(t^{-1}C)$, where C is the Clifford operator introduced in Lecture 1.

Lemma 2.12. *The asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{C}(\mathbb{R}^n)$ given by the above family of $*$-homomorphisms $\beta_t\colon \mathcal{S} \to \mathcal{C}(\mathbb{R}^n)$ is $\mathbb{Z}^n$-equivariant.*

Proof. We must show that if $f \in \mathcal{S}$ and $g \in \mathbb{Z}^n$ then

$$\lim_{t\to\infty} \|f(t^{-1}C) - g(f(t^{-1}C)))\| = 0.$$

Since the set of all $f \in \mathcal{S}$ for which this holds (for all g) is a C^*-subalgebra of $\mathcal{S}$ it suffices to prove the limit formula for the generators $f = (x \pm i)^{-1}$ of $\mathcal{S}$. For these we have

$$\begin{aligned}\|f(t^{-1}C) - g(f(t^{-1}C))\| &= \|(t^{-1}C \pm i)^{-1} - (t^{-1}g(C) \pm i)^{-1}\| \\ &\leq t^{-1}\|C - g(C)\|\end{aligned}$$

by the resolvent identity. Since the Clifford algebra-valued function $C - g(C)$ is bounded on $\mathbb{R}^n$ the lemma is proved.

Definition 2.34. *Denote by $\beta \in E_{\mathbb{Z}^n}(\mathbb{C}, \mathcal{C}(\mathbb{R}^n))$ the class of the asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{C}(\mathbb{R}^n)$.*

Definition 2.35. *If $g \in \mathbb{Z}^n$ and $v \in \mathbb{R}^n$, and if $s \in [0,1]$, then denote by $g \cdot_s v$ the translation of v by $sg \in \mathbb{R}^n$. Denote by $g \cdot_s f$ the corresponding action of $g \in \mathbb{Z}^n$ on elements of the C^*-algebra $\mathcal{C}(\mathbb{R}^n)$ and also on operators on the Hilbert space $\mathcal{H}(\mathbb{R}^n)$ that was introduced in Definition 1.27.*

To define the class $\alpha \in E_{\mathbb{Z}^n}(\mathcal{C}(\mathbb{R}^n), \mathbb{C})$ that we require we shall use the asymptotic morphism $\alpha\colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathbb{R}^n) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathbb{R}^n))$ that we defined in Proposition 1.5, but we shall interpret it as an *equivariant* asymptotic morphism in the following way:

Lemma 2.13. *If $f\widehat{\otimes}h \in \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathbb{R}^n)$, $g \in \mathbb{Z}^n$, and $t \in [1,\infty)$ then*

$$\lim_{t\to\infty} \|\alpha_t(f\widehat{\otimes}g\cdot h) - g \cdot_{t^{-1}} \alpha_t(f\widehat{\otimes}h)\| = 0.$$

Proof. The Dirac operator D is translation invariant, and so $g \cdot_{t^{-1}} f(t^{-1}D) = f(t^{-1}D)$ for all t. But $g \cdot_{t^{-1}} M_{h_t} = M_{(g\cdot h)_t}$ for all t. The lemma therefore follows from the formula

$$\alpha_t(f\widehat{\otimes}h) = f(t^{-1}D)M_{h_t}$$

for the asymptotic morphism α.

Definition 2.36. *Denote by $\alpha \in E_{\mathbb{Z}^n}(\mathcal{C}(\mathbb{R}^n), \mathbb{C})$ the E-theory class of the equivariant asymptotic morphism $\alpha\colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathbb{R}^n) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathbb{R}^n))$, where $\mathcal{K}(\mathcal{H}(\mathbb{R}^n))$ is equipped with the family of actions $(g,k) \mapsto g \cdot_{t^{-1}} k$ (compare Remark 2.6).*

Proposition 2.21 *Continuing with the notation above, $\alpha \circ \beta = 1 \in E_{\mathbb{Z}^n}(\mathbb{C}, \mathbb{C})$.*

Proof. Let $s \in [0,1]$ and denote by $\mathcal{C}_s(\mathbb{R}^n)$ the C^*-algebra $\mathcal{C}(\mathbb{R}^n)$, but with the scaled $\mathbb{Z}^n$-action $(g,h) \mapsto g \cdot_s h$. The algebras $\mathcal{C}_s(\mathbb{R}^n)$ form a continuous field of $\mathbb{Z}^n$-C^*-algebras over the unit interval (since the algebras are all the same this just means that the $\mathbb{Z}^n$-actions vary continuously). Denote by $\mathcal{C}_{[0,1]}(\mathbb{R}^n)$ the $\mathbb{Z}^N$-C^*-algebra of continuous sections of this field (namely the continuous functions from $[0,1]$ into $\mathcal{C}(\mathbb{R}^n)$, equipped with the $\mathbb{Z}^N$-action $(g \cdot h)(s) = g \cdot_s h(s))$. In a similar way, form the continuous field of $\mathbb{Z}^N$-C^*-algebras $\mathcal{K}_s(\mathcal{H}(\mathbb{R}^n))$ and denote by $\mathcal{K}_{[0,1]}(\mathcal{H}(\mathbb{R}^n))$ the $\mathbb{Z}^N$-C^*-algebra of continuous sections. With this notation, what we want to prove is that the composition

$$\mathbb{C} \xrightarrow{\beta} \mathcal{C}_1(\mathbb{R}^n) \xrightarrow{\alpha} \mathbb{C}$$

is the identity in equivariant E-theory.

The asymptotic morphism $\alpha\colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathbb{R}^n) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathbb{R}^n))$ induces an asymptotic morphism

$$\bar{\alpha}\colon \mathcal{S}\widehat{\otimes}\mathcal{C}_{[0,1]}(\mathbb{R}^n) \dashrightarrow \mathcal{K}_{[0,1]}(\mathcal{H}(\mathbb{R}^n)) \ ,$$

and similarly the asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{C}(\mathbb{R}^n)$ determines an asymptotic morphism

$$\bar{\beta}\colon \mathcal{S} \dashrightarrow \mathcal{C}_{[0,1]}(\mathbb{R}^n)$$

by forming the tensor product of β with the identity on $C[0,1]$ and then composing with the inclusion $\mathcal{S} \subseteq \mathcal{S}[0,1]$ as constant functions. Consider then the diagram of equivariant E-theory morphisms

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\bar{\beta}"] \arrow[d, "="'] & \mathcal{C}_{[0,1]}(\mathbb{R}^n) \arrow[r, "\bar{\alpha}"] \arrow[d, "\varepsilon_s"] & C[0,1] \arrow[d, "\varepsilon_s"] \\ \mathbb{C} \arrow[r, "\beta"'] & \mathcal{C}_s(\mathbb{R}^n) \arrow[r, "\alpha"'] & \mathbb{C} \end{tikzcd} \ ,$$

where ε_s denotes the element induced from evaluation at $s \in [0,1]$. Observe that ε_s is an isomorphism in equivariant E-theory, for every s (indeed, ε_s, considered as a $*$-homomorphism, is an equivariant homotopy equivalence). Set $s = 0$. In this case the bottom composition is the identity element of $E_{\mathbb{Z}^n}(\mathbb{C},\mathbb{C})$. This is because when $s = 0$ the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ is trivial and the asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{C}_0(\mathbb{R}^n)$ is homotopic to the (trivially equivariant) $*$-homomorphism $\beta\colon \mathcal{S} \to \mathcal{C}(\mathbb{R}^n)$ of Definition 1.26. So the required formula $\alpha \circ \beta = 1$ follows from Proposition 2.4. Since the bottom composition in the diagram is the identity it follows that the top composition is an isomorphism too.[11] Now set $s = 1$. Since, as we just showed, the top composition in the diagram is the identity, it follows that the bottom composition is the identity too. The proposition is proved.

3 Groups with the Haagerup Property

3.1 Affine Euclidean Spaces

Recall that we are using the term *Euclidean vector space* to refer to a real vector space equipped with a positive-definite inner product. In this lecture we shall be studying Euclidean spaces of possibly infinite dimension.

Definition 3.1. *An* affine Euclidean space *is a set E equipped with a simply-transitive action of the additive group underlying a Euclidean vector space V. An* affine subspace *of E is an orbit in E of a vector subspace of V. A subset X of E* generates *E if the smallest affine subspace of E which contains X is E itself.*

[11] It is the identity once $C[0,1]$ is identified with $\mathbb{C}$ via evaluation at any s, or equivalently once $C[0,1]$ is identified with $\mathbb{C}$ via the inclusion of $\mathbb{C}$ into $C[0,1]$ as constants.

Remark 3.1. Note that even if E is infinite-dimensional we are not assuming any completeness here (and moreover affine subspaces need not be closed).

Example 3.1. Every Euclidean vector space is of course an affine Euclidean space (over itself).

The following prescription makes E into a metric space.

Definition 3.2. *Let E be an affine Euclidean space over the Euclidean vector space V. If $e_1, e_2 \in E$, and if v is the unique vector in V such that $e_1 + v = e_2$, then we define the* distance *between e_1 and e_2 to be $d(e_1, e_2) = \|v\|$.*

Let Z be a subset of an affine Euclidean space E and let $b\colon Z \times Z \to \mathbb{R}$ be the square of the distance function: $b(z_1, z_2) = d^2(z_1, z_2)$. This function has the following properties:

(a) $b(z, z) = 0$, for all $z \in Z$,
(b) $b(z_1, z_2) = b(z_2, z_1)$, for all $(z_1, z_2) \in Z \times Z$, and
(c) for all n, all $z_1, \ldots, z_n \in Z$, and all $a_1, \ldots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i = 0$,

$$\sum_{i,j=1}^{n} a_i b(z_i, z_j) a_j \leq 0.$$

(To prove the inequality, identify E with V and identify the sum with the quantity $-2\| \sum_{i=1}^n a_i z_i \|^2$.)

Proposition 3.1. *Let Z be a set and let $b\colon Z \times Z \to \mathbb{R}$ be a function with the above three properties. There is a map $\Phi\colon Z \to E$ of Z into an affine Euclidean space such that the image of f generates E and such that*

$$b(z_1, z_2) = d^2(\Phi(z_1), \Phi(z_2)),$$

for all $z_1, z_2 \in Z$. If $\Phi'\colon Z \to E'$ is another such map into another Euclidean space then there is a unique isometry $h\colon E \to E'$ such that $h(\Phi(z)) = \Phi'(z)$, for every $z \in Z$.

Proof. Denote by $\mathbb{R}_0[Z]$ the vector space of finitely supported, real-valued functions on Z which sum to zero:

$$\mathbb{R}_0[Z] = \{ f \in \mathbb{R}[Z] \ : \ \sum f(z) = 0 \}$$

If we equip $\mathbb{R}_0[Z]$ with the positive semidefinite form

$$\langle f_1, f_2 \rangle = -\frac{1}{2} \sum_{z_1, z_2 \in \mathbb{Z}} f(z_1) b(z_1, z_2) f(z_2)$$

then the set of all $f \in \mathbb{R}_0[Z]$ for which $\langle f, f \rangle = 0$ is a vector subspace $\mathbb{R}_0^0[Z]$ of $\mathbb{R}_0[Z]$ (this is thanks to the Cauchy-Schwarz inequality) and the quotient

$$V = \mathbb{R}_0[Z]/\mathbb{R}_0^0[Z]$$

has the structure of a Euclidean vector space. Consider now the set of all finitely supported functions on Z which sum to 1. Let us say that two functions in this set are equivalent if their difference belongs to $\mathbb{R}_0^0[Z]$. The set of equivalence classes is then an affine Euclidean space E over V. If $\Phi\colon Z \to E$ is defined by $\Phi(z) = \delta_z$ then $d^2(\Phi(z_1), \Phi(z_2)) = b(z_1, z_2)$, as required. If $\Phi'\colon Z \to E'$ is another such map then the unique isometry h as in the statement of the lemma is given by the formula

$$h(f) = \sum f(z)\Phi'(z)$$

(note that in an affine space one can form linear combinations so long as the coefficients sum to 1).

Exercise 3.1. Justify the parenthetical assertion at the end of the proof. Prove that if h is an isometry of affine Euclidean spaces then

$$\sum a_i = 1 \quad \Rightarrow \quad h(\sum a_i e_i) = \sum a_i h(e_i).$$

This completes the uniqueness argument above.

Definition 3.3. *Let Z be a set. A function $b\colon Z \times Z \to \mathbb{R}$ is a* negative-type kernel *if b has the properties (a), (b) and (c) listed prior to Proposition 3.1.*

Thus, according to the proposition, maps into affine Euclidean spaces are classified, up to isometry, by negative-type kernels.

3.2 Isometric Group Actions

Let E be an affine Euclidean space and suppose that a group G acts on E by isometries. If e is any point of E then the function $g \mapsto g \cdot e$ maps G into E, and there is an associated negative-type function

$$b(g_1, g_2) = d^2(g_1 \cdot e, g_2 \cdot e).$$

Since G acts by isometries the function b is G-invariant, in the sense that

$$b(g_1, g_2) = b(gg_1, gg_2), \qquad \forall g, g_1, g_2 \in G,$$

and as a result it is determined by the one-variable function $b(g) = b(e, g)$, which is a negative-type function on G in the sense of the following definition.

Definition 3.4. *Let G be a group. A function $b\colon G \to \mathbb{R}$ is a* negative-type function *on G if it has the following three properties:*

(a) $b(e) = 0$,
(b) $b(g) = b(g^{-1})$, *for all* $g \in G$, *and*

(c) $\sum_{i,j=1}^{n} a_i b(g_i^{-1} g_j) a_j \leq 0$, *for all* n, *all* $g_1, \ldots, g_n \in X$, $s_i \in G$ *and all* $a_1, \ldots a_n \in \mathbb{R}$ *such that* $\sum_{i=1}^{n} a_i = 0$.

Proposition 3.2. *Let* G *be a set and let* b *be a negative-type function on* G. *There is an isometric action of* G *on an affine Euclidean space* E *and a point* $e \in E$ *such that the orbit of* e *generates* E, *and such that*

$$b(g) = d^2(e, g \cdot e),$$

for all $g \in G$.

Proof. Let E be the affine space associated to the kernel $b(g_1, g_2) = b(g_1^{-1} g_2)$, as in the statement of Proposition 3.1. There is therefore a map from G into E, which we shall write as $g \mapsto \bar{g}$, whose image generates E, and for which

$$b(g_1, g_2) = d^2(\bar{g}_1, \bar{g}_2).$$

Fix $h \in G$ and consider now the map $g \mapsto \overline{hg}$. Since

$$b(g_1, g_2) = d^2(\bar{g}_1, \bar{g}_2) = d^2(\overline{hg_1}, \overline{hg_2})$$

it follows from the uniqueness part of Proposition 3.1 that there is a (unique) isometry of E mapping $\bar{g}$ to $\overline{hg}$. The map which associates to $h \in G$ this isometry is the required action, and $e = \bar{e}$ is the required point in E.

Remark 3.2. There is also a uniqueness assertion: if E' is a second affine Euclidean space equipped with an isometric G-action, and if $e' \in E$ is a point such that $b(g) = d(e', g \cdot e')^2$, for all $g \in G$, then there is a G-equivariant isometry $h \colon E \to E'$ such that $h(e) = e'$.

Remark 3.3. Proposition 3.2 is of course reminiscent of the GNS construction in C^*-algebra theory, which associates to each state of a C^*-algebra a Hilbert space representation and a unit vector in the representation space.

Exercise 3.2. Let E be an affine Euclidean space over the Euclidean vector space V. Suppose that a group G acts on E by isometries. Show that there is a linear representation π of G by orthogonal transformations on V such that

$$g \cdot (e + v) = g \cdot e + \pi(g) v,$$

for all $g \in G$, all $e \in E$, and all $v \in V$.

Exercise 3.3. According to the previous exercise, if V is viewed as an affine space over itself then for every isometric action of G on V there is a linear representation π of G by orthogonal transformations on V such that

$$g \cdot v = g \cdot 0 + \pi(g) v.$$

Show that for every $s \in [0, 1]$ the ‘scaled’ actions $g \cdot_s v = s(g \cdot 0) + \pi(g)(v)$ are also isometric actions of G on E.

3.3 The Haagerup Property

Definition 3.5. *Let G be a countable discrete group. An isometric action of G on an affine Euclidean space E is* metrically proper *if for some (and hence for every) point e of E,*

$$\lim_{g\to\infty} d(e, g\cdot e) = \infty.$$

In other words,an action is metrically proper if for every $R > 0$ there are only finitely many $g \in G$ such that $d(e, g\cdot e) \leq R$.

Definition 3.6. *A countable discrete group G has the Haagerup property if it admits a metrically proper isometric action on an affine Euclidean space.*

In view of Proposition 3.2, the Haagerup property may characterized as follows:

Proposition 3.3. *A group G has the Haagerup property if and only if there exists on G a proper, negative-type function $b\colon G \to \mathbb{R}$ (that is, a negative-type function for which the inverse image of each bounded set of real numbers is a finite subset of G).* □

Groups with the Haagerup property are also called (by Gromov [5]) *a-T-menable*. This terminology is justified by the following two results. The first is due to Bekka, Cherix and Valette [8].

Theorem 3.1. *Every countable amenable group has the Haagerup property.*

Proof. A function $\varphi\colon G \to \mathbb{C}$ is said to be *positive-definite* if $\varphi(e) = 1$,[12] if $\varphi(g) = \varphi(g^{-1})$, and if for all $g_1, \ldots, g_n \in G$, and all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} \bar{\lambda}_i \varphi(g_i^{-1} g_j)\lambda_j \geq 0.$$

Observe that if φ is positive-definite then $1 - \operatorname{Re}\varphi$ is a negative-type function. Now one of the many characterizations of amenability is that G is amenable if and only if there exists a sequence $\{\varphi_n\}$ of finitely supported positive-definite functions on G which converges pointwise to the constant function 1. Given such a sequence we can find a subsequence such that the series $\sum_k (1 - \operatorname{Re}\varphi_{n_k})$ converges at every point of G. The limit is a proper, negative-type function.

The next result is essentially due to Delorme [18].

Theorem 3.2. *If G is a discrete group with Kazhdan's property T, and if G has in addition the Haagerup property, then G is finite.*

[12] This normalization is not always incorporated into the definition, but it is convenient here. We should also remark that the next condition $\varphi(g) = \varphi(g^{-1})$ is actually implied by the condition $\sum_{i,j=1}^{n} \bar{\lambda}_i \varphi(g_i^{-1} g_j)\lambda_j \geq 0$.

