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A Bott Periodicity Theorem for Infinite Dimensional Euclidean Space*

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We formulate and prove an equivariant Bott periodicity theorem for infinite dimensional Euclidean vector spaces. The main features of our argument are (i) the construction of a non-commutative C^*-algebra to play the role of the algebra of functions on infinite dimensional Euclidean space; and (ii) the construction of a certain index one elliptic partial differential operator which provides the basis for an inverse to the Bott periodicity map. These tools have applications to index theory and the Novikov conjecture, notably a proof of the Novikov conjecture for amenable groups (the applications will be considered elsewhere).

1. INTRODUCTION

If G is a compact group and X is a compact space then the equivariant K-theory group $K_G^0(X)$ is the Grothendieck group of complex G-vector bundles on X. There is an associated group $K_G^1(X)$, defined using the suspension of X, and if X is a locally compact G-space then its equivariant

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K-theory is defined by means of the one-point compactification, in such a way that

$$K_G^*(\tilde{X}) \cong K_G^*(X) \oplus K_G^*(\mathrm{pt}).$$

The main result in equivariant K-theory is the Bott periodicity theorem [17, 1], which asserts that if E is a finite dimensional Euclidean vector space, on which G acts continuously, linearly and isometrically, then there is an isomorphism

$$K_G^*(TE) \cong K_G^*(\mathrm{pt}).$$

The purpose of this article is to prove a version of the Bott periodicity theorem in the case where the Euclidean space E is possibly infinite dimensional. Our initial impetus to do so came from index theory, more precisely the equivariant index theory of elliptic operators on proper and co-compact G-manifolds, but as we shall point out in a moment, our work has at least one other significant application.

Because we are interested in infinite dimensional spaces, which are not locally compact, and because we wish to consider noncompact groups G, some care must be taken even in the formulation of a Bott periodicity theorem. In particular:

- If E is infinite dimensional then it is not locally compact, and so it lies outside of the ordinary scope of equivariant K-theory, even for compact groups.
- If G is noncompact then the definition of equivariant K-theory as a Grothendieck group of equivariant vector bundles is no longer appropriate.

Both problems are dealt with by means of C^*-algebra K-theory. If G is a compact group, acting on a locally compact space X, then it is known [12] that the equivariant K-theory $K_G^*(X)$ is isomorphic to the C^*-algebra K-theory $K_*(C_0(X) \rtimes G)$, where $C_0(X) \rtimes G$ denotes the crossed product C^*-algebra. So if G is locally compact, and X is any locally compact G-space, then we shall define $K_G^*(X)$ to be $K_*(C_0(X) \rtimes G)$.[1] More generally, if A is any C^*-algebra, equipped with a continuous action of G by C^*-algebra automorphisms, then we shall define $K_*^G(A)$ to be $K_*(A \rtimes G)$. If E is a finite dimensional Euclidean vector space then denote by $\mathscr{C}(E)$ the C^*-algebra of continuous functions, vanishing at infinity, from E into the complexified Clifford algebra of E. If G acts linearly and isometrically on E then it also

[1] If G is noncompact then there is an issue of which version of the C^*-algebra $C_0(X) \rtimes G$ to use—we shall choose the full crossed product [15].

acts by C^*-algebra automorphisms on $\mathscr{C}(E)$, and if we take into account the natural $\mathbb{Z}/2$-grading on $\mathscr{C}(E)$, then this C^*-algebra serves as a non-commutative analogue of $C_0(TE)$. In particular, the Bott periodicity theorem may be reformulated as follows:

$$K_*^G(\mathscr{C}(E)) \cong K_*^G(\mathscr{C}(0)).$$

If E is infinite dimensional then there seems to be no suitable substitute for either $C_0(TE)$ or $\mathscr{C}(E)$. However if we introduce an operation of suspension, and form the ($\mathbb{Z}/2$-graded) tensor product

$$\mathscr{S}\mathscr{C}(E) = C_0(\mathbb{R}) \hat{\otimes} \mathscr{C}(E),$$

then an interesting C^*-algebra may be devised in the infinite dimensional case. To describe it, we will show in Section 2 that if E_1 is a subspace of a finite dimensional Euclidean space E_2 then there is a canonical homomorphism of C^*-algebras, $\mathscr{S}\mathscr{C}(E_1) \to \mathscr{S}\mathscr{C}(E_2)$ which induces the Bott periodicity isomorphism at the level of K-theory. Our construction is functorial, not only at the level of K-theory, but at the level of the C^*-algebras themselves, and this allows us to form a direct limit

$$\mathscr{S}\mathscr{C}(E) = \varinjlim_{E_a \subset E} \mathscr{S}\mathscr{C}(E_a),$$

taken over the directed system of all finite dimensional linear subspaces of an infinite dimensional Euclidean vector space E.

The inclusion of the zero dimensional subspace into E induces a homomorphism $\mathscr{S}\mathscr{C}(0) \to \mathscr{S}\mathscr{C}(E)$, and our main theorem is this:

THEOREM. *If G is a countable discrete group and E is a countably infinite dimensional Euclidean vector space then the map $K_*^G(\mathscr{S}\mathscr{C}(0)) \to K_*^G(\mathscr{S}\mathscr{C}(E))$ is an isomorphism of abelian groups.*

The hypothesis that E be of countably infinite dimensions is not really important. It streamlines a key technical argument while being adequate for the applications we have in mind. The periodicity map makes perfect sense when G is any locally compact group, not necessarily discrete, and our proof of the periodicity theorem extends to this case, at least when G is second countable. But one or two extra calculations are required—they will be outlined in an appendix.

To prove our theorem we shall adapt the well known argument of Atiyah [1], which involves the construction of a left inverse map, followed by a quite formal "rotation" argument to show that the left inverse is also a right inverse. Our construction of the left inverse uses the theory of asymptotic morphisms, and the construction of a certain index one operator on the infinite dimensional Euclidean space E.