Proof. If G has property T then every isometric action of G on an affine Hilbert space has a fixed point (this is Delorme's theorem).[13] But if an isometric action has a fixed point it cannot be metrically proper, unless G is finite.

Remark 3.4. The reader is referred to [17] for a comprehensive introduction to the theory of property T groups. We shall also return to the subject in the last lecture.

Various classes of discrete groups are known to have the Haagerup property. Here is an incomplete list.

- Amenable groups (see above),
- Finitely generated free groups [30], or more generally, groups which act properly on locally finite trees.
- Coxeter groups [9],
- Discrete subgroups of $SO(n,1)$ and $SU(n,1)$ [56, 55],
- Thompson's groups [20, 51].

For more information about the Haagerup property consult [12].

3.4 The Baum-Connes Conjecture

The main objective of this lecture is to discuss the proof of the following theorem:

Theorem 3.3. *Let G be a countable discrete group with the Haagerup property. There exists a proper G-C^*-algebra B and E_G-theory elements $\alpha \in E_G(B,\mathbb{C})$ and $\beta \in E_G(\mathbb{C},B)$ such that $\alpha \circ \beta = 1 \in E_G(\mathbb{C},\mathbb{C})$.*

Thanks to the theory developed in the last lecture this has the following consequence:

Corollary 3.1. *Let G be a countable discrete group with the Haagerup property and let D be a G-C^*-algebra. The maximal Baum-Connes assembly map with coefficients in D is an isomorphism. Moreover if G is exact then the reduced Baum-Connes assembly map with coefficients in D is also an isomorphism*

Remark 3.5. The theorem and its corollary are also true for locally compact groups with the Haagerup property.

Remark 3.6. In fact the final conclusion is known to hold whether or not G is exact, but the proof involves supplementary arguments which we shall not develop here. In any case, perhaps the most striking application of the corollary is to amenable groups, and here of course the full and reduced assembly maps are one and the same (since the full and reduced crossed product C^*-algebras are one and the same).

In connection with the last remark it is perhaps worth noting that the following problem remains unsolved:

Problem 3.4 *Is every countable discrete group with the Haagerup property C^*-exact?*

[13] In fact the converse is true as well.

3.5 Proof of the Main Theorem, Part One

Let E be an affine Euclidean space equipped with a metrically proper, isometric action of a countable group G. In this section we shall build from E a proper G-C^*-algebra $\mathcal{A}(E)$. In the next section we shall construct equivariant E-theory elements α and β, as in Theorem 3.3, and in Section 3.7 we shall prove that $\alpha \circ \beta = 1$.

Notation 3.5 *From here on we shall fix an affine Euclidean space E over a Euclidean vector space V. We shall be working extensively with* finite-dimensional *affine subspaces of E, and we shall denote these by E_a, E_b and so on. We shall denote by V_a the vector subspace of V corresponding to the finite-dimensional affine subspace E_a. If $E_a \subseteq E_b$ then we shall denote by V_{ba} the* orthogonal complement *of E_a in E_b. This is the orthogonal complement of V_a in V_b. Note that*

$$E_b = V_{ba} + E_a,$$

and that this is a direct sum decomposition in the sense that every point of E_b has a unique decomposition $e_b = v_{ba} + e_a$.

The following definition extends to affine spaces a definition we previously made for linear spaces. The change is only very minor.

Definition 3.7. *Let E_a be a finite-dimensional affine Euclidean subspace of E. Let $\mathcal{C}(E_a) = C_0(E_a, \mathrm{Cliff}(V_a))$.*

Here is the counterpart of Proposition 1.13:

Lemma 3.1. *Let $E_a \subseteq E_b$ be a nested pair of finite-dimensional subspaces of E. The correspondence $h \leftrightarrow h_1 \widehat{\otimes} h_2$, where $h(v + e) = h_1(v)h_2(e)$ determines an isomorphism of graded C^*-algebras*

$$\mathcal{C}(E_b) \cong \mathcal{C}(V_{ba}) \widehat{\otimes} \mathcal{C}(E_a). \qquad \square$$

In Lecture 1 we made extensive use of the Clifford operator C. Recall that this was the function $C(v) = v$ from the Euclidean vector space V into the Clifford algebra $\mathrm{Cliff}(V)$. In the present context of affine spaces the Cliiford operator is not generally available since to define it we have to identify affine spaces with their underlying vector spaces, and we want to avoid doing this, at least for now. But we shall work with Clifford operators associated to various vector spaces which appear as orthogonal complements.

The following is a minor variation on Definition 1.26.

Definition 3.8. *Let V_a be a finite-dimensional linear subspace of V and denote by C_a the corresponding Clifford operator. Define a $*$-homomorphism*

$$\beta_a \colon \mathcal{S} \to \mathcal{S} \widehat{\otimes} \mathcal{C}(\mathcal{V}_{\dashv})$$

by the formula

$$\beta_a(f) = f(X \widehat{\otimes} 1 + 1 \widehat{\otimes} C_a).$$

Remark 3.7. The definition uses the language of unbounded multipliers. An alternative formulation, using the 'comultiplication' Δ, is that β_a is the composition

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S}\widehat{\otimes}\mathcal{S} \xrightarrow{1\widehat{\otimes}\beta} \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{V}_{\dashv}),$$

where $\beta\colon \mathcal{S} \to \mathcal{C}(\mathcal{V}_{\dashv})$ is the $*$-homomorphism $\beta(f) = f(C_a)$ of Definition 1.26.

We are now going to construct a C^*-algebra $\mathcal{A}(E)$ as a direct limit of C^*-algebras $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$ associated to finite-dimensional affine subspaces E_a of E.

Definition 3.9. *Let $E_a \subseteq E_b$ be a nested pair of finite-dimensional affine subspaces of E. Define a $*$-homomorphism*

$$\beta_{b,a} : \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \to \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\lfloor})$$

by using the identification $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\lfloor}) \cong \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{V}_{\lfloor\dashv})\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$ and the formula

$$\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \ni \ \{\widehat{\otimes}\langle \longmapsto \beta_{\lfloor\dashv}(\{)\widehat{\otimes}\langle \ \in \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{V}_{\lfloor\dashv})\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \cong \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\lfloor}),$$

where $\beta_{ba}\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{V}_{\lfloor\dashv})$ is the $*$-homomorphism of Definition 3.8.

Lemma 3.6 *Let $E_a \subseteq E_b \subseteq E_c$ be finite-dimensional affine subspaces of E. We have $\beta_{c,b} \circ \beta_{b,a} = \beta_{c,a}$.*

Proof. Compute using the generators $u(x) = e^{-x^2}$ and $v(x) = xe^{-x^2}$ of $\mathcal{S}$.

As a result the graded C^*-algebras $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$, where E_a ranges over the finite-dimensional affine subspaces of E, form a directed system, as required, and we can make the following definition:

Definition 3.10. *Let E be an affine Euclidean space. The C^*-*algebra of E*, denoted $\mathcal{A}(E)$, is the direct limit C^*-algebra*

$$\mathcal{A}(E) = \varinjlim_{\substack{E_a \subseteq E \\ \textit{fin. dim.} \\ \textit{affine sbsp.}}} \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}).$$

An action of G by isometries on E makes $\mathcal{A}(E)$ into a G-C^*-algebra. To see this, first define $*$-isomorphisms

$$g_{**}\colon \mathcal{C}(E_a) \to \mathcal{C}(gE_a)$$

by $(g_{**}f)(e) = g_*((f(g^{-1}e))$, where here $g_*\colon \operatorname{Cliff}(V_a) \to \operatorname{Cliff}(gV_a)$ is induced from the linear isometry of V associated to $g\colon E \to E$ (see Exercise 3.2).

Lemma 3.7 *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_\dashv) \arrow[r, "\beta_{b,a}"] \arrow[d, "1\widehat{\otimes}g_{**}"'] & \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_\lfloor) \arrow[d, "1\widehat{\otimes}g_{**}"] \\ \mathcal{S}\widehat{\otimes}\mathcal{C}(\}\mathcal{E}_\dashv) \arrow[r, "\beta_{gb,ga}"'] & \mathcal{S}\widehat{\otimes}\mathcal{C}(\}\mathcal{E}_\lfloor). \end{tikzcd}$$

□

The lemma asserts that the maps g_{**} are compatible with the maps in the directed system which is used to define $\mathcal{A}(E)$. Consequently, we obtain a map g_{**} on the direct limit. In this way $\mathcal{A}(E)$ is made into a G-C^*-algebra, as required.

Theorem 3.8. *Let E be an affine Euclidean space equipped equipped with a metrically proper action of a countable discrete group G. Then the C^*-algebra $\mathcal{A}(E)$ is a proper G-C^*-algebra.*

Proof. Denote by $\mathcal{Z}(E_a)$ the grading-degree zero part of the center of the C^*-algebra $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_\dashv)$. It is isomorphic to the algebra of continuous functions, vanishing at infinity, on the locally compact space $[0,\infty) \times E_a$. The linking map $\beta_{b,a}$ embeds $\mathcal{Z}(E_a)$ into $\mathcal{Z}(E_b)$, and so we can form the direct limit $\mathcal{Z}(E)$, which is a C^*-subalgebra of $\mathcal{A}(E)$, and is contained in the grading-degree zero part of the center of $\mathcal{A}(E)$ (in fact it is the entire degree zero part of the center). The C^*-subalgebra $\mathcal{Z}(E)$ has the property that $\mathcal{Z}(E) \cdot \mathcal{A}(E)$ is dense in $\mathcal{A}(E)$. The Gelfand spectrum of $\mathcal{Z}(E)$ is the locally compact space $Z = [0,\infty) \times \overline{E}$, where $\overline{E}$ is the metric space completion of E and Z is given the weakest topology for which the projection to $\overline{E}$ is weakly continuous[14] and the function $t^2 + d^2(e_0, e)$ is continuous, for some (hence any) fixed $e_0 \in E$. If G acts metrically properly on V then the induced action on the locally compact space Z is proper.

Remark 3.8. The above elegant argument is due to G. Skandalis.

3.6 Proof of the Main Theorem, Part Two

In this section we shall assume that E is a *countably* infinite-dimensional affine Euclidean space on which G acts by isometries (this simplifies one or two points of our presentation). For later purposes it will be important to work with actions which are not necessarily proper. Note however that if G has the Haagerup property then G will act properly and isometrically on some countably infinite-dimensional affine space E.

We are going to construct classes $\alpha \in E_G(\mathcal{A}(E), \mathbb{C})$ and $\beta \in E_G(\mathbb{C}, \mathcal{A}(E))$. We shall begin with the construction of β, and for this purpose we fix a point $e_0 \in E$.

[14] Observe that $\overline{E}$ is an affine space over the Hilbert space $\overline{V}$; by identifying $\overline{E}$ as an orbit of $\overline{V}$ we can transfer the weak topology of the Hilbert space $\overline{V}$ to $\overline{E}$.

This point is, by itself, an affine subspace of E, and there is therefore an inclusion $*$-homomorphism

$$\beta\colon \mathcal{S} \to \mathcal{A}(\mathcal{E}).$$

The image of β lies in all those subalgebras $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$ for which $e_0 \in E_a$, and considered as a map into $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$ the $*$-homomorphism β is given by the formula

$$\beta\colon f \mapsto f(C_{a,0}),$$

where $C_{a,0}\colon E_a \to \mathrm{Cliff}(V_a)$ is defined by $C_{a,0}(e) = e - e_0 \in V_a$.

Lemma 3.2. *If $\beta\colon \mathcal{S} \dashrightarrow \mathcal{A}(\mathcal{E})$ is the asymptotic morphism defined by*

$$\beta_t(f) = \beta(f_t),$$

where $f_t(x) = f(t^{-1}x)$, then β is G-equivariant.

Proof. We must show that if e_0 and e_1 are two points in a finite-dimensional affine space E_a, then for every $f \in \mathcal{S}$,

$$\lim_{t\to\infty} \|f(t^{-1}C_{a,0}) - f(t^{-1}C_{a,1})\| = 0,$$

where $C_{a,0}$ is as above and similarly $C_{a,1}(e) = e - e_1$. It suffices to compute the limit for the functions $f(x) = (x \pm i)^{-1}$. For these one has

$$\|f(t^{-1}C_{a,0}) - f(t^{-1}C_{a,1})\| = t^{-1}\|C_{a,0} - C_{a,1}\| = t^{-1}d(e_0, e_1).$$

The proof is complete.

Definition 3.11. *The element $\beta \in E_G(\mathbb{C}, \mathcal{A}(E))$ is the E-theory class of the equivariant asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{A}(\mathcal{E})$ defined by $\beta_t\colon f \mapsto \beta(f_t)$.*

The definition of α is a bit more involved. It will be the E-theory class of an asymptotic morphism

$$\alpha\colon \mathcal{A}(E) \dashrightarrow \mathcal{K}(\mathcal{H}(E)),$$

and our first task is to associate a Hilbert space $\mathcal{H}(E)$ to the infinite-dimensional affine Euclidean space E. We begin by broadening Definition 1.27 to the context of affine spaces.

Definition 3.12. *Let E_a be a finite-dimensional affine subspace of E, with associated linear subspace V_a. The Hilbert space of E_a is the space of square integrable* $\mathrm{Cliff}(V_a)$*-valued functions on E_a:*

$$\mathcal{H}(E_a) = L^2(E_a, \mathrm{Cliff}(V_a)).$$

This is a graded Hilbert space, with grading inherited from that of $\mathrm{Cliff}(V_a)$.

The following is the Hilbert space counterpart of Lemma 3.1.

Lemma 3.3. *Let $E_a \subset E_b$ be a nested pair of finite-dimensional subspaces of the affine space E and let V_{ba} be the orthogonal complement of E_a in E_b. The correspondence $h \leftrightarrow h_1 \widehat{\otimes} h_2$, where $h(v+e) = h_1(v)h_2(e)$ determines an isomorphism of graded Hilbert spaces $\mathcal{H}(E_b) \cong \mathcal{H}(V_{ba}) \widehat{\otimes} \mathcal{H}(E_a)$.* $\square$

Following the same path that we took in the last section, the next step is to assemble the spaces $\mathcal{H}(E_a)$ into a directed system.

Definition 3.13. *If W is a finite-dimensional Euclidean vector space V then the* basic vector $f_W \in \mathcal{H}(W)$ *is defined by*

$$f_W(w) = \pi^{-\frac{1}{4}\dim(W)}\, e^{-\frac{1}{2}\|w\|^2}.$$

Thus f_W maps $w \in W$ to the multiple $\pi^{-\frac{1}{4}\dim(W)}\, e^{-\frac{1}{2}\|w\|^2}$ of the identity element in $\mathrm{Cliff}(W)$.

Remark 3.9. The constant $\pi^{-\frac{1}{4}\dim(W)}$ is chosen so that $\|f_W\| = 1$.

Using the basic vectors $f_{ba} \in \mathcal{H}(V_{ba})$ we can organize the Hilbert spaces $\mathcal{H}(E_a)$ into a directed system as follows.

Definition 3.14. *If $E_a \subseteq E_b$ then define an isometry of graded Hilbert spaces $V_{ba} \colon \mathcal{H}(E_a) \to \mathcal{H}(E_b)$ by*

$$\mathcal{H}(E_a) \ni f \longmapsto f_{ba} \widehat{\otimes} f \in \mathcal{H}(V_{ba}) \widehat{\otimes} \mathcal{H}(E_a) \cong \mathcal{H}(E_b).$$

Lemma 3.9 *Let $E_a \subseteq E_b \subseteq E_c$ be finite-dimensional affine subspaces of E. Then $V_{ca} = V_{cb} V_{ba}$.* $\square$

We therefore obtain a directed system, as required, and we can make the following definition:

Definition 3.15. *Let E be an affine Euclidean space. The graded Hilbert space $\mathcal{H}(E)$ is the direct limit*

$$\mathcal{H}(E) = \varinjlim_{\substack{E_a \subseteq E \\ \text{fin. dim.} \\ \text{affine sbsp.}}} \mathcal{H}(E_a),$$

in the category of Hilbert spaces and graded isometric inclusions.

If G acts isometrically on E then $\mathcal{H}(E)$ is equipped with a unitary representation of G, just as $\mathcal{A}(E)$ is equipped with a G-action.

We are now almost ready to begin the definition of the asymptotic morphism $\alpha \colon \mathcal{A}(E) \dashrightarrow \mathcal{K}(\mathcal{H}(E))$. What we are going to do is construct a family of asymptotic morphisms,

$$\alpha^a \colon \mathcal{S} \widehat{\otimes} \mathcal{C}(\mathcal{E}_\dashv) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E})),$$

one for each finite-dimensional subspace of E, and then prove that if $E_a \subseteq E_b$ then the diagram

$$\begin{tikzcd} \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \arrow[r, dashed, "\alpha^a"] \arrow[d, "\beta_{ba}"'] & \mathcal{K}(\mathcal{H}(E)) \arrow[d, "="] \\ \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\lfloor}) \arrow[r, dashed, "\alpha^b"'] & \mathcal{K}(\mathcal{H}(E)) \end{tikzcd}$$

is asymptotically commutative. Once we have done that we shall obtain a asymptotic morphism defined on the direct limit $\varinjlim \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$, as required.

To give the basic ideas we shall consider first a simpler 'toy model', as follows. Suppose for a moment that E is itself a *finite-dimensional* space. Fix a point in E; call it $0 \in E$; use it to identify E with its underlying linear space V; and use this identification to define scaling maps $e \mapsto t^{-1}e$ on E, for $t \geq 1$, with the common fiexed point $0 \in E$. If $h \in \mathcal{C}(E_a)$ and if $0 \in E_a$ then define $h_t \in \mathcal{C}(E_a)$ by the usual formula $h_t(e) = h(t^{-1}e)$.

Lemma 3.4. *Let E_a be an affine subspace of a* finite-dimensional *affine Euclidean space E. Denote by D_a the Dirac operator for E_a and denote by $B_{a^\perp} = C_{a^\perp} + D_{a^\perp}$ the Clifford-plus-Dirac operator for $E_a^\perp$. The formula*

$$\alpha_t^a \colon f\widehat{\otimes}h \mapsto f_t(B_{a^\perp}\widehat{\otimes}1 + 1\widehat{\otimes}D_a)(1\widehat{\otimes}M_{h_t})$$

defines an asymptotic morphism

$$\alpha^a \colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E})).$$

Proof. The operator $B_{a^\perp}$ is essentially self-adjoint and has compact resolvent (see Section 1.13). So we can define $*$-homomorphisms $\gamma_t \colon \mathcal{S} \to \mathcal{K}(\mathcal{H}(\mathcal{E}_{\dashv}^{\perp}))$ by $\gamma_t(f) = f_t(B_{a^\perp})$. Moreover we saw in Section 1.12 that the formula

$$\alpha_t \colon f\widehat{\otimes}h \mapsto f_t(D_a)M_{h_t}$$

defines an asymptotic morphism $\alpha \colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E}_{\dashv}))$. The formula for α^a in the statement of the lemma is nothing but the formula for the composition

$$\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \xrightarrow{\Delta\widehat{\otimes}1} \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \overset{\gamma\widehat{\otimes}\alpha}{\dashrightarrow} \mathcal{K}(\mathcal{H}(E_a^{\perp}))\widehat{\otimes}\mathcal{K}(\mathcal{H}(E_a)).$$

So α^a is an asymptotic morphism, as required.

Lemma 3.5. *Let $E_a \subseteq E_b$ be a nested pair of affine subspaces of a finite-dimensional affine Euclidean space E. Denote by D_a and D_b the Dirac operators for E_a and E_b, and denote by*

$$\alpha^a \colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E})) \quad \textit{and} \quad \alpha^{\lfloor} \colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\lfloor}) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E}))$$

the asymptotic morphisms of Lemma 3.4. The diagram

$$\begin{tikzcd} \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \arrow[r, dashed, "\alpha^a"] \arrow[d, "\beta_{ba}"'] & \mathcal{K}(\mathcal{H}(E)) \arrow[d, "="] \\ \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\lfloor}) \arrow[r, dashed, "\alpha^b"'] & \mathcal{K}(\mathcal{H}(E)) \end{tikzcd}$$

is asymptotically commutative.

Proof. We shall do a computation using the generators $u(x) = e^{-x^2}$ and $v(x) = xe^{-x^2}$ of $\mathcal{S}$. Denote by E_{ba} the orthogonal complement of E_a in E_b, so that

$$E = E_b^{\perp} \oplus E_{ba} \oplus E_a$$

and

$$\mathcal{H}(E) \cong \mathcal{H}(E_b^{\perp})\widehat{\otimes}\mathcal{H}(E_{ba})\widehat{\otimes}\mathcal{H}(E_a).$$

To do the computation we need to note that under the isomorphism of Hilbert spaces $\mathcal{H}(E_b) \cong \mathcal{H}(E_{ba})\widehat{\otimes}\mathcal{H}(E_a)$ the Dirac operator D_b corresponds to $D_{ba}\widehat{\otimes}1 + 1\widehat{\otimes}D_a$ (to be precise, the self-adjoint closures of these essentially self-adjoint operators correspond to one another). Similarly $B_{a^\perp}$ corresponds to $B_{b^\perp}\widehat{\otimes}1 + 1\widehat{\otimes}B_{ba}$ under the isomorphism $\mathcal{H}(E_a^{\perp}) \cong \mathcal{H}(E_b^{\perp})\widehat{\otimes}\mathcal{H}(E_{ba})$. Hence by making these identifications of Hilbert spaces we get

$$\exp(-t^{-2}D_b^2) = \exp(-t^{-2}D_{ba}^2)\widehat{\otimes}\exp(-t^{-2}D_a^2)$$

and

$$\exp(-t^{-2}B_{a^\perp}^2) = \exp(-t^{-2}B_{b^\perp}^2)\widehat{\otimes}\exp(-t^{-2}B_{ba}^2).$$

Now, applying α_t^a to the element $u\widehat{\otimes}h \in \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$ we get

$$\exp(-t^{-2}B_{b^\perp}^2)\widehat{\otimes}\exp(-t^{-2}B_{ba}^2)\widehat{\otimes}\exp(-t^{-2}D_a^2)M_{h_t}$$

in $\mathcal{K}(\mathcal{H}(E_b^{\perp}))\widehat{\otimes}\mathcal{K}(\mathcal{H}(E_{ba}))\widehat{\otimes}\mathcal{K}(\mathcal{H}(E_a))$, while applying $\alpha_t^b \circ \beta_{ba}$ to $u\widehat{\otimes}h$ we get

$$\exp(-t^{-2}B_{b^\perp}^2)\widehat{\otimes}\exp(-t^{-2}D_{ba}^2)\exp(-t^{-2}C_{ba}^2)\widehat{\otimes}\exp(-t^{-2}D_a^2)M_{h_t}.$$

But we saw in Section 1.13 that the two families of operators $\exp(-t^2B_{ba}^2)$ and $\exp(-t^{-2}D_{ba}^2)\exp(-t^{-2}C_{ba}^2)$ are asymptotic to one another, as $t \to \infty$. It follows that $\alpha_t^a(u\widehat{\otimes}h)$ is asymptotic to $\alpha_t^b(\beta_{ba}(u\widehat{\otimes}h))$, as required. The calculation for $v\widehat{\otimes}h$ is similar.

Turning to the infinite-dimensional case, it is clear that the major problem is to construct a suitable operator $B_{a^\perp}$. We begin by assembling some preliminary facts. Suppose that we fix for a moment a finite-dimensional affine subspace E_a of E. Denote by $E_a^{\perp}$ its orthogonal complement in E. This is an infinite-dimensional subspace of V, but in particular it is a Euclidean space in its own right, and we can form the direct limit Hilbert space $\mathcal{H}(E_a^{\perp})$ as in Definition 3.15.

Lemma 3.6. *Let E_a be a finite-dimensional affine subspace of E and let $E_a^\perp$ be its orthogonal complement in E. The isomorphisms*

$$\mathcal{H}(E_b) \cong \mathcal{H}(V_{ba}) \widehat{\otimes} \mathcal{H}(E_a) \qquad (E_a \subseteq E_b)$$

of Lemma 3.3 combine to provide an isomorphism

$$\mathcal{H}(E) \cong \mathcal{H}(E_a^\perp) \widehat{\otimes} \mathcal{H}(E_a). \quad \square$$

Definition 3.16. *Let E_a be a finite-dimensional subspace of an affine Euclidean space E. The Schwartz space of E_a, denoted $\mathfrak{s}(E_a)$ is*

$$\mathfrak{s}(E_a) = \{ \text{ Schwartz-class } \operatorname{Cliff}(V_a)\text{-valued functions on } E_a \}.$$

The Schwartz space $\mathfrak{s}(E)$ is the algebraic direct limit of the Schwartz spaces $\mathfrak{s}(E_a)$:

$$\mathfrak{s}(E) = \varinjlim_{\substack{E_a \subseteq E \\ \text{fin. dim.} \\ \text{affine sbsp.}}} \mathfrak{s}(E_a),$$

using the inclusions $V_{ba} \colon \mathfrak{s}(E_a) \to \mathfrak{s}(E_b)$.

We now want to define a suitable operator $B_{a^\perp}$ on $\mathcal{H}(E_a^\perp)$ with domain $\mathfrak{s}(E_a^\perp)$. A very interesting possibility is as follows. If $V \subseteq E_a^\perp$ is a finite-dimensional subspace then the operator $B_V = C_V + D_V$ acts on every Schwartz space $\mathfrak{s}(W)$, where $V \subseteq W$: just use the formula

$$(B_V f)(w) = \sum_1^n x_i e_i(f(w)) + \sum_1^n \widehat{e}_i(\frac{\partial f}{\partial x_i}(w)),$$

from Lecture 1, where $e_1, \ldots, e_n$ is an orthonormal basis for V and $x_1, \ldots, x_n$ are the dual coordinates on V, extended to coordinates on W by orthogonal projection. The actions on the Schwartz spaces $\mathfrak{s}(W)$ are compatible with the inclusions used to define the direct limit $\mathfrak{s}(E_a^\perp) = \varinjlim \mathfrak{s}(W)$, and we obtain an unbounded, essentially self-adjoint operator on $\mathcal{H}(E_a^\perp)$ with domain $\mathfrak{s}(E_a^\perp)$. Let us now make the following key observation:

Lemma 3.7. *Suppose that $E_a^\perp$ is decomposed as an algebraic direct sum of pairwise orthogonal, finite-dimensional subspaces,*

$$E_a^\perp = V_0 \oplus V_1 \oplus V_2 \oplus \cdots .$$

If $f \in \mathfrak{s}(E_a^\perp)$ then the sum

$$B_{a^\perp} f = B_0 f + B_1 f + B_2 f + \cdots ,$$

where $B_j = C_j + D_j$ is the Clifford-Dirac operator on V_j, has only finitely many nonzero terms. The operator defined by the sum is essentially self-adjoint on $\mathfrak{s}(E_a^\perp)$ and is independent of the direct sum decomposition of $E_a^\perp$ used in its construction.

Proof. Observe that

$$\mathfrak{s}(E_a^{\perp}) = \varinjlim_n \mathfrak{s}(V_0 \oplus \cdots \oplus V_n).$$

Therefore if $f \in \mathfrak{s}(E_a^{\perp})$ then f belongs to some $\mathfrak{s}(V_0 \oplus \cdots \oplus V_n)$. Its image in $\mathfrak{s}(V_0 \oplus \cdots \oplus V_{n+k})$ under the linking map in the directed system is a function of the form

$$f_k(v_0 + \cdots v_{n+k}) = \text{constant} \cdot f(v_0 + \cdots + v_n) e^{-\frac{1}{2}\|v_{n+1}\|} \cdot \ldots \cdot e^{-\frac{1}{2}\|v_{n+k}\|^2}.$$

Since $e^{-\frac{1}{2}\|v_{n+k}\|^2}$ is in the kernel of B_{n+k} we see that $B_{n+k} f = 0$ for all $k \geq 1$. This proves the first part of the lemma. Essential self-adjointness follows from the existence of an eigenbasis for $B_{a^{\perp}}$, which in turn follows immediately from the existence of eigenbases in the finite-dimensional case (see Corollary 1.1). The fact that $B_a^{\perp}$ is independent of the choice of direct sum decomposition follows from the formula

$$B_{a^{\perp}} f = B_W f \qquad \text{if } f \in \mathfrak{s}(W) \subseteq \mathfrak{s}(E_a^{\perp}),$$

which in turn follows from the formula $B_{W_1} + B_{W_2} = B_{W_1 \oplus W_2}$ in finite dimensions.

Unfortunately the operator $B_{a^{\perp}}$ above does *not* have compact resolvent. Indeed

$$B_{a^{\perp}}^2 = B_0^2 + B_1^2 + B_2^2 + \cdots ,$$

from which it follows that the eigenvalues for $B_{a^{\perp}}^2$ are the sums

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \cdots ,$$

where λ_j is an eigenvalue for B_j^2 and where almost all λ_j are zero. It therefore follows from Proposition 1.16 that while the eigenvalue 0 occurs with multiplicity 1, each positive integer is an eigenvalue of $B_{a^{\perp}}^2$ of infinite multiplicity.

Because $B_{a^{\perp}}$ fails to have compact resolvent we cannot immediately follow Lemma 3.4 to obtain our asymptotic morphisms α^a. Instead we first have to 'perturb' the operators $B_{a^{\perp}}$ in a certain way.

Notation 3.10 *We are now going to* fix *an increasing sequence $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ of finite-dimensional affine subspaces of E whose union is E. We shall denote by V_n the orthogonal complement of E_{n-1} in E_n (and write $V_0 = E_0$), so that there is an algebraic orthogonal direct sum decomposition*

$$E = V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots .$$

Later on we shall want to arrange matters so that this decomposition is compatible with the action of G on E, but for now any decomposition will do.

Having chosen a direct sum decomposition as above, letus make the following definitions:

Definition 3.17. *Let E_a be a finite-dimensional affine subspace of E. An algebraic orthogonal direct sum decomposition*

$$E_a^{\perp} = W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

is standard *if it is of the form*

$$E_a^{\perp} = V_a \oplus V_n \oplus V_{n+1} \oplus \cdots ,$$

for some finite-dimensional linear space V_a and some $n \geq 1$, where the spaces V_n are the members of the fixed decomposition of E given above.

Definition 3.18. *Let E_a be a finite-dimensional affine subspace of E. An algebraic orthogonal direct sum decomposition*

$$E_a^{\perp} = Z_0 \oplus Z_1 \oplus Z_2 \oplus \cdots$$

into finite-dimensional linear subspaces is acceptable *if there is a standard decomposition*

$$E_a^{\perp} = W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

such that

$$W_0 \oplus \cdots \oplus W_n \subseteq Z_n \oplus \cdots \oplus Z_n \subseteq W_0 \oplus \cdots \oplus W_{n+1}$$

for all sufficiently large n.

We are now going to define perturbed operators $B_{a^{\perp},t}$ which depend on a choice of acceptable decomposition, as well as on a parameter $t \in [1, \infty)$.

Definition 3.19. *Let E_a be a finite-dimensional affine subspace of E and let*

$$E_a^{\perp} = Z_0 + Z_1 + Z_2 + \cdots$$

be an acceptable decomposition of $E_a^{\perp}$ as an orthogonal direct sum of finite-dimensional linear subspaces. For each $t \geq 1$ define an unbounded operator $B_{a^{\perp},t}$ on $\mathcal{H}(E_a^{\perp})$, with domain $\mathfrak{s}(E_a^{\perp})$, by the formula

$$B_{a^{\perp},t} = t_0 B_0 + t_1 B_1 + t_2 B_2 + \cdots$$

where $t_j = 1 + t^{-1} j$, where $B_b = C_n + D_n$, and where C_n and D_n are the Clifford and Dirac operators on the finite-dimensional spaces Z_n.

It follows from Lemma 3.7 that the infinite sum actually defines an operator with domain $\mathfrak{s}(E_a^{\perp})$. The perturbed operators $B_{a^{\perp},t}$ have the key compact resolvent property that we need:

Lemma 3.8. *Let E_a be a finite-dimensional affine subspace of E and let*

$$E_a^{\perp} = Z_0 \oplus Z_1 \oplus Z_2 \oplus \cdots$$

be an acceptable decomposition of $E_a^{\perp}$ as an orthogonal direct sum of finite-dimensional linear subspaces. The operator

$$B_{a^{\perp},t} = t_0 B_0 + t_1 B_1 + t_2 B_2 + \cdots$$

is essentially self-adjoint and has compact resolvent.

Proof. The proof of self-adjointness follows the same argument as the proof in Lemma 3.7: one shows that there is an orthonormal eigenbasis for $B_{a^{\perp},t}$ in $\mathfrak{s}(E_a^{\perp})$. As for compactness of the resolvent, the formula

$$B_{a^{\perp},t}^2 = t_0^2 B_0^2 + t_1^2 B_1^2 + t_2^2 B_2^2 +$$

implies that the eigenvalues of $B_{a^{\perp},t}^2$ are the sums

$$\lambda = t_0^2 \lambda_0 + t_1^2 \lambda_1 + t_2^2 \lambda_2 + \cdots ,$$

where λ_j is an eigenvalue for B_j^2 and where almost all λ_j are zero. Since the lowest positive eigenvalue for B_j is 1, and since $t_j \to \infty$ as $j \to \infty$ (for fixed t), it follows that for any R there are only finitely many eigenvalues for ${B_{a^{\perp},t}}^2$ of size R or less. This proves that $B_{a^{\perp},t}$ has compact resolvent, as required.

We can now define the asymptotic morphisms $\alpha^a \colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E}))$ that we need. Fix a point 0 in E and use it to define scaling automorphisms $h \mapsto h_t$ on each $\mathcal{C}(E_a)$ for which $0 \in E_a$.

Proposition 3.4. *Let E_a be a finite-dimensional affine subspace of E for which $0 \in E_a$ and let $B_{a^{\perp},t}$ be the operator associated to some acceptable decomposition of $E_a^{\perp}$. The formula*

$$\alpha_t^a \colon f\widehat{\otimes}h \mapsto f_t(B_{a^{\perp},t}\widehat{\otimes}1 + 1\widehat{\otimes}D_a)(1\widehat{\otimes}M_{h_t})$$

defines an asymptotic morphism $\alpha^a \colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E}_{\dashv}^{\perp}))\widehat{\otimes}\mathcal{K}(\mathcal{H}(\mathcal{E}_{\dashv}))$, and hence, thanks to the isomorphism of Lemma 3.6, an asymptotic morphism

$$\alpha^a \colon \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \dashrightarrow \mathcal{K}(\mathcal{H}(\mathcal{E})).$$

Proof. This is proved in exactly the same way as was Lemma 3.4.

It should be pointed out that operator $B_{a^{\perp},t}$ *does* depend on the choice of accepable decomposition, and so our definition of α^a appears to depend on quite a bit of extraneous data. But the situation improves in the limit as $t \to \infty$. The basic calculation here is as follows:

Lemma 3.9. *Let E_a be a finite-dimensional affine subspace of E and denote by $B_t = B_{a^\perp,t}$ and $B'_t = B'_{a^\perp,t}$ be the operators associated to two acceptable decompositions of $E_a^\perp$. Then for every $f \in \mathcal{S}$,*

$$\lim_{t\to\infty} \|f(B_t) - f(B'_t)\| = 0.$$

Proof. We shall prove the following special case: we shall show that if the summands in the acceptable decompositions are Z_n and Z'_n, and if

$$Z_0 \oplus \cdots \oplus Z_n \subseteq Z'_0 \oplus \cdots \oplus Z'_n \subseteq Z_0 \oplus \cdots \oplus Z_{n+1}$$

for *all* n, then $\lim_{t\to\infty} \|f(B_t) - f(B'_t)\| = 0$. (For the general case, which is not really any harder, see [32] and [34].)

Denote by X_n the orthogonal complement of Z_n in Z'_n, and by Y_n the orthogonal complement of Z'_{n-1} in Z_n (set $Y_0 = Z_0$). There is then a direct sum decomposition

$$E_a^\perp = Y_0 \oplus X_0 \oplus Y_1 \oplus X_1 \oplus \cdots ,$$

with respect to which the operators $B_{a^\perp,t}$ and $B'_{a^\perp,t}$ can be written as infinite sums

$$B_t = t_0 B_{Y_0} + t_1 B_{X_0} + t_1 B_{Y_1} + t_2 B_{X_1} + \cdots$$

and

$$B'_t = t_0 B_{Y_0} + t_0 B_{X_0} + t_1 B_{Y_1} + t_1 B_{X_1} + \cdots .$$

Since $t_j - t_{j-1} = t^{-1}$ it follows that

$$B_t - B'_t = t^{-1} B_{X_0} + t^{-1} B_{X_1} + \cdots ,$$

and therefore that

$$(B_t - B'_t)^2 = t^{-2} B_{X_0}^2 + t^{-2} B_{X_1}^2 + \cdots .$$

In contrast,

$$B_t^2 = t_0^2 B_{Y_0}^2 + t_1^2 B_{X_0}^2 + t_1^2 B_{Y_1}^2 + t_2^2 B_{X_1}^2 + \cdots ,$$

and since $t_j^2 \geq 1$ it follows that $\|(B_t - B'_t)f\| \leq t^{-1}\|B_t f\|$ for every $f \in \mathfrak{s}(E_a^\perp)$. This implies that if $f(x) = (x \pm i)^{-1}$ then

$$\|f(B_t) - f(B'_t)\| = \|(B'_t \pm i)^{-1}(B'_t - B_t)(B_t \pm i)^{-1}\| \leq \|(B'_t - B_t)(B_t \pm i)^{-1}\| \leq t^{-1}.$$

An approximation argument involving the Stone-Weierstrass theorem (which we have seen before) now finishes the proof.

For later purposes we note the following simple strengthening of Lemma 3.9. It is proved by following exactly the same argument.

Lemma 3.10. *With the hypotheses of the previous lemma, is $s \in [1,\infty)$*

$$\lim_{t\to\infty} \|f(sB_t) - f(sB'_t)\| = 0,$$

uniformly in s. □

It follows from Lemma 3.9 that our definition of the asymptotic morphism α^a is independent, up to asymptotic equivalence, of the choice of acceptable decomposition of $E_a^{\perp}$ (compare the proof of Lemma 3.4).[15]

Proposition 3.5. *The diagram*

$$\begin{tikzcd}
\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv}) \arrow[r, dashed, "\alpha^a"] \arrow[d, "\beta_{ba}"'] & \mathcal{K}(\mathcal{H}(E)) \arrow[d, "="] \\
\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\lfloor}) \arrow[r, dashed, "\alpha^b"'] & \mathcal{K}(\mathcal{H}(E))
\end{tikzcd}$$

is asymptotically commutative.

Proof. Using the computations we made in Section 1.13, as we did in the proof of Lemma 3.5, we see that the composition $\alpha^b \circ \beta_{ba}$ is asymptotic to the asymptotic morphism

$$f\widehat{\otimes}h \mapsto f_t(B_t'\widehat{\otimes}1 + 1\widehat{\otimes}D_a)(1\widehat{\otimes}M_{h_t}),$$

where, if α^b is computed using the acceptable decomposition

$$E_b^{\perp} = Z_0 \oplus Z_1 \oplus Z_2 \oplus \cdots ,$$

then B_t' is the operator of Definition 3.19 associated to the decomposition

$$E_a^{\perp} = (E_{ba} \oplus Z_0) \oplus Z_1 \oplus Z_2 \oplus \cdots .$$

But this is an acceptable decomposition for $E_a^{\perp}$, and so $\alpha^b \circ \beta_{ba}$ is asymptotic to α^a, as required.

It follows that the asymptotic morphisms α^a combine to form a single asymptotic morphism

$$\alpha\colon \mathcal{A}(E) \dashrightarrow \mathcal{K}(\mathcal{H}(E)).$$

Our definition of the class $\alpha \in E_G(\mathfrak{A}(E), \mathbb{C})$ is therefore almost complete. It remains only to discuss the equivariance of α.

Suppose that the countable group G acts isometrically on E. Using the point $0 \in E$ that we chose prior to the proof of Proposition 3.4, indentify E with its underlying Euclidean vector space V, and thereby define a family of actions on E, parametrized by $s \in [0, 1]$ by

$$g \cdot_s e = s(g \cdot 0) + \pi(g)v, \qquad (0 + v = e)$$

(see Exercise 3.3). Thus the action $g \cdot_1 e$ is the original action, while $g \cdot_0 e$ has a global fixed point (namely $0 \in E$).

[15] It should be added however that α^a *does* depend on the choice of initial direct sum decomposition, as in 3.10.

Lemma 3.11. *There exists a direct sum decompostion*

$$E = V_0 \oplus V_1 \oplus V_2 \oplus + \cdots$$

as in 3.10 such that, if $E_n = V_0 \oplus \cdots \oplus V_n$*, then for every* $g \in G$ *there is an* $N \in \mathbb{N}$ *for which*

$$n > N \quad \Rightarrow \quad g \cdot E_n \subseteq E_{n+1}, \quad \text{for all } s \in [0,1]. \quad \square$$

Proposition 3.6. *If the direct sum decomposition*

$$E = V_0 \oplus V_1 \oplus V_2 \oplus + \cdots$$

is chosen as in Lemma 3.11 then the asymptotic morphism $\alpha \colon \mathcal{A}(E) \dashrightarrow \mathcal{K}(\mathcal{H}(E))$ *is equivariant in the sense that*

$$\lim_{t\to\infty} \|\alpha_t(g \cdot a) - g \cdot_{t^{-1}} (\alpha_t(x))\| = 0,$$

for all $a \in \mathcal{A}(E)$ *and all* $g \in G$.

Proof. Examining the definitions, we see that on $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathcal{E}_{\dashv})$ the asymptotic morphism $a \mapsto g^{-1} \cdot_{t^{-1}} (\alpha_t(g \cdot a))$ is given by exactly the same formula used to define α^a, except for the choice of acceptable direct sum decomposition of $E_a^{\perp}$. But we already noted that different choices of acceptable direct sum decomposition give rise to asymptotically equivalent asymptotic morphisms, so the proposition is proved.

Definition 3.20. *Denote by* $\alpha \in E_G(\mathcal{A}(E), \mathbb{C})$ *the class of the asymptotic morphism* $\alpha \colon \mathcal{A}(E) \dashrightarrow \mathcal{K}(\mathcal{H}(E))$.

3.7 Proof of the Main Theorem, Part Three

Here we show that $\alpha \circ \beta = 1 \in E_G(\mathbb{C}, \mathbb{C})$. The proof is almost exactly the same as the proof of Proposition 2.21 in the last lecture.

Lemma 3.12. *Suppose that the action of* G *on the affine Euclidean space* E *has a fixed point. Then the composition*

$$\mathbb{C} \xrightarrow{\beta} \mathcal{A}(E) \xrightarrow{\alpha} \mathbb{C}$$

in equivariant E*-theory is the identity morphism on* $\mathbb{C}$.

Proof. The proof has three parts. First, recall that in the definition of the asymptotic morphism β we began by fixing a point of E. It is clear from the proof of Lemma 3.2 that different choices of point give rise to asymptotically equivalent asymptotic morphisms, so we might as well choose a point which is fixed for the action of G on E. But having done so each $*$-homomorphism $\beta_t(f) = \beta(f_t)$ in the asymptotic morphism β is individually G-equivariant. It follows that the equivariant asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{A}(\mathcal{E})$ is equivariantly homotopy equivalent to the equivariant $*$-homomorphism $\beta\colon \mathcal{S} \to \mathcal{A}(\mathcal{E})$. Using this fact, it follows that we may compute the composition $\alpha \circ \beta$ in equivariant E-theory by computing the composition of the asymptotic morphism α with the $*$-homomorphism β. But the results in Section 1.13 show that this composition is asymptotic to $\gamma\colon \mathcal{S} \to \mathcal{K}(\mathcal{H}(\mathcal{E}))$, where

$$\gamma_t(f) = f_t(B_t),$$

and B_t is the operator of Definition 3.19 associated to any acceptable decomposition of E. This in turn is homotopic to the asymptotic morphism $f \mapsto f(B_t)$. Finally this is homotopic to the asymptotic morphism defining $1 \in E_G(\mathbb{C}, \mathbb{C})$ by the homotopy

$$f \mapsto \begin{cases} f(sB_t) & s \in [1, \infty) \\ f(0)P & s = \infty, \end{cases}$$

where P is the projection onto the kernel of B_t (note that all the B_t have the same 1-dimensional, G-fixed kernel).

Theorem 3.11. *The composition $\alpha \circ \beta \in E_G(\mathbb{C}, \mathbb{C})$ is the identity.*

Proof. Let $s \in [0,1]$ and denote by $\mathcal{A}_s(E)$ the C^*-algebra $\mathcal{A}(E)$, but with the scaled G-action $(g,h) \mapsto g \cdot_s h$. The algebras $\mathcal{A}_s(E)$ form a continuous field of G-C^*-algebras over the unit interval. Denote by $\mathcal{A}_{[0,1]}(E)$ the G-C^*-algebra of continuous sections of this field. In a similar way, form the continuous field of G-C^*-algebras $\mathcal{K}_s(\mathcal{H}(E))$ and denote by $\mathcal{K}_{[0,1]}(\mathcal{H}(E))$ the G-C^*-algebra of continuous sections. The asymptotic morphism $\alpha\colon \mathcal{A}(E) \dashrightarrow \mathcal{K}(\mathcal{H}(E))$ induces an asymptotic morphism

$$\bar{\alpha}\colon \mathcal{A}_{[0,1]}(E) \dashrightarrow \mathcal{K}_{[0,1]}(\mathcal{H}(E)) \ ,$$

and similarly the asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{A}(E)$ determines an asymptotic morphism

$$\bar{\beta}\colon \mathcal{S} \dashrightarrow \mathcal{A}_{[0,1]}(E).$$

From the diagram of equivariant E-theory morphisms

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\bar{\beta}"] \arrow[d, "="'] & \mathcal{A}_{[0,1]}(E) \arrow[r, "\bar{\alpha}"] \arrow[d, "\varepsilon_s"] & C[0,1] \arrow[d, "\cong"', "\varepsilon_s"] \\ \mathbb{C} \arrow[r, "\beta"'] & \mathcal{A}_s(E) \arrow[r, "\alpha"'] & \mathbb{C} \end{tikzcd} \ ,$$

where ε_s denotes the element induced from evaluation at $s \in [0,1]$, we see that if the bottom composition is the identity for some $s \in [0,1]$ then it is the identity for all $s \in [0,1]$. But by Lemma 3.12 the composition is the identity when $s = 0$ since the action $(g,e) \mapsto g \cdot_0 e$ has a fixed point. It follows that the composition is the identity when $s = 1$, which is what we wanted to prove.

3.8 Generalization to Fields

We conclude this lecture by quickly sketching a simple extension of the main theorem to a situation involving fields of affine spaces over a compact parameter space. This generalization will be used in the next lecture to prove injectivity results about the Baum-Connes assembly map.

Definition 3.21. *Let Z be a set. Denote by $\mathrm{NT}(Z)$ the set of negative-type kernels $b\colon Z \times Z \to \mathbb{R}$. Equip $\mathrm{NT}(Z)$ with the topology of pointwise convergence, so that $b_\alpha \to b$ in $\mathrm{NT}(Z)$ if and only if $b_\alpha(z_1, z_2) \to b(z_1, z_2)$ for all $z_1, z_2 \in Z$.*

Suppose now that X is a compact Hausdorff space and that we are given a continuous map $x \mapsto b_x$ from X into $\mathrm{NT}(Z)$. For each $x \in X$ we can construct a Euclidean vector space V_x and an affine space E_x over V_x following the prescription laid out in the proof of Proposition 3.1 (thus for example V_x is a quotient of the space of finitely supported functions on Z which sum to 0, and E_x is a quotient of the space of finitely supported functions on Z which sum to 1). We obtain in this way some sort of 'continuous field' of affine Euclidean spaces over X (we shall not need to make this notion precise).

The C^*-algebras $\mathcal{A}(E_x)$ may be put together to form a *continuous field* of C^*-algebras over X. (See [19] for a proper discussion of continuous fields.) To do so we must specify which sections $x \mapsto f_x \in \mathcal{A}(E_x)$ are to be deemed continuous, and for this purpose we begin by deeming to be continuous certain families of isometries from $\mathbb{R}^n$ into the affine spaces E_x.

Definition 3.22. *Let U be an open subset of X and let $h_u\colon \mathbb{R}^n \to E_u$ be a family of isometries of $\mathbb{R}^n$ into the affine Euclidean spaces E_u ($u \in U$) defined above. We shall say that the family is* continuous *if there is a finite subset $F \subseteq Z$ and if there are functions $f_{j,u}\colon Z \to \mathbb{R}$ (where $j = 1, \ldots, n$, and $u \in U$) such that*

(a) *Each function $f_{j,u}$ sums to one, and is supported in F (and therefore each $f_{j,u}$ determines a point of E_u).*
(b) *For each $z \in Z$ and each j, the value $f_{j,u}(z)$ is a continuous function of $u \in U$.*
(c) *The isometry h_u maps the standard basis element e_j of $\mathbb{R}^n$ to the point of E_u determined by $f_{j,u}$.*

Definition 3.23. *Let us say that a function $x \mapsto f_x$ which assigns to each point $x \in X$ an element of the C^*-algebra $\mathcal{A}(E_x)$ is a* continuous section *if for every $x \in X$ and every $\varepsilon > 0$ there is an open set U containing x, a continuous family of isometries $h_u\colon \mathbb{R}^n \to E_u$ as above, and an element $f \in \mathcal{S}\widehat{\otimes}\mathcal{C}(\mathbb{R}^{\setminus})$ such that*

$$\|h_{u,**}(f) - f_u\| < \varepsilon \quad \forall u \in U.$$

Here by $h_{u,**}$ we are using an abbreviated notation for the inclusion of $\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathbb{R}^{\setminus})$ into $\mathcal{A}(E_u)$ induced from the isometry $h_u\colon \mathbb{R}^n \to h_u[\mathbb{R}^n] \subseteq E_u$ by forming the composition

$$\mathcal{S}\widehat{\otimes}\mathcal{C}(\mathbb{R}^{\setminus}) \xrightarrow{1\otimes h_{u,**}} \mathcal{S}\widehat{\otimes}\mathcal{C}(\langle_{\sqcap}[\mathbb{R}^{\setminus}]) \xrightarrow{\subseteq} \mathcal{A}(E).$$

Lemma 3.13. *With the above definition of continuous section the collection of C^*-algebras $\{\mathcal{A}(E_x)\}_{x\in X}$ is given the structure of a continuous field of C^*-algebras over the space X.* $\square$

Definition 3.24. *Denote by $\mathcal{A}(X,E)$ the C^*-algebra of continuous sections of the above continuous field.*

If a group G acts on the set Z then G acts on $\mathrm{NT}(Z)$ by the formula $(g \cdot b)(z_1, z_2) = b(g^{-1}z_1, g^{-1}z_2)$. In what follows we shall be solely interested in the case where $Z = G$ and the action is by left translation.

Definition 3.25. *Let X be a compact space equipped with an action of a countable discrete group G by homeomorphisms. An equivariant map $x \mapsto b_x$ from X into $\mathrm{NT}(G)$ is* proper-valued *if for every $S \geq 0$ there is a finite set $F \subseteq G$ such that*

$$b_x(g_1, g_2) \leq S \quad \Rightarrow \quad g_1^{-1}g_2 \in F.$$

The following is a generalization of Theorem 3.8.

Proposition 3.7. *If $b\colon X \to \mathrm{NT}(G)$ is a G-equivariant and proper-valued map then the G-C^*-algebra $\mathcal{A}(X,E)$ is proper.* $\square$

By carrying out the constructions of the previous sections fiberwise we obtain the following result (which is basically due to Tu [18]).

Theorem 3.12. *Let G be a countable discrete group and let X be a compact metrizable G-space. Assume that there exists a proper-valued, equivariant map from X into $\mathrm{NT}(G)$. Then $\mathcal{A}(X,E)$ is a proper G-C^*-algebra and there are E-theory classes*

$$\alpha \in E_G(\mathcal{A}(X,E), C(X)) \quad \textit{and} \quad \beta \in E_G(C(X), \mathcal{A}(X,E))$$

for which the composition $\alpha \circ \beta$ is the identity in $E_G(C(X), C(X))$. $\square$

By trivially adapting the simple argument used to prove Theorem 2.20 we obtain the following important consequence of the above:

Corollary 3.2. *Let G be a countable discrete group and let X be a compact metrizable G-space. If there exists a proper-valued, equivariant map from X into $\mathrm{NT}(G)$ then for every G-C^*-algebra D the Baum-Connes assembly map*

$$\mu : K^{top}(G, D(X)) \to K(C^*(G, D(X)))$$

is an isomorphism. If G is exact then the same is true for the assembly map into $K(C^*_\lambda(G, D(X)))$. $\square$

4 Injectivity Arguments

The purpose of this lecture is to prove that in various cases the Baum-Connes assembly map

$$K^{top}(G, D) \to K(C^*(G, D))$$

is injective. A great deal more is known about the injectivity of the assembly map than its surjectivity. In a number of cases, injectivity is implied by a *geometric* property of G, whereas surjectivity seems to require the understanding of more subtle issues in harmonic analysis.

In all but the last section we shall work with the full crossed product $C^*(G, D)$, but all the results have counterparts for $C^*_\lambda(G, D)$. If G is exact then arguments below applied in the reduced case; otherwise different arguments are needed.

4.1 Geometry of Groups

Let G be a discrete group which is generated by a finite set S. The *word-length* of an element $g \in G$ is the minimal length $\ell(g)$ of a string of elements from S and S^{-1} whose product is g. The (left-invariant) distance function on G associated to the length function ℓ is defined by

$$d(g_1, g_2) = \ell(g_1^{-1} g_2).$$

The word-length metric depends on the choice of generating set S. Nevertheless, the 'large-scale' geometric structure of G endowed with a word-length metric is independent of the generating set: the metrics associated to two finite generating sets S and T are related by inequalities

$$\frac{1}{C} \cdot d_S(g_1 g_2) - C \leq d_T(g_1, g_2) \leq C \cdot d_S(g_1, g_2) + C,$$

where the constant $C > 0$ depends on S and T but not of course on g_1 and g_2.

Definition 4.1. *Let Z be a set and let d and δ be two distance functions on Z. They are* coarsely equivalent *if for every $R > 0$ there exists a constant $S > 0$ such that*

$$d(z_1, z_2) < R \quad \Rightarrow \quad \delta(z_1, z_2) < S$$

and

$$\delta(z_1, z_2) < R \quad \Rightarrow \quad d_1(z_1, z_2) < S$$

Thus any two word-length metrics on a finitely generated group are coarsely equivalent. When we speak of 'geometric' properties of a finitely generated group we shall mean (in this lecture) properties shared by all metrics on G which are coarsely equivalent to a word-length metric. This geometry may often be visualized using Theorem 4.1 below.

Definition 4.2. *A* curve *in a metric space X is a continuous map from a closed interval into X. The length of a curve $\gamma\colon [a,b] \to X$ is the quantity*

$$\operatorname{length}(\gamma) = \sup_{a=t_0<t_1<\cdots<t_n=b} \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})).$$

A metric space X is a length space if for all $x_1, x_2 \in X$, $d(x_1,x_2)$ is the infimum of the lengths of curves joining x_1 and x_2.

Theorem 4.1. *Let G be a finitely generated discrete group acting properly and cocompactly by isometries on a length space X. Let x be a point of X which is fixed by no nontrivial element of G. Then the distance function*

$$\delta(g_1, g_2) = d(g_1 \cdot x, g_2 \cdot x)$$

on G is coarsely equivalent to the word-length metric on G. $\square$

See [48, 64] for the original version of this theorem and [10] for an up to date treatment. In the context of the above theorem we shall say that the *space* G is coarsely equivalent to the space X (see [57, 58] for a development of the notion of coarse equivalence between metric spaces, of which our notion of coarse equivalence between two metrics on a single space is a special case).

Example 4.1. If G is the fundamental group of a closed Riemannian manifold M then G is coarsely equivalent to the universal covering space $\widetilde{M}$.

Example 4.2. Any finitely generated group is coarsely equivalent to its Cayley graph. For example free groups are coarsely equivalent to trees.

4.2 Hyperbolic Groups

Gromov's hyperbolic groups provide a good example of how geometric hypotheses on groups lead to theorems in C^*-algebra K-theory. In this section we shall sketch very briefly the rudiments of the theory of hyperbolic groups. Later on in the lecture we shall prove the injectivity of the Baum-Connes assembly map

$$\mu\colon K^{top}(G,D) \to K(C^*(G,D)$$

for hyperbolic groups.[16] The first injectivity result in this direction is due to Connes and Moscovici [15] who essentially proved rational injectivity of the assembly map in the case $D = \mathbb{C}$. Our arguments here will however be quite different.

Definition 4.3. *Let X be a metric space. A geodesic segment in X is a curve $\gamma\colon [a,b] \to X$ such that*

$$d(\gamma(s), \gamma(t)) = |s-t|$$

for all $a \le s \le t \le b$.

[16] We shall see that every hyperbolic group is exact; hence the reduced assembly map μ_λ is injective too.

Observe that if γ is a geodesic segment from x_1 to x_2 then the length of γ is precisely $d(x_1, x_2)$.

Definition 4.4. *A* geodesic metric space *is a metric space in which each two points are joined by a geodesic segment.*

Definition 4.5. *A* geodesic triangle *in a metric space X is a triple of points of X, together with three geodesic segments in X connecting the points pairwise. A geodesic triangle is D-*slim *for some $D \geq 0$ if each point on each edge lies within a distance D of some point on one of the other two edges.*

Example 4.3. Geodesic triangles in trees are 0-slim. An equilateral triangle of side R in Euclidean space is $\frac{\sqrt{3}}{4}R$-slim.

Definition 4.6. *A geodesic metric space X is D-hyperbolic if every geodesic triangle Δ in X is D-slim and hyperbolic if it is D-hyperbolic for some $D \geq 0$.*

Thus trees are hyperbolic metric spaces but Euclidean spaces of dimension 2 or more are not.

Definition 4.6 is attributed by Gromov to Rips [24]. It is equivalent to a wide variety of other conditions, for which we refer to the original work of Gromov [24] or one of a number of later expositions, for example [22, 10]. (The reader is also referred to these sources for further information on everything else in this section.)

Definition 4.7. *A finitely generated discrete group G is* word-hyperbolic*, or just* hyperbolic*, if its Cayley graph is a hyperbolic metric space.*

This definition leaves open the possibility that the Cayley graph of G constructed with respect to one finite set of generators is hyperbolic while that constructed with respect to another is not. But the following theorem asserts that this is impossible:

Theorem 4.2. *If a finitely generated group G is hyperbolic for one finite generating set then it is hyperbolic for any other.* $\square$

Examples 4.3 *Every tree is a 0-hyperbolic space and a finitely generated free group is 0-hyperbolic. The Poincaré disk Δ is a hyperbolic metric space. If G is a proper and cocompact group of isometries of Δ then G is a hyperbolic group. In particular, the fundamental group of a Riemann surface of genus 2 or more is hyperbolic.*

If G is a finitely generated group and if $K \geq 0$ then the *Rips complex* $\mathrm{Rips}(G, K)$ is the simplicial complex with vertex set G, for which a $(p+1)$-tuple $(g_0, \ldots, g_p)$ is a p-simplex if and only if $d(g_i, g_j) \leq K$, for all i and j. In the case of hyperbolic groups the Rips complex provides a simple model for the universal space $\mathrm{e}G$:

Theorem 4.4. [7, 47] *Let G be a hyperbolic group. If $K \gg 0$ then the Rips complex $\mathrm{Rips}(G, K)$ is a universal proper G-space.* $\square$

In the following sections we shall need one additional construction, as follows:

Definition 4.8. *A geodesic ray in a hyperbolic space X is a continuous function*

$$c : [0, \infty) \to X$$

such that the restriction of c to every closed interval $[0, l]$ is a geodesic segment. Two geodesic rays in X are equivalent *if*

$$\limsup_{t \to \infty} d(c_1(t), c_2(t)) < \infty.$$

The Gromov boundary *of a hyperbolic metric space X is the set of equivalence classes of geodesic rays in X. The* Gromov boundary ∂G *of a hyperbolic group G is the Gromov boundary of its Cayley graph.*

The Gromov boundary ∂G does not depend on the choice of generating set. It is equipped in the obvious way with an action of G. It may also be equipped with a compact metrizable topology, on which G acts by homeomorphisms. Moreover the disjoint union $\overline{G} = G \cup \partial G$ may be equipped with a compact metrizable topology in such a way that G acts by homeomorphisms, that G is an open dense subset of $\overline{G}$, and that a sequence of points $g_n \in G$ converges to a point $x \in \partial G$ iff $g_n \to \infty$ in G and there is a geodesic ray c representing x such that

$$\sup_n d(g_n, c) < \infty.$$

From our point of view, a key feature of $\overline{G} = G \cup \partial G$ is that the action of G on $\overline{G}$ is *amenable*. We shall discuss this notion in Section 4.5.

4.3 Injectivity Theorems

In this section we shall formulate several results which assert the injectivity of the Baum-Connes map μ under various hypotheses.

Our first injectivity result is a theorem which is essentially due to Kasparov (an improved version of it, which invokes his RKK-theory, underlies his approach to the Novikov conjecture).

Theorem 4.5. *Let G be a countable discrete group. Suppose there exists a proper G-C^*-algebra B and elements $\alpha \in E_G(B, \mathbb{C})$ and $\beta \in E_G(\mathbb{C}, B)$ such that for every finite subgroup H of G the composition $\gamma = \alpha \circ \beta \in E_G(\mathbb{C}, \mathbb{C})$ restricts to the identity in $E_H(\mathbb{C}, \mathbb{C})$. Then for every G-C^*-algebra D the Baum-Connes assembly map*

$$\mu\colon K^{top}(G, D) \to K(C^*(G, D))$$

is injective.

Proof. We begin by considering the same diagram we introduced in the proof of Theorem 2.20:

$$\begin{tikzcd}
K^{top}(G,\mathbb{C}\widehat{\otimes}D) \arrow[r, "\mu"] \arrow[d, "\beta_*"'] & K(C^*(G,\mathbb{C}\widehat{\otimes}D)) \arrow[d, "\beta_*"] \\
K^{top}(G,B\widehat{\otimes}D) \arrow[r, "\mu", "\cong"'] \arrow[d, "\alpha_*"'] & K(C^*(G,B\widehat{\otimes}D)) \arrow[d, "\alpha_*"] \\
K^{top}(G,\mathbb{C}\widehat{\otimes}D) \arrow[r, "\mu"'] & K(C^*(G,\mathbb{C}\widehat{\otimes}D)).
\end{tikzcd}$$

The middle assembly map is an isomorphism since $B\widehat{\otimes}D$ is a proper G-C^*-algebra. We want to show that the top assembly map is injective, and for this it suffices to show that the top left-hand vertical map $\beta_*\colon K^{top}(G,D) \to K^{top}(G,D\widehat{\otimes}B)$ is injective. For this we shall show that the composition

$$K^{top}(G,\mathbb{C}\widehat{\otimes}D) \xrightarrow{\beta_*} K^{top}(G,B\widehat{\otimes}D) \xrightarrow{\alpha_*} K^{top}(G,\mathbb{C}\widehat{\otimes}D)$$

is an isomorphism. In view of the definition of K^{top} it suffices to show that if Z is a G-compact proper G-space then the map

$$\gamma_* = \alpha_* \circ \beta_*\colon E_G(Z,D) \longrightarrow E_G(Z,D)$$

is an isomorphism. The proof of this is an induction argument on the number n of G-invariant open sets U needed to cover Z, each of which admits a map to a proper homogeneous space G/H. If $n=1$, so that Z itself admits such a map, then $Z = G\times_H W$, where W is a compact space equipped with an action of H. There is then a commuting diagram of restriction isomorphisms

$$\begin{tikzcd}
E_G(G\times_H W,D) \arrow[r, "\gamma_*"] \arrow[d, "\mathrm{Res}"', "\cong"] & E_G(G\times_H W,D) \arrow[d, "\cong"', "\mathrm{Res}"] \\
E_H(W,D) \arrow[r, "\gamma_*"'] & E_H(W,D),
\end{tikzcd}$$

(see Proposition 2.8), and the bottom map is an isomorphism (in fact the identity) since $\gamma = 1$ in $E_H(\mathbb{C},\mathbb{C})$. If $n>1$ then choose a G-invariant open set $U\subseteq Z$ which admits a map to a proper homogeneous space, and for which the space $Z_1 = Z\setminus U$ may be covered by $n-1$ G-invariant open sets, each admitting a map to a proper homogeneous space. By induction we may assume that the map γ_* is an isomorphism for Z_1. Applying the five lemma to the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E_G(Z_1,D) \arrow[r] \arrow[d, "\gamma_*"', "\cong"] & E_G(Z,D) \arrow[r] \arrow[d, "\gamma_*"'] & E_G(U,D) \arrow[r] \arrow[d, "\gamma_*"', "\cong"] & \cdots \\
\cdots \arrow[r] & E_G(Z_1,D) \arrow[r] & E_G(Z,D) \arrow[r] & E_G(U,D) \arrow[r] & \cdots
\end{tikzcd}$$

we conclude that γ_* is an isomorphism for Z too.

Remark 4.1. The proof actually shows that the assembly map is *split* injective.

The second result is taken from [10] and is as follows.

Theorem 4.6. *Let X be a compact, metrizable G-space and assume that X is H-equivariantly contractible, for every finite subgroup H of G. Let D be a separable G-C^*-algebra. If the Baum-Connes assembly map*

$$\mu : K^{top}(G, D(X)) \to K(C^*(G, D(X)))$$

is an isomorphism then the Baum-Connes assembly map

$$\mu : K^{top}(G, D) \to K(C^*(G, D))$$

is split injective.

Proof. The inclusion ι of D into $D(X)$ as constant functions gives rise to a commutative diagram

$$\begin{tikzcd} K^{top}(G, D(X)) \arrow[r, "\mu"] & K(C^*(G, D(X))) \\ K^{top}(G, D) \arrow[u, "\iota_*"] \arrow[r, "\mu"'] & K(C^*(G, D)). \arrow[u, "\iota_*"'] \end{tikzcd}$$

We shall prove the theorem by showing that the left vertical map is an isomorphism. In view of the definition of K^{top} it suffices to show that if Z is any G-compact proper G-space then the map

$$\iota_* : E_G(C_0(Z), D) \to E_G(C_0(Z), D(X))$$

is an isomorphism. By a Mayer-Vietoris argument like the one we used in the proof of Theorem 4.6 it actually suffices to consider the case where Z admits a map to a proper homogeneous space G/H. In this case there is a compact space W equipped with an action of H such that $Z = G \times_H W$. Consider now the following commuting diagram of restriction isomorphisms:

$$\begin{tikzcd} E_G(C_0(G \times_H W), D) \arrow[r, "\iota_*"] \arrow[d, "\mathrm{Res}"', "\cong"] & E_G(C_0(G \times_H W), D(X)) \arrow[d, "\cong"', "\mathrm{Res}"] \\ E_H(C(W), D) \arrow[r, "\iota_*"'] & E_H(C(W), DX)). \end{tikzcd}$$

The bottom horizontal map is an isomorphism (since ι is a homotopy equivalence of H-C^*-algebras) and therefore the top horizontal map is an isomorphism too.

The last injectivity result is an analytic version of a result of Carlsson-Pedersen [11]. We will not discuss the proof, but refer the reader to the original paper of Higson [10, Thm. 1.2 & 5.2]. We include it only because it applies more or less directly to the case of hyperbolic groups.

Definition 4.9. *Let G be a discrete group, let X be a G-compact, proper G-space, and let $\overline{X}$ be a metrizable compactification of G to which the action of G on X extends to an action by homeomorphisms. The extended action is* small at infinity *if for every compact set $K \subseteq X$,*

$$\lim_{g\to\infty} \operatorname{diameter}(gK) = 0,$$

where the diameters are computed using a metric on $\overline{X}$.

Theorem 4.7. *Let G be a countable discrete group. Suppose there is a G-compact model for $\mathrm{e}G$ having a metrizable compactification $\overline{\mathrm{e}G}$ satisfying*

(a) *the G action on $\mathrm{e}G$ extends continuously to $\overline{\mathrm{e}G}$,*
(b) *the action of G on $\overline{\mathrm{e}G}$ is small, and*
(c) *$\overline{\mathrm{e}G}$ is H-equivariantly contractible, for every finite subgroup H of G.*

Then for every separable G-C^*-algebra D the Baum-Connes assembly map

$$\mu : K^{top}(G, D) \to K(C^*(G, D))$$

is injective. $\square$

4.4 Uniform Embeddings in Hilbert Space

We are now going to apply the second theorem of the previous section to prove injectivity of the Baum-Connes assembly map for a quite broad class of groups.

Definition 4.10. *Let X and Y be metric spaces. A* uniform embedding *of X into Y is a function $f : X \to Y$ with the following two properties:*

(a) For every $R \geq 0$ there exists some $S \geq 0$ such that

$$d(x_1, x_2) \leq R \quad \Rightarrow \quad d(f(x_1), f(x_2)) \leq S.$$

(b) For every $S \geq 0$ there exists $R \geq 0$ such that

$$d(x_1, x_2) \geq R \quad \Rightarrow \quad d(f(x_1), f(x_2)) \geq S.$$

Example 4.4. If f is a bi-Lipschitz homeomorphism from X onto its image in Y then f is a uniform embedding. But note however that if the metric space X is bounded then *any* function from X to Y is a uniform embedding (in particular, uniform embeddings need not be one-to-one).

Exercise 4.1. Let G a finitely generated group and let H be a finitely generated subgroup of G. If G and H are equipped with their word-length metrics then the inclusion $H \subseteq G$ is a uniform embedding.

Remark 4.2. In the context of groups, any function satisfying condition (a) of Definition 4.10 is in fact a Lipschitz function. But condition (b) is more delicate. For example it is easy to find examples for which the optimal inequality in (b) is something like

$$d(x_1, x_2) \geq e^S \quad \Rightarrow \quad d(f(x_1), f(x_2)) \geq S$$

If a finitely generated group G acts metrically-properly on an affine Euclidean space E, and if $e \in E$, then the map $g \mapsto g \cdot e$ is a uniform embedding of G into E. We are going to prove the following result, which partially extends the main theorem of the last lecture:

This theorem is due to Tu [18] and Yu [68] (in both cases, in a somewhat disguised form).

Theorem 4.8. *Let G be a countable discrete group. If G is uniformly embeddable in a Euclidean space then for every G-C^*-algebra D the Baum-Connes assembly map*

$$\mu : K^{top}(G, D) \to K(C^*(G, D))$$

is (split) injective.

The first step of the proof is to convert a uniform embedding, which is something purely metric in nature, into something more G-equivariant. For this purpose let us recall the following object from general topology:

Definition 4.11. *Let X be a discrete set. The* Stone-Cech compactification *of X is the set βX of all nonzero, finitely additive, $\{0, 1\}$-valued probability measures on the algebra of all subsets of X. We equip βX with the topology of pointwise convergence (with respect to which it is a compact Hausdorff space).*

Thus a point of βX is a function μ from the subsets of X into $\{0, 1\}$ which is additive on finite disjoint unions and which is not identically zero. A net μ_α converges to μ if and only if $\mu_\alpha(E)$ converges to $\mu(E)$, for every $E \subseteq X$.

Example 4.5. If x is a point of X then the measure μ_x, defined by the formula

$$\mu_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

is a point of βX. In this way X is embedded into βX as a dense open subset.

Remark 4.3. The fact that the measures μ assume only the values 0 and 1 will matter little in what follows, and we could equally well work with arbitrary, finitely additive measures for which $\mu(X) = 1$.

If b is a *bounded* complex function on X, and if μ is a finitely additive measure on X, then we may form the integral

$$\int_X b(x)\, d\mu(x)$$

as follows. First, if b assumes only finitely many values (in other words if b is a *simple* function) then define

$$\int_X b(x)\, d\mu(x) = \sum_\lambda \lambda \cdot \mu\{x : b(x) = \lambda\}.$$

Second, if b is a general bounded function, write b as a uniform limit of simple functions and define the integral of b to be the limit of the integrals of the approximants.

Exercise 4.2. If b is a bounded function then the map $\mu \mapsto \int_X b(x)\, d\mu(x)$ is a continuous function from βX into $\mathbb{C}$.

Remark 4.4. The virtue of $\{0,1\}$-valued measures is that this integration process makes sense in very great generality — it is possible to integrate any function from X into *any* compact space.

Suppose now that G is a finitely generated discrete group. The compact space βG is equipped with a continuous action of G by the formula

$$(g \cdot \mu)(E) = \mu(Eg).$$

Let $f\colon G \to E$ be a uniform embedding into an affine Euclidean space and let $b\colon G \times G \to \mathbb{R}$ be the associated negative type kernel:

$$b(g_1, g_2) = d^2(f(g_1), f(g_2)).$$

According to part (a) of Definition 4.10 the function $g \mapsto b(gg_1, gg_2)$ is bounded, for every $g_1, g_2 \in G$. As a result, we may define negative type kernels b_μ, for $\mu \in \beta G$, by integration:

$$b_\mu(g_1, g_2) = \int_G b(gg_1, gg_2) d\mu(g).$$

Observe that $b_{g\cdot\mu}(g_1, g_2) = b_\mu(g^{-1}g_1, g^{-1}g_2)$, so that our integration construction defines an equivariant map from βG into the negative type kernels on G.

Lemma 4.1. *For every $S \geq 0$ there exists $R \geq 0$ so that if $d(g_1, g_2) \geq R$ then $b_\mu(g_1, g_2) \geq S$, for every $\mu \in \beta G$.*

Proof. This is a consequence of part (b) of Definition 4.10.

Since G is finitely generated, for every R there is a finite set F so that if $d(g_1, g_2) < R$ then $g_1^{-1}g_2 \in F$. The map $\mu \mapsto b_\mu$ is therefore *proper-valued* in the sense of Definition 3.25, and we have proved the following result:

Proposition 4.1. *If a finitely generated group G may be uniformly embedded into an affine Euclidean space then there is an equivariant, proper-valued continuous map from βG into the space* $\mathrm{NT}(G)$ *of negative-type kernels on G.* □

It will now be clear that to prove Theorem 4.8 we mean to apply Theorems 3.12 and 4.6. To do so we must replace βG by a compact G-space which is smaller (second countable) and more connected (in fact contractible) than βG. This is done as follows.

Lemma 4.2. *Let G be a countable group, let X be a compact G-space and let $b\colon X \to \mathrm{NT}(G)$ be a continuous and G-equivariant map from X into the negative type kernels on G. There is a metrizable compact G-space Y and a G-map from X to Y through which the map b factors.*

Proof. Take Y to be the Gelfand dual of the separable C^*-algebra of functions on X generated by the functions $x \mapsto b_{gx}(g_1, g_2)$, for all $g, g_1, g_2 \in G$.

Lemma 4.3. *Let G be a countable group, let Y be a compact metrizable G-space and let $b\colon Y \to \mathrm{NT}(G)$ be a proper-valued, G-equivariant continuous map. There is a metrizable compact G-space Z which is H-equivariantly contractible, for every finite subgroup H of G, and a proper-valued, G-equivariant continuous map from Z into $\mathrm{NT}(G)$.*

Proof. Let Z be the compact space of Borel probability measures on Y (we give Z the weak* topology it inherits as a subset of the dual of $C(Y)$; note that we are speaking now of *countably additive* measures defined on the Borel σ-algebra). If $\mu \in Z$ then define $b_\mu \in \mathrm{NT}(G)$ by integration:

$$b_\mu(g_1, g_2) = \int_Y b_y(g_1, g_2)\, d\mu(y).$$

The map $\mu \mapsto b_\mu$ has the required properties.

Proof (Proof of Theorem 4.8). Proposition 4.1 and the lemmas above show that the hypotheses of Theorem 3.12 and Corollary 3.2 are met. Theorem 4.6 then implies injectivity of the assembly map, as required.

4.5 Amenable Actions

In this section we shall discuss a means of constructing uniform embeddings of groups into affine Hilbert spaces.

Definition 4.12. *Let G be a discrete group. Denote by $\mathrm{prob}(G)$ the set of functions $f\colon G \to [0,1]$ such that $\sum_{g\in G} f(g) = 1$. Equip $\mathrm{prob}(G)$ with the topology of pointwise convergence, so that $f_\alpha \to f$ if and only if $f_\alpha(g) \to f(g)$ for every $g \in G$. Equip $\mathrm{prob}(G)$ with an action of G by homeomorphisms via the formula $(g \cdot f)(h) = f(g^{-1}h)$.*

Definition 4.13. *Let G be a countable discrete group. An action of G by homeomorphisms on a compact Hausdorff space X is amenable if there is a sequence of continuous maps $f_n : X \to \mathrm{prob}(G)$ such that for every $g \in G$*

$$\lim_{n\to\infty} \sup_{x\in X} \|f_n(g \cdot x) - g \cdot (f_n(x))\|_1 = 0.$$

Here, if k is a function on G, then we define $\|k\|_1 = \sum_{g\in G} |k(g)|$.

We are going to prove the following result:

Proposition 4.9 *If a finitely generated group G acts amenably on a compact space X then G is uniformly embeddable in a Hilbert space.*

Remark 4.5. The method below can easily be modified to show that if a countable group G (which is not necessarily finitely generated) acts amenably on some compact space X then there is an equivariant, proper-valued map from X into the negative-type kernels on G. The methods of the previous section then show that the Baum-Connes assembly map is injective for G.

Examples 4.10 *Every hyperbolic group acts amenably on its Gromov boundary. If G is a discrete subgroup of a connected Lie group H then G acts amenably some compact homogeneous space H/P. If G is a discrete group of finite asymptotic dimension then G acts amenably on the Stone-Cech compactification βG. See [33] for a discussion of all these cases (along with references to proofs).*

Definition 4.14. *Let Z be a set. A function $\varphi\colon Z \times Z \to \mathbb{C}$ is a* positive-definite kernel *on the set Z if $\varphi(z,z) = 1$ for all z, if $\varphi(z_1, z_2) = \overline{\varphi(z_2, z_1)}$, for all $z_1, z_2 \in Z$, and if*

$$\sum_{i,j=1}^{k} \overline{\lambda_i}\varphi(z_i, z_j)\lambda_j \geq 0$$

for all positive integers k, all $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and all $z_1, \ldots, z_k \in Z$.

Remark 4.6. The normalization $\varphi(z,z) = 1$ is not always made, but it is useful here. As is the case with positive-definite functions on groups (which we discussed in Lecture 4), the condition $\varphi(z_1, z_2) = \overline{\varphi(z_2, z_1)}$ is actually implied by the last condition.

Comparing definitions, the following is immediate:

Lemma 4.4. *If φ is a positive-definite kernel on a set Z if $\operatorname{Re}\varphi$ denotes its real part, then $1 - \operatorname{Re}\varphi$ is a negative type kernel on Z.* $\square$

Proof (Proof of Proposition 4.9). Suppose that G acts amenably on a compact space X, and let $f_n\colon X \to \operatorname{prob}(G)$ be a sequence of functions as in Definition 4.13. After making suitable approximations to the f_n we may assume that for each n there is a finite set $F \subseteq G$ such that for every $x \in X$ the function $f_n(x) \in \operatorname{prob}(G)$ is supported in F. Now let $h_n(x,g) = f_n(x)(g)^{1/2}$. Then fix a point $x \in X$ and define functions $\varphi_n\colon G \times G \to \mathbb{C}$ by

$$\varphi_n(g_1, g_2) = \sum_{g \in G} h_n(g_1 x, g_1 g) h_n(g_2 x, g_2 g).$$

These are positive definite kernels on $G \times G$. For every finite subset $F \subseteq G$ and every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$n > N \quad \text{and} \quad g_1^{-1}g_2 \in F \quad \Rightarrow \quad |\varphi_n(g_1, g_2) - 1| < \varepsilon.$$

In addition, for every $n \in \mathbb{N}$ there exists a finite subset $F \subseteq G$ such that

$$g_1^{-1}g_2 \notin F \quad \Rightarrow \quad \varphi_n(g_1, g_2) = 0.$$

It follows that for a suitable subsequence the series $\sum_j (1 - \mathrm{Re}\,\varphi_{n_j})$ is pointwise convergent everywhere on $G \times G$. But each function $1 - \mathrm{Re}\,\varphi_{n_j}$ is a negative type kernel, and therefore so is the sum. The map into affine Euclidean space which is associated to the sum is a uniform embedding.

Remark 4.7. In fact it is possible to characterize the amenability of a group action in terms of positive definite kernels. See [2] for a clear and rapid presentation of the facts relevant here, and [3] for a comprehensive account of amenability. The existence of a sequence of positive definite kernels on G which have the two properties displayed in the proof of the lemma is equivalent to the amenability of the action of G on its Stone-Cech compactification βG. See [2] again, and see also Section 5.6 for more on this topic.

Remark 4.8. The theory of amenable actions is very closely connected to the theory of exact groups. To see why, suppose that G admits an amenable action on some compact space X. Then using the theory of positive-definite kernels it may be shown that

$$C^*(G, X) = C^*_\lambda(G, X)$$

and moreover that the crossed product C^*-algebra is *nuclear*. This means that the for any C^*-algebra D,

$$C^*(G, X) \otimes_{max} D = C^*(G, D) \otimes_{min} D.$$

See [2] for a discussion of these results. It follows of course that the crossed product C^*-algebra is *exact*, in the sense of Definition 2.10. But then it follows that $C^*_\lambda(G)$, which is a subalgebra of $C^*(G, X) = C^*_\lambda(G, X)$, is exact too. Therefore, by Proposition 2.6 the group G is exact. To summarize: *if G acts amenably on some compact space then G is exact.* In fact the converse to this is true too: see Section 5.6.

4.6 Poincaré Duality

We conclude this lecture with a few remarks concerning a 'dual' formulation of the Baum-Connes conjecture for certain groups. With an application to Lecture 6 in mind we shall formulate the following theorem in the context of reduced crossed products.

Theorem 4.11. *Let G be a countable exact group and let A be a separable proper G-C^*-algebra. Suppose that there is a class $\alpha \in E_G(A, \mathbb{C})$ with the property that for every finite subgroup H of G the restricted class $\alpha\,|_H \in E_H(A, \mathbb{C})$ is invertible. Then the Baum-Connes assembly map*

$$\mu_\lambda \colon K^{top}(G, D) \to K(C^*_\lambda(G, D))$$

is an isomorphism for a given separable G-C^*-algebra D if and only if the map

$$\alpha_*\colon K(C^*_\lambda(G, A\widehat{\otimes}D)) \to K(C^*_\lambda(G, D))$$

induced from α is an isomorphism.

Remark 4.9. In the proof we shall identify μ with α_*, so that μ will be for example injective if and only if α_* is injective. As usual, analogous statements may be proved for reduced crossed products, either in the same way if G is exact, or with some additional arguments otherwise.

Proof. Consider the diagram

$$\begin{tikzcd}
K^{top}(G, A\widehat{\otimes}D) \arrow[r, "\mu_\lambda"] \arrow[d, "\alpha_*"'] & K(C^*_\lambda(G, A\widehat{\otimes}D) \arrow[d, "\alpha_*"] \\
K^{top}(G, D) \arrow[r, "\mu_\lambda"'] & K(C^*_\lambda(G, D)),
\end{tikzcd}$$

in which the horizontal arrows are the Baum-Connes assembly maps and the vertical arrows are induced by composition with α in E_G-theory and by composition with the element descended from α in nonequivariant E-theory. The diagram commutes. By Theorem 2.20 the top horizontal map is an isomorphism. Furthermore, an argument like the ones used in Section 4.3 shows that the left hand vertical map is an isomorphism. Therefore the bottom horizontal map is an isomorphism if and only if the right vertical map is an isomorphism, as required.

The theorem in effect reformulates the Baum-Connes conjecture entirely in the framework of K-theory (hence the term 'Poincaré duality', since we have replaced the K-homological functor K^{top} of G with K-theory). It has an important application to groups which act isometrically on Riemannian manifolds. We shall not go into details, but here is a rapid summary of the relevant facts. The Clifford algebra constructions we developed in Lecture 1 may be generalized to complete Riemannian manifolds M. We denote by $\mathcal{C}(M)$ the C^*-algebra of sections of the bundle of Clifford algebras $\mathrm{Cliff}(T_xM)$ associated to the tangent spaces of M. There is a Dirac operator on M (an unbounded self-adjoint operator acting on the Hilbert space of L^2-sections of the Clifford algebra bundle on M), and it defines a class

$$\alpha \in E(\mathcal{C}(M), \mathbb{C})$$

in almost exactly the same way that we defined α for linear spaces. Moreover if a group G acts isometrically on M then the Dirac operator defines an equivariant class

$$\alpha \in E_G(\mathcal{C}(M), \mathbb{C}).$$

Now if M happens to be a universal proper G-space then the hypotheses of Theorem 4.11 are met:

Proposition 4.12 *Let M be a complete Riemannian manifold and suppose that a countable group G acts on M by isometries. Assume further that M is a universal proper G-space. The Dirac operator on M defines an equivariant E-theory class*

$$[D] \in E_G(\mathcal{C}(M), \mathbb{C}),$$

which, restricting from G to and finite subgroup $H \subseteq G$, determines invertible elements

$$[D] \mid_H \in E_H(\mathcal{C}(M), \mathbb{C}). \qquad \square$$

The proposition applies for example when G is a lattice in a semisimple group (take M to be the associated symmetric space), and in this case (which is perhaps the most important case of the Baum-Connes conjecture yet to be resolved) the conjecture reduces to a statement which can be formulated purely within K-theory.

5 Counterexamples

In this lecture we shall present a miscellany of examples and counterexamples. Together they show that the Baum-Connes conjecture is the weakest conjecture of its type which one can reasonably formulate. They also point to shortcomings in the machinery we have developed in these lectures. The counterexamples involve Kazhdan's property T and expander graphs.

5.1 Property T

Definition 5.1. *A discrete group G has property T if the trivial representation is an isolated point in the unitary dual of G.*

See the monograph [17] for an extensive discussion of property T. We shall use the following equivalent formulations of property T:

Theorem 5.1. *Let G be a discrete group. The following are equivalent:*

(a) *G has property T.*
(b) *Every isometric action of G on an affine Euclidean space has a fixed point.*
(c) *There is a central projection $p \in C^*(G)$ with the property that in any unitary representation of G, on a Hilbert space $\mathcal{H}$ the operator p acts as the orthogonal projection onto the G-fixed vectors in $\mathcal{H}$.* $\square$

The projection $p \in C^*(G)$ will be called the *Kazhdan projection* for the property T group G.

Remark 5.1. If G is finite then the Kazhdan projection p is the sum

$$p = \frac{1}{|G|} \sum_{g \in G} g$$

in the group algebra $\mathbb{C}[G] = C^*(G)$ (in the formula we are regarding G as a unitary subgroup of $C^*(G)$). If G is infinite then p is a very mysterious object. For example if we (mistakenly) regard p as an infinite formal series of group elements, $p = \sum a_g \cdot g$, then from the easily proved relation $g \cdot p = p$ we conclude that all the scalars a_g are equal, while from the fact that p acts as 1 in the trivial representation we conclude that the scalars a_g sum to 1. Thus we arrive at a formula for p like the one displayed above, where the sum is infinitely large and the normalizing constant $1/|G|$ is infinitely small.

It is quite difficult to exhibit infinite property T groups, but they do exist. For example Kazhdan proved that lattices in semisimple groups of real rank 2 or more have property T. It is also known that there are many hyperbolic groups with property T.

Lemma 5.1. *If G is an infinite property T group then the quotient mapping from $C^*(G)$ onto $C^*_\lambda(G)$ does not induce an isomorphism in K-theory.*

Proof. The central projection p generates a cyclic direct summand of $K(C^*(G))$ which is mapped to zero in $K(C^*_\lambda(G))$.

It follows immediately that if G is an infinite property T group then the Baum-Connes assembly maps into $K(C^*(G))$ and $K(C^*_\lambda(G))$ cannot both be isomorphisms. We shall not go into the matter in detail here but in fact it is the assembly map into $K(C^*(G))$ which is the problem. This can be seen quite easily in some examples. For instance it is not hard to show that if G has property T then associated to *each* irreducible, finite-dimensional and unitary representation of G is a distinct central projection in $C^*(G)$ (the Kazhdan projection is the one associated to the trivial representation). So if a property T group G has infinitely many irreducible, finite-dimensional and unitary representations (this will happen if G is an infinite linear group) then $K(C^*(G))$ will contain a free abelian subgroup of infinite rank, whereas $K^{top}(G)$ will very often be finitely generated.

Unfortunately the main method we have applied to prove cases of the Baum-Connes conjecture treats the full and reduced C^*-algebra more or less equally. Hence property T causes the method to fail:

Proposition 5.1. *If G is an exact, infinite property T group then G does not satisfy the hypotheses of Theorem 2.20.*

Proof. If G did satisfy the hypotheses then by Theorem 2.20 the quotient mapping $K(C^*(G))$ to $K(C^*_\lambda(G))$ would be an isomorphism.

5.2 Property T and Descent

Proposition 5.1 indicates that our basic strategy for proving the Baum-Connes conjecture for a group G, which involves proving an identity in equivariant, bivariant K-theory, will not work for infinite property T groups (at least if these groups are exact).

However one can ask whether it is possible to prove the conjecture for a given group G by proving an identity in bivariant K-theory for crossed product algebras. We noted in Lecture 5 that if eG is a complete manifold M then the Baum-Connes conjecture for G is equivalent to the assertion that the map

$$\alpha_* \colon K_*(C^*_\lambda(G, \mathcal{C}(M))) \to K_*(C^*_\lambda(G)),$$

induced from the Dirac operator class $\alpha \in E_G(\mathcal{C}(M), \mathbb{C})$, is an isomorphism. One might hope that in fact the descended class

$$\alpha \in E(C^*_\lambda(G, \mathcal{C}(M)), C^*_\lambda(G))$$

is an isomorphism. This is not (always) the case, as the following theorem of Skandalis [60] shows:

Theorem 5.2. *Let G be an infinite, hyperbolic property T group. Then $C^*_\lambda(G)$ is not equivalent in E-theory to any nuclear C^*-algebra.*

Recall from the last lecture that a C^*-algebra A is *nuclear* if $A\widehat{\otimes}_{min}D = A\widehat{\otimes}_{max}D$, for all D. Since the C^*-algebra $C^*_\lambda(G, \mathcal{C}(M))$ is easily proved to be nuclear we obtain the following result:

Corollary 5.1. *Let G be an infinite, hyperbolic, property T group and assume that G acts on a complete Riemannian manifold M by isometries. The Dirac operator class*

$$\alpha \in E(C^*_\lambda(G, \mathcal{C}(M)), C^*_\lambda(G))$$

is not *invertible.* □

Remark 5.2. The corollary applies to discrete, cocompact subgroups of the Lie groups $Sp(n, 1)$ (M is quaternionic hyperbolic space). See [17]. Despite this, it follows from the work of Lafforgue [44] that in this case α as above *does* induce an isomorphism on K-theory. This shows that E-theory is not a perfect weapon with which to attack the Baum-Connes conjecture.[17]

To prove Theorem 5.2 we shall use the following result.

Theorem 5.3. *Let G be a hyperbolic group and let ∂G be its Gromov boundary. There is a compact, metrizable topology on the disjoint union $\overline{G} = G \cup \partial G$ with the following properties:*

(a) *The set G is an open, discrete subset of $\overline{G}$.*
(b) *The left action of G on itself extends continuously to an amenable action of G on $\overline{G}$.*
(c) *The right action of G on itself extends continuously to an action on $\overline{G}$ which is trivial on $\overline{G}$.* □

[17] Exactly the same remarks apply here to KK-theory.

Remark 5.3. Item (c) is essentially the assertion that the natural action on the Gromov compactification is small at infinity, in the sense of Definition 4.9.

We shall also require a few simple representation-theoretic ideas.

Definition 5.2. *Let G be a discrete group. The* left regular, right regular *and* adjoint representations *of G on $\ell^2(G)$ are defined by the formulas*

$$
\begin{aligned}
(\lambda_g \xi)(h) &= \xi(g^{-1}h) \\
(\rho_g \xi)(h) &= \xi(hg) \\
(\alpha_g \xi)(h) &= \xi(g^{-1}hg)
\end{aligned}
$$

for all $g, h \in G$ and $\xi \in \ell^2(G)$. The biregular representation *of $G \times G$ on $\ell^2(G)$ is defined by the formula*

$$(\alpha_{g_1 g_2} \xi)(h) = \xi(g_1^{-1} h g_2)$$

for all $g_1, g_2, h \in G$ and $\xi \in \ell^2(G)$.

The left and right regular representations determine representations λ and ρ of $C^*_\lambda(G)$ in $\mathcal{B}(\ell^2(G))$. Since these representations commute with one another, together they determine a C^*-algebra representation

$$\beta \colon C^*_\lambda(G) \otimes_{max} C^*_\lambda(G) \longrightarrow \mathcal{B}(\ell^2(G)),$$

which is of course the biregular representation on $G \times G \subseteq C^*_\lambda(G) \otimes_{max} C^*_\lambda(G)$.

Definition 5.3. *Denote by J the kernel of the quotient homomorphism from $C^*_\lambda(G) \otimes_{max} C^*_\lambda(G)$ onto $C^*_\lambda(G) \otimes_{min} C^*_\lambda(G)$, so that there is an exact sequence*

$$0 \longrightarrow J \longrightarrow C^*_\lambda(G) \otimes_{max} C^*_\lambda(G) \longrightarrow C^*_\lambda(G) \otimes_{min} C^*_\lambda(G) \longrightarrow 0.$$

Lemma 5.2. *The C^*-algebra representation β maps the ideal J of $C^*_\lambda(G) \otimes_{max} C^*_\lambda(G)$ into the ideal $\mathcal{K}(\ell^2(G))$ of $\mathcal{B}(\ell^2(G))$.*

Proof. Denote by $\mathcal{Q}(\ell^2(G))$ the Calkin algebra for $\ell^2(G)$ — the quotient of the bounded operators by the ideal of compact operators. We are going to construct a $*$-homomorphism from $C^*_\lambda(G) \otimes_{min} C^*_\lambda(G)$ into $\mathcal{Q}(\ell^2(G))$ which makes the following diagram commute:

$$
\begin{tikzcd}
C^*_\lambda(G) \otimes_{min} C^*_\lambda(G) \arrow[r, dotted] & \mathcal{Q}(\ell^2(G)) \\
C^*_\lambda(G) \otimes_{max} C^*_\lambda(G) \arrow[u] \arrow[r, "\lambda\otimes\rho"'] & \mathcal{B}(\ell^2(G)). \arrow[u]
\end{tikzcd}
$$

Here the vertical arrows are the quotient mappings. Commutativity of the diagram will prove the lemma.

We begin by constructing a $*$-homomorphism from $C(\partial G)$ into $\mathcal{Q}(\ell^2(G))$, as follows. If $f \in C(\partial G)$ then extend f to a continuous function on $\overline{G}$, restrict the extension to the open set $G \subseteq \overline{G}$, and then let the restriction act on $\ell^2(G)$ by pointwise multiplication. Two different extensions of $f \in C(\partial G)$ will determine two pointwise multiplication operators which differ by a compact operator. Hence our procedure defines a $*$-homomorphism $\varphi\colon C(\partial G) \to \mathcal{Q}(\ell^2(G))$, as required. Now let G act on $C(\partial G)$ via the (nontrivial) left action of G (see Theorem 5.3) and define a $*$-homomorphism

$$\varphi\colon C^*(G, \partial G) \to \mathcal{Q}(\ell^2(G))$$

by the formula

$$\varphi(\sum_{g \in G} f_g \cdot g) = \sum_{g \in G} \varphi(f_g)\lambda(g)$$

(we are using $\lambda(g)$ to denote both the unitary operator on $\ell^2(G)$ and its image in the Calkin algebra). Next, thanks to part (c) of Theorem 5.3 the *right* regular representation commutes with the algebra $\varphi[C(\partial G)] \subseteq \mathcal{Q}(\ell^2(G))$. We therefore obtain a $*$-homomorphism

$$C^*(G, \partial G) \otimes_{max} C^*_\lambda(G) \longrightarrow \mathcal{Q}(\ell^2(G)).$$

But since the action of G on ∂G is amenable the C^*-algebra $C^*(G, \partial G)$ is nuclear, so that the maximal tensor product above is the same as the minimal one. Moreover amenability also implies that $C^*(G, \partial G)$ agrees with $C^*_\lambda(G, \partial G)$. See Remark 4.8. It follows that the $*$-homomorphism displayed above is the same thing as a $*$-homomorphism

$$C^*_\lambda(G, \partial G) \otimes_{min} C^*_\lambda(G) \longrightarrow \mathcal{Q}(\ell^2(G)).$$

The lemma now follows by restricting this $*$-homomorphism to the subalgebra $C^*_\lambda(G) \otimes_{min} C^*_\lambda(G)$ of $C^*_\lambda(G, \partial G) \otimes_{min} C^*_\lambda(G)$.

Lemma 5.3. *The K-theory group $K(J)$ is nonzero.*

Proof. Let $\Delta\colon C^*(G) \to C^*_\lambda(G) \otimes_{max} C^*_\lambda(G)$ be the $*$-homomorphism $g \mapsto g \otimes g$. Let $p \in C^*(G)$ be the Kazhdan projection and let $q = \Delta(p)$. Then $q \in J$. To see this, observe that the composition

$$C^*(G) \xrightarrow{\Delta} C^*_\lambda(G) \otimes_{max} C^*_\lambda(G) \longrightarrow C^*_\lambda(G) \otimes_{min} C^*_\lambda(G) \subseteq \mathcal{B}(\ell^2(G) \otimes \ell^2(G))$$

corresponds to the tensor product of two copies of the regular representation, and observe also that this representation has no nonzero G-fixed vectors. Hence, the image of the Kazhdan projection in $C^*_\lambda(G) \otimes_{min} C^*_\lambda(G)$ is zero. We shall now prove that $[q] \neq 0$ in $K(J)$. Note first that the representation

$$\beta\colon C^*_\lambda(G) \otimes_{max} C^*_\lambda(G) \longrightarrow \mathcal{B}(\ell^2(G))$$

maps q to a nonzero projection operator. Indeed the composition

$$C^*(G) \xrightarrow{\Delta} C^*_\lambda(G) \otimes_{max} C^*_\lambda(G) \xrightarrow{\beta} \mathcal{B}(\ell^2(G))$$

is the representation α of $C^*(G)$ associated to the adjoint representation of G, which *does* have nonzero G-fixed vectors, and β maps q to the orthogonal projection onto these fixed vectors. But by Lemma 5.2 the representation β maps J into the compact operators, and every nonzero projection in $\mathcal{K}(\ell^2(G))$ determines a nonzero K-theory class. Hence the map from $K(J)$ to $K(\mathcal{K}(\ell^2(G)))$ takes $[q]$ to a nonzero element, and therefore the class $[q] \in K(J)$ is itself nonzero.

Proof (Proof of Theorem 5.2). Let us suppose that there is a separable nuclear C^*-algebra A and an invertible E-theory element $\varphi \in E(C^*_\lambda(G), A)$. Since $C^*_\lambda(G)$ is an exact C^*-algebra there are invertible elements

$$\varphi \otimes_{max} 1 \in E(C^*_\lambda(G) \otimes_{max} C^*_\lambda(G), A \otimes_{max} C^*_\lambda(G))$$

and

$$\varphi \otimes_{min} 1 \in E(C^*_\lambda(G) \otimes_{min} C^*_\lambda(G), A \otimes_{min} C^*_\lambda(G)).$$

We therefore arrive at the following commuting diagram in the E-theory category:

$$\begin{tikzcd}
C^*_\lambda(G) \otimes_{max} C^*_\lambda(G) \arrow[r, "\varphi\otimes_{max}1", "\cong"'] \arrow[d] & A \otimes_{max} C^*_\lambda(G)) \arrow[d] \\
C^*_\lambda(G) \otimes_{min} C^*_\lambda(G) \arrow[r, "\cong", "\varphi\otimes_{min}1"'] & A \otimes_{min} C^*_\lambda(G))
\end{tikzcd}$$

But since A is nuclear the right hand vertical map is an isomorphism (even at the level of C^*-algebras). It follows that the left hand vertical map is an isomorphism in the E-theory category too. As a result, the K-theory map

$$K(C^*_\lambda(G) \otimes_{max} C^*_\lambda(G)) \longrightarrow K(C^*_\lambda(G) \otimes_{min} C^*_\lambda(G))$$

is an isomorphism of abelian groups. But thanks to the K-theory long exact sequence this contradicts Lemma 5.3.

5.3 Bivariant Theories

In the previous section we showed that it is not possible to prove the Baum-Connes conjecture for certain groups (for example uniform lattices in $Sp(n, 1)$) by working purely within E-theory (or for that matter within KK-theory). In this section we shall prove a theorem, also due to Skandalis [61], which points to another sort of weakness of bivariant K-theory. Recall that the bivariant theory we constructed — namely E-theory — has long exact sequences in both variables but that we could not equip it with a minimal tensor product operation (since the operation of minimal tensor product does not in general preserve exact sequences). Kasparov's KK-theory has minimal tensor products but the long exact sequences are only constructed under some hypothesis or other related to C^*-algebra nuclearity. One might ask whether or not there is an 'ideal' theory which has both desirable properties. The answer is no:

Theorem 5.4. *There is no bivariant K-theory functor on separable C^*-algebras which has both a minimal tensor product operation and long exact sequences in both variables.*

Remark 5.4. By the term 'bivariant K-theory functor' we mean a bifunctor which, like E-theory and KK-theory, is equipped with an associative product allowing us to create from it an additive category. The homotopy category of separable C^*-algebras should map to this category, and the ordinary one-variable K-theory functor should factor through it.

To prove Theorem 5.4 we shall need one additional computation from representation theory.

Lemma 5.4. [41] *Let G be a residually finite discrete group. The biregular representation β of $G\times G$ on $\ell^2(G)$ extends to a representation of the minimal tensor product $C^*(G) \otimes_{min} C^*(G)$.*

Proof. Let $\{G_n\}$ be a decreasing family of finite index normal subgroups of G for which the intersection $\cap G_n$ is the trivial one-element subgroup of G. If $x \in \mathbb{C}[G] \odot \mathbb{C}[G]$ then denote by x_n the corresponding 'quotient' element of $\mathbb{C}[G/G_n] \odot \mathbb{C}[G/G_n]$ and denote by β_n the biregular representation of $G/G_n \times G/G_n$ on $\ell^2(G/G_n)$. Thanks to the functoriality of $\otimes_{min}$ it is certainly the case that

$$\|x\|_{C^*(G)\otimes_{min}C^*(G)} \geq \sup_n \|x_n\|_{C^*(G/G_n)\otimes_{min}C^*(G/G_n)}.$$

In addition

$$\|x_n\|_{C^*(G/G_n)\otimes_{min}C^*(G/G_n)} \geq \|\beta_n(x_n)\|_{\mathcal{B}(\ell^2(G/G_n))}$$

(observe that since $C^*(G/G_n)$ is finite-dimensional the minimal tensor product here is equal to the maximal one). Now, it is easily checked that

$$\sup_n \|\beta_n(x_n)\|_{\mathcal{B}(\ell^2(G/G_n))} \geq \|\beta(x)\|_{\mathcal{B}(\ell^2(G))}.$$

Putting together all the inequalities we conclude that

$$\|x\|_{C^*(G)\otimes_{min}C^*(G)} \geq \|\beta(x)\|_{\mathcal{B}(\ell^2(G))},$$

as required.

Lemma 5.5. *Let I be the kernel of the quotient map π from $C^*(G)$ onto $C^*_\lambda(G)$, so that there is a short exact sequence*

$$0 \longrightarrow I \longrightarrow C^*(G) \xrightarrow{\pi} C^*_\lambda(G) \longrightarrow 0.$$

If there is a bivariant theory $F(A,B)$ which has long exact sequences in both variables, and if C_π is the mapping cone of π, then the inclusion $I \subseteq C_\pi$ determines an invertible element of $F(I, C_\pi)$.

Proof. Consider the commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & C^*(G) \arrow[r, "\pi"] \arrow[d] & C^*_\lambda(G) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & C_\pi \arrow[r] & Z_\pi \arrow[r] & C^*_\lambda(G) \arrow[r] & 0, \end{tikzcd}$$

where $Z_\pi = \{\, a \oplus f \in C^*(G) \oplus C^*_\lambda(G)[0,1] \ : \pi(a) = f(0)\,\}$. The inclusion of $C^*(G)$ into Z_π (as constant functions) is a homotopy equivalence, and therefore by applying F-theory to the diagram and then the five lemma we see that the inclusion $I \subseteq C_\pi$ induces isomorphisms

$$F(A, I) \xrightarrow{\cong} F(A, C_\pi) \quad \text{ and } F(C_\pi, B) \xrightarrow{\cong} F(I, B)$$

for every A and B. It follows that the inclusion determines an invertible element of $F(I, C_\pi)$ as required.

Proof (Proof of Theorem 5.4). If the bivariant 'F-theory' has a minimal tensor product then it follows from the lemma above that the inclusion

$$I \otimes_{min} C^*(G) \subseteq C_\pi \otimes_{min} C^*(G)$$

determines an invertible element in F-theory and therefore an isomorphism on K-theory groups. We shall prove the theorem by showing that the map on K-theory induced from the above inclusion fails to be surjective.

Consider the short exact sequence

$$0 \longrightarrow L \longrightarrow C^*(G) \otimes_{min} C^*(G) \xrightarrow{\pi \otimes 1} C^*_\lambda(G) \otimes_{min} C^*(G) \longrightarrow 0,$$

where the ideal L is by definition the kernel of the quotient mapping $\pi \otimes 1$. The mapping cone of $\pi \otimes 1$ is (canonically isomorphic to) $C_\pi \otimes C^*(G)$, and therefore the inclusion $L \subseteq C_\pi \otimes_{min} C^*(G)$ induces an isomorphism in K-theory. Observe now that we have a sequence of inclusions

$$I \otimes_{min} C^*(G) \subseteq L \subseteq C_\pi \otimes_{min} C^*(G).$$

We wish to prove that the overall inclusion fails to be surjective in K-theory, and since the second inclusion is an *isomorphism* in K-theory it suffices to prove that the first inclusion fails to be surjective. From here the proof is more or less the same as the proof of Lemma 5.3, and we shall be very brief. There is a diagonal map

$$\Delta \colon C^*(G) \to C^*(G) \otimes_{min} C^*(G)$$

and we denote by $q \in C^*(G) \otimes_{min} C^*(G)$ the image under Δ of the Kazhdan projection. It is an element of the ideal L. According to Lemma 5.4 the biregular representation of $G \times G$ on $\ell^2(G)$ extends to $C^*(G) \otimes_{min} C^*(G)$. From the proof of Lemma 5.2 we obtain a commuting diagram

$$\begin{tikzcd} C^*_\lambda(G) \otimes_{min} C^*_\lambda(G) \arrow[r] & \mathcal{Q}(\ell^2(G)) \\ C^*(G) \otimes_{min} C^*(G) \arrow[r, "\beta"'] \arrow[u] & \mathcal{B}(\ell^2(G)), \arrow[u] \end{tikzcd}$$

which shows that the C^*-algebra representation β maps the ideal L into the compact operators. Consider now the sequence of maps

$$I \otimes_{min} C^*(G) \xrightarrow{\subseteq} L \xrightarrow{\beta} \mathcal{K}(\ell^2(G)) \ .$$

The composition is zero. But the projection p is mapped to a nonzero element in $\mathcal{K}(\ell^2(G))$, and the K-theory class of $[q]$ is mapped to a nonzero element in the K-theory of $\mathcal{K}(\ell^2(G))$. This shows that the class $[q] \in K(L)$ is not the image of any K-theory class for $I \otimes_{min} C^*(G)$, and this completes the proof of the theorem.

5.4 Expander Graphs

The purpose of this section and the next is to present a counterexample to the Baum-Connes conjecture with coefficients, contingent on some assertions of Gromov.

Definition 5.4. *Let Γ be a finite graph (a finite, 1-dimensional simplicial complex) and let $V = V(\Gamma)$ be the set of vertices of Γ. The* Laplace operator $\Delta\colon \ell^2(V) \to \ell^2(V)$ *is the linear operator defined by the quadratic form*

$$\langle f, \Delta f\rangle = \sum_{d(v,v')=1} |f(v) - f(v')|^2.$$

The sum is over all (unordered) pairs of adjacent vertices, or in other words over the edges of Γ. We shall denote by $\lambda_1(\Gamma)$ the first nonzero eigenvalue of Δ.

If the graph Γ is connected then the kernel of Δ consists precisely of the constant functions on V. In this case

$$\left.\begin{array}{l} f \in \ell^2(V) \\ \sum_{x\in V} f(v) = 0 \end{array}\right\} \Longrightarrow \|f\|^2 \leq \frac{1}{\lambda_1(\Gamma)}\langle\, \Delta f, f\,\rangle. \qquad (1)$$

Definition 5.5. *Let k be a positive integer and let $\varepsilon > 0$. A finite graph Γ is a (k, ε)-expander if it is connected, if no vertex of Γ is incident to more than k edges, and if $\lambda_1(\Gamma) \geq \varepsilon$.*

See [46] for an extensive discussion of the theory of expander graphs.

The following observation of Gromov shows that expander graphs give rise to examples of metric spaces which cannot be uniformly embedded in affine Euclidean spaces.

Proposition 5.2. *Let k be a positive integer, let $\varepsilon > 0$, and let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of (k,ε)-expander graphs for which $\lim_{n\to\infty} |V(\Gamma_n)| = \infty$. Let V be the disjoint union of the sets $V_n = V(\Gamma_n)$ and suppose that V is equipped with a distance function which restricts to the path-distance function on each V_n. Then the metric space V may not be embedded in an affine Euclidean space.*

Proof. Suppose that f is a uniform embedding into an affine Euclidean space E. We may assume that E is complete and separable, and we may then identify it isometrically with $\ell^2(\mathbb{N})$. By restricting f to each V_n, and by adjusting each f_n by a translation in $\ell^2(\mathbb{N})$ (that is, by adding suitable constant vector-valued functions to each f_n) we can arrange that each f_n is orthogonal to every constant function in the Hilbert space of functions from V_n to $\ell^2(\mathbb{N})$ (we just have to arrange that $\sum_{x\in X_n} f(x) = 0$). Now the Laplace operator can be defined on $\ell^2(\mathbb{N})$-valued functions just as it was on scalar functions, and the expander property (1) carries over to the vector-Laplacian (compute using coordinates in $\ell^2(\mathbb{N})$). However

$$\begin{aligned}\langle \Delta f_n, f_n\rangle &= \sum_{d(v,v')=1} |f_n(v) - f_n(v')|^2 \\ &\le \sum_{d(v,v')=1} 1 \\ &\le \frac{k}{2}|V_n|.\end{aligned}$$

It therefore follows from the expander property that

$$\sum_{v\in V_n} \|f_n(v)\|^2 = \|f_n\|^2 \le \frac{1}{\varepsilon}\langle \Delta f_n, f_n\rangle \le \frac{k}{2\varepsilon}|V_n|.$$

Thus for all n, and for at least half of the points $v \in V_n$, we have $\|f_n(v)\|^2 \le \frac{k}{\varepsilon}$. This contradicts the definition of uniform embedding since among this half there must be points v_n and v_n' with $\lim_{n\to\infty} d(v_n, v_n') = \infty$.

In a recent paper [26], M. Gromov has announced the existence of finitely generated groups which do not uniformly embed into Hilbert space. Complete details of the construction have not yet appeared, but the idea is to construct within the Cayley graph of a group a sequence of images of expander graphs. Let us make this a little more precise, as follows.

Definition 5.6. *Let us say that a finitely generated discrete group G is a* Gromov group *if for some positive integer k and some $\varepsilon > 0$ there is a sequence of (k,ε)-expander graphs Γ_n and a sequence of maps $\varphi_n\colon V(\Gamma_n) \to G$ such that :*

(a) There is a constant R, such that if v and v' are adjacent vertices in some graph Γ_n then $d(\varphi_n(v), \varphi_n(v')) \le R$.
(b) $\lim_{n\to\infty}\Big(\max\{ \frac{|\varphi_n^{-1}[g]|}{|V(\Gamma_n)|} \,:\, g \in G\,\}\Big) = 0.$

Remark 5.5. The second condition implies that $\lim_{n\to\infty} |V(\Gamma_n)| = \infty$.

It appears that Gromov's ideas prove that Gromov groups, as above, exist. In any case, we shall explore below some of the properties of Gromov groups. We conclude this section with a simple extension of Proposition 5.2, the proof of which is left to the reader.

Proposition 5.5 *If G is a Gromov group then G cannot be uniformly embedded in an affine Euclidean space.* □

5.5 The Baum-Connes Conjecture with Coefficients

We shall prove that, contingent on the existence of a Gromov group as in the last section, there exists a separable, commutative C^*-algebra D, and an action of a countable group G on D, for which the Baum-Connes map

$$\mu_\lambda \colon K^{top}(G, D) \to K(C^*_\lambda(G, D))$$

fails to be an isomorphism.

Lemma 5.6. *Let G be a countable group and let J be an ideal in a G-C^*-algebra A. If the Baum-Connes assembly map μ_λ is an isomorphism for G, with coefficients all the separable C^*-subalgebras A and A/J, then the K-theory sequence*

$$K(C^*_\lambda(G, J)) \longrightarrow K(C^*_\lambda(G, A)) \longrightarrow K(C^*_\lambda(G, A/J))$$

is exact in the middle.

Proof. Since exactness of the sequence is preserved by direct limits it suffices to consider the case in which A itself is separable. The proof then follows from a chase around the diagram of assembly maps

$$\begin{tikzcd}
 & K^{top}(G, A) \arrow[r] \arrow[d] & K^{top}(G, A/J) \arrow[d] \\
K(C^*(G, J)) \arrow[r] \arrow[d] & K(C^*(G, A)) \arrow[r] \arrow[d] & K(C^*(G, A/J)) \arrow[d] \\
K(C^*_\lambda(G, J)) \arrow[r] & K(C^*_\lambda(G, A)) \arrow[r] & K(C^*_\lambda(G, A/J))
\end{tikzcd}$$

and the fact that the middle row is exact in the middle.

We shall prove that if G is a Gromov group then for a suitable A and J the conclusion of the lemma fails.

Definition 5.7. *Let A be the C^*-algebra of bounded complex-valued functions on $G \times \mathbb{N}$ for which the restriction to each subset $G \times \{n\}$ is a c_0-function. Denote by J the ideal in A consisting of c_0-functions on $G \times \mathbb{N}$.*

Thus $A \cong \ell^\infty(\mathbb{N}, c_0(G))$ and $J \cong c_0(\mathbb{N}, c_0(G))$.

Now let G act on A be the *right* translation action of G on $G \times \mathbb{N}$.

Lemma 5.7. *The (right regular) covariant representation of A on $\ell^2(G \times \mathbb{N})$ determines a faithful representation of the reduced crossed product algebra $C^*_\lambda(G, A)$ as operators on $\ell^2(G)$.* □

From here on we shall assume that G is a Gromov group. For simplicity we shall now assume that the maps $\varphi_n \colon V_n \to G$ which appear in Definition 5.6 are *injective*. For the general case see [11].

Let V be the disjoint union of the V_n. Let us map V_n via φ_n to the nth copy of G in $G \times \mathbb{N}$, and thereby embed V into $G \times \mathbb{N}$. We can now identify $\ell^2(V)$ with a closed subspace of $\ell^2(G \times \mathbb{N})$.

Definition 5.8. *Denote by $\Delta \colon \ell^2(G \times \mathbb{N}) \to \ell^2(G \times \mathbb{N})$ the direct sum of the Laplace operators on each $\ell^2(V_n) \subseteq \ell^2(V)$ with the identity on the orthogonal complement of $\ell^2(V) \subseteq \ell^2(G \times \mathbb{N})$.*

Lemma 5.8. *The operator $\Delta - I$ belongs to $C^*_\lambda(G, A) \subseteq \mathcal{B}(\ell^2(G \times \mathbb{N}))$ (it is in fact in the algebraic crossed product).*

Proof. First, some notation. Let us continue to identify the vertex set $V_n = V(\Gamma_n)$, via φ_n, with a subset of G. We shall write $[g, g'] \in E(\Gamma_n)$ if the group elements g and g' correspond to vertices in V_n which are adjacent in the graph Γ_n. Finally if g corresponds to a vertex of Γ_n we shall write $k_n(g)$ for its valence, minus 1.

The Hilbert space $\ell^2(G \times \mathbb{N})$ has canonical basis elements f_{gn} and in this basis the formula for Δ is

$$\begin{cases} (\Delta - I) \colon f_{gn} \mapsto k(g) f_{gn} - \displaystyle\sum_{[g,g'] \in E(\Gamma_n)} f_{g'n} & \text{if } g \in V_n \\ (\Delta - I) \colon f_{gn} \mapsto 0 & \text{if } g \notin V_n. \end{cases}$$

We can therefore write $\Delta - I$ as a finite sum

$$\Delta - I = e \cdot a_e + \sum_{h \neq e} h \cdot a_h,$$

where the coefficient functions $a_g \in A$ are defined by

$$a_e(g, n) = \begin{cases} k_n(g) & \text{if } g \in V_n \\ 0 & \text{if } g \notin V_n \end{cases}$$

and, for $h \neq e$,

$$a_h(g, n) = \begin{cases} -1 & \text{if } [gh^{-1}, g] \in E(\Gamma_n) \\ 0 & \text{if } [gh^{-1}, g] \in E(\Gamma_n). \end{cases}$$

(The sum is finite thanks to the first item in Definition 5.6.)

Since the graphs Γ_n are (k,ε)-expanders the point 0 is isolated in the spectrum of Δ, and therefore we can make the following definition:

Definition 5.9. *Let G be a Gromov group and assume that the maps $\varphi_n\colon V_n \to G$ are injective. Denote by $E \in C^*_\lambda(G,A)$ orthogonal projection onto the kernel of Δ.*

The operator E is the orthogonal projection onto the ℓ^2-functions on $G\times\mathbb{N}$ which are constant on each V_n and zero on the complement of V.

Lemma 5.9. *The class of E in $K(C^*_\lambda(G,A))$ is not in the image of the map $K(C^*_\lambda(G,J)) \to K(C^*_\lambda(G,A))$.*

Proof. Let $A_n = c_0(G\times\{n\})$, which is a quotient of A, and denote by

$$\pi_n : C^*_\lambda(G,A) \to C^*_\lambda(G,A_n)$$

the quotient mapping. We get maps

$$\pi_{n*}\colon K(C^*_\lambda(G,A)) \to K(C^*_\lambda(G,A_n)) = \mathbb{Z}.$$

Since $\pi_n(E)$ is a rank one projection, we find $\pi_n([E]) = 1$, for all n. Therefore the K-theory class of p in $K(C^*_\lambda(G,A))$ does not come from $K(C^*_\lambda(G,J))$ (which maps to the direct sum $\oplus_{n\in\mathcal{N}}\mathbb{Z}$ under $\oplus\pi_n$).

Lemma 5.10. *The image of E in $C^*_\lambda(G,A/J)$ is zero.*

To prove the lemma we shall need some means of determining when elements in reduced crossed product algebras $C^*_\lambda(G,D)$ are zero. For this purpose, recall that the C^*-algebra $C^*_\lambda(G,D)$ is faithfully represented as operators on the Hilbert D-module $\ell^2(G,D)$.

Exercise 5.1. If P_g denotes the orthogonal projection onto the functions in $\ell^2(G,D)$ supported on $\{g\}$, and if $T\in C^*_\lambda(G,D)$, then P_gTP_e is an operator from functions supported on $\{e\}$ to functions supported on $\{g\}$. If all the elements P_gTP_e are equal to 0 then $T=0$.

Exercise 5.2. The operator P_gTP_e can be identified with an element $T_g\in D$ via the formula

$$(P_gTP_e\xi)(g) = T_g\cdot\xi(e), \qquad \forall\xi\in\ell^2(G,D).$$

If T is a finite sum $T=\sum d_g\cdot g$ in the algebraic crossed product (where $d_g\in D$) then $T_g = d_g$. If $\varphi\colon D\to D'$ is a G-equivariant $*$-homomorphism and if Φ is the induced map on crossed products then $\Phi(T)_g = \varphi(T_g)$.

By checking on finite sums we see that if an operator $T\in C^*_\lambda(G,A)$ has matrix coefficients $T_{gn,g'n'}$ for the canonical basis of $\ell^2(G,\mathbb{N})$ then the functions $T_g\in A$ associated to T are defined by

$$T_g(h,n) = T_{hgn,hn}.$$

Proof (Proof of Lemma 5.10). The projection $E\colon \ell^2(G\times\mathbb{N})\to\ell^2(G\times\mathbb{N})$ is comprised of the sequence projections E_n onto the constant functions in $\ell^2(V_n)$. The matrix coefficients of E are therefore described by the formula

$$\begin{cases} E\colon f_{gn}\mapsto \displaystyle\sum_{g'\in V_n}\frac{1}{|V_n|}f_{g'n} & \text{if } g\in V_n \\ E\colon f_{gn}\mapsto 0 & \text{if } g\notin V_n. \end{cases}$$

As a result, the functions $E_g\in A$ associated to the projection E, as in the exercises, are given by the formula

$$E_g(h,n)=\begin{cases}\dfrac{1}{|V_n|} & \text{if } hg,h\in V_n \\ 0 & \text{if } hg\notin V_n \text{ or } h\notin V_n.\end{cases}$$

This shows that $E_g\in J$, for all $g\in G$. It follows that the elements $E_n\in A/J$ associated to the image of E in $C^*_\lambda(G,A/J)$ are 0, and so the projection E is itself 0 in $C^*_\lambda(G,A/J)$.

The two lemmas show that the K-theory sequence

$$K(C^*_\lambda(G,J))\longrightarrow K(C^*_\lambda(G,A))\longrightarrow K(C^*_\lambda(G,A/J))$$

fails to be exact in the middle. Hence:

Theorem 5.6. *Let G be a Gromov group. There is a separable, commutative G-C^*-algebra D for which the Baum-Connes assembly map*

$$\mu_\lambda\colon K^{top}(G,D)\to K(C^*_\lambda(G,D))$$

fails to be an isomorphism. □

5.6 Inexact Groups

The following result (see [28, 29, 15]) shows that Gromov groups fail to be exact.

Theorem 5.7. *If a finitely generated discrete group G is exact then G embeds uniformly in a Hilbert space.*

To prove the theorem we shall use a difficult characterization of separable exact C^*-algebras, due to Kirchberg [42] (see also [66] for an exposition). It involves the following notion:

Definition 5.10. *Let A and B be unital C^*-algebras. A unital linear map $\Phi: A\to B$ of C^*-algebras is completely positive if for all $k\in\mathbb{N}$ the linear map $\Phi_k: M_k(A)\to M_k(B)$ defined by applying Φ entrywise to a matrix of elements of A is positive (meaning it maps positive matrices to positive matrices).*

Theorem 5.8. *A separable C^*-algebra A is exact if and only if every injective $*$-homomorphism $A \to \mathcal{B}(\mathcal{H})$ can be approximated in the point norm topology by a sequence of unital completely positive maps, each of which factors, via unital, completely positive maps, through a matrix algebra.* $\square$

Kirchberg's theorem has the following consequence:

Corollary 5.2. *If G is a countable exact group then there exists a sequence of completely positive maps $\Phi_n \colon C^*_\lambda(G) \to \mathcal{B}(\ell^2(G))$ which converge pointwise in norm to the natural inclusion of $C^*_\lambda(G)$ into $\mathcal{B}(\ell^2(G))$ and which have the property that for every $n \in \mathbb{N}$ the operator valued function $g \mapsto \Phi_n(g)$ is supported on a finite subset of G.*

Proof. By Theorem 5.8 there exists a sequence of unital completely positive maps which converge pointwise in norm to the natural inclusion of $C^*_\lambda(G)$ into $\mathcal{B}(\ell^2(G))$, and which individually factor through matrix algebras. Let us write these maps as compositions

$$C^*_\lambda(G) \xrightarrow{\Theta_n} M_{k_n}(\mathbb{C}) \xrightarrow{\Psi_n} \mathcal{B}(\ell^2(G)) \ . \tag{2}$$

Now, a linear map $\Theta \colon C^*_\lambda(G) \to M_k(\mathbb{C})$ is completely positive if and only if the linear map $\theta \colon M_k(C^*_\lambda(G)) \to \mathbb{C}$ defined by the formula

$$\theta([f_{ij}]) = \frac{1}{k} \sum_{i,j=1}^{k} \Theta(f_{ij})_{ij}$$

is a state. Moreover the correspondence $\Theta \leftrightarrow \theta$ is a bijection between completely positive maps and states. In addition, if $h_1, \ldots, h_k$ are finitely supported functions on G which determine a unit vector in the k-fold direct sum $\ell^2(G) \oplus \cdots \oplus \ell^2(G)$, then the vector state

$$\theta([f_{ij}]) = \sum_{i,j=1}^{k} \langle h_i, \lambda(f_{ij}) h_j \rangle$$

on $M_k(C^*_\lambda(G))$ corresponds to a completely positive map Θ which is finitely supported, as a function on G, as in the statement of the lemma. But the convex hull of the vector states associated to a faithful representation of a C^*-algebra is always weak*-dense in the set of all states (this is a version of the Hahn-Banach theorem). It follows that the set of those completely positive maps from $C^*_\lambda(G)$ into $M_k(\mathbb{C})$ which are finitely supported as functions on G is dense, in the topology of pointwise norm-convergence, in the set of all completely positive maps from $C^*_\lambda(G)$ into $M_k(\mathbb{C})$. By approximating the maps Θ_n in the compositions (2) we obtain completely positive maps from $C^*_\lambda(G)$ into $\mathcal{B}(\ell^2(G))$ with the required properties.

Proof (Proof of Theorem 5.7). According to Corollary 5.2 there exists a sequence of unital completely positive maps $\Phi_n \colon C^*_\lambda(G) \to \mathcal{B}(\ell^2(G))$ which converge pointwise in norm to the natural inclusion of $C^*_\lambda(G)$ into $\mathcal{B}(\ell^2(G))$ and which are individually finitely supported as functions on G. Define a sequence of functions

$$\varphi_n : G \times G \to \mathbb{C}$$

by

$$\varphi_n(g_1, g_2) = \langle [g_1^{-1}], \Phi_n(g_1^{-1} g_2)[g_2^{-1}] \rangle.$$

The functions φ_n are *positive-definite kernels* on the set G, in the sense of Definition 4.14. (To prove the inequality $\sum \overline{\lambda_i} \varphi_n(g_i, g_j)\lambda_j \geq 0$ write the sum as a matrix product

$$\begin{bmatrix} g_1 & \dots & g_k \end{bmatrix} \begin{bmatrix} \overline{\lambda_1}\Phi_n(g_1^{-1}g_1)\lambda_1 & \dots & \overline{\lambda_1}\Phi_n(g_1^{-1}g_k)\lambda_k \\ \vdots & \ddots & \vdots \\ \overline{\lambda_k}\Phi_n(g_k^{-1}g_1)\lambda_1 & \dots & \overline{\lambda_k}\Phi_n(g_k^{-1}g_k)\lambda_k \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}$$

and apply the definition of complete positivity.) The functions φ_n converge pointwise to 1, and moreover for every finite subset $F \subseteq G$ and every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$n > N \quad \text{and} \quad g_1^{-1} g_2 \in F \quad \Rightarrow \quad |\varphi_n(g_1, g_2) - 1| < \varepsilon.$$

In addition, for every $n \in \mathbb{N}$ there exists a finite subset $F \subset G$ such that

$$g_1^{-1} g_2 \notin F \quad \Rightarrow \quad \varphi_n(g_1, g_2) = 0.$$

It follows that for a suitable subsequence the series $\sum_j (1 - \varphi_{n_j})$ is pointwise convergent everywhere on $G \times G$. But each function $1 - \varphi_{n_j}$ is a negative type kernel, and therefore so is the sum. The map into affine Euclidean space which is associated to the sum is a uniform embedding.

Remark 5.6. This proof is obviously very similar to that of Proposition 4.9. In fact, according to Remark 4.7 the above argument shows that if a countable group G is exact then G acts amenably on its Stone-Cech compactification βG [28, 29, 15]. As a result: *if a countable group G is exact then the Baum-Connes assembly map*

$$\mu_\lambda : K^{top}(G, D) \to K(C^*_\lambda(G, D))$$

is injective, for every D.

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