We have in mind two applications of our result, both of which will be considered elsewhere (see [18] and [11]). The first is the construction of a topological index for equivariant elliptic operators on proper, G-compact, G-manifolds. Recall that if G is compact and if M is a smooth, closed G-manifold then associated to an equivariant embedding $e\colon M \hookrightarrow \mathbb{R}^n$ (for some linear action of G on $\mathbb{R}^n$) there is a Gysin homomorphism $e_!\colon K_G^*(TM) \to K_G^*(T\mathbb{R}^n)$, and Bott periodicity allows us to define a *topological index map*

$$\begin{tikzcd} K_G^*(TM) \arrow[r, "\text{topological index}"] \arrow[d, "e_!"'] & K_G^*(\mathrm{pt}) \\ K_G^*(T\mathbb{R}^n) \arrow[r, "\cong", "\text{Bott periodicity}"'] & K_G^*(\mathrm{pt}). \arrow[u, "="'] \end{tikzcd}$$

The equivariant K-theory group of a point identifies with the complex representation ring of G, and according to the Atiyah–Singer index theorem [2], if D is an equivariant elliptic operator on M, with symbol class $[\sigma_D] \in K_G(TM)$, then the analytic index of D is equal to the image of $[\sigma_D]$ under the topological index map. Our Bott periodicity theorem allows us to make essentially the same construction in the case where G is locally compact and M is a proper and G-compact G-manifold.

The second application is quite different, and is a development of the relationship, explored in [13], between Bott periodicity and the Novikov higher signature conjecture, in which a version of the Bott periodicity argument, applied to symmetric spaces of non-compact type (which are diffeomorphic to Euclidean space, in a controlled fashion), is used to prove the Novikov conjecture for discrete subgroups of connected Lie groups. In a similar fashion, our Bott periodicity argument may be used to verify the Novikov conjecture (as well as the stronger Baum–Connes conjecture [4]) for groups which act both isometrically and metrically properly on Euclidean space, in the sense of [10]. This applies, for example, to countable amenable groups [5].

The contents of the paper are as follows. In Section 2 we shall give an account of the standard Bott periodicity theorem, from the point of view of C^*-algebra theory, and with a view to our infinite dimensional generalization. In Section 3 we shall construct the C^*-algebra $\mathcal{SC}(E)$ associated to an infinite dimensional Euclidean space and formulate our periodicity theorem for it. Sections 4 and 5 contain the substance of our proof. Finally there are three appendices which review some aspects of $\mathbb{Z}/2$-graded C^*-algebra theory; complete a calculation involving Mehler's formula; and provide some results relevant to the periodicity theorem for continuous groups.

2. CLIFFORD ALGEBRAS AND BOTT PERIODICITY

Let E be a finite dimensional Euclidean vector space (i.e., a real inner product space).

1. Definition. Denote by *Cliff*(E) the Clifford algebra of E, that is, the universal complex algebra with unit which contains E as a real linear subspace in such a way that $e^2 = \|e\|^2\, 1$, for every e in E.

If $\{e_1, ..., e_n\}$ is an orthonormal basis for E then the monomials $e_{i_1} \cdots e_{i_k}$, for $i_1 < \cdots < i_k$, form a linear basis for *Cliff*(E). We give *Cliff*(E) a Hermitian inner product by deeming these monomials to be orthonormal; the inner product does not depend on the choice of basis.

By thinking of the *algebra Cliff*(E) as acting by left multiplication on the *Hilbert space Cliff*(E), we endow the Clifford algebra with the structure of a C^*-algebra, in such a way that each $e \in E$ is self-adjoint. It is a $\mathbb{Z}/2$-graded C^*-algebra, with each $e \in E$ having grading degree one. The structure of *Cliff*(E) is well known. See [14] for a discussion of Clifford algebras, and [6] for a general discussion of graded C^*-algebras, along with Appendix A for a few supplementary remarks.

2. Definition. Denote by $\mathscr{C}(E)$ the $\mathbb{Z}/2$-graded C^*-algebra $C_0(E, \mathit{Cliff}(E))$ of continuous, *Cliff*(E)-valued functions on E which vanish at infinity, with $\mathbb{Z}/2$-grading induced from *Cliff*(E).

3. Definition. Denote by $\mathscr{S} = C_0(\mathbb{R})$ the C^*-algebra of continuous complex-valued functions on $\mathbb{R}$ which vanish at infinity. Grade $\mathscr{S}$ according to even and odd functions. If A is any $\mathbb{Z}/2$-graded C^*-algebra then let $\mathscr{S}A$ be the graded tensor product $\mathscr{S} \hat{\otimes} A$. In particular, let $\mathscr{S}\mathscr{C}(E) = \mathscr{S} \hat{\otimes} \mathscr{C}(E)$.

The C^*-algebra $\mathscr{S}\mathscr{C}(E)$ carries a natural $\mathbb{Z}/2$-grading, as does any graded tensor product, but when we come to consider its K-theory, we shall ignore the grading. To make this matter clear, *from now on, if A is a C^*-algebra—graded or not—then $K_*(A)$ will denote the K-theory of the underlying C^*-algebra, forgetting the grading*. The Bott periodicity isomorphism we wish to address is the isomorphism

$$K_*(\mathscr{S}\mathscr{C}(0)) \cong K_*(\mathscr{S}\mathscr{C}(E)).$$

In the remainder of this section we shall describe the construction of a $*$-homomorphism

$$\beta\colon \mathscr{S}\mathscr{C}(0) \to \mathscr{S}\mathscr{C}(E)$$

which implements the Bott periodicity isomorphism, and then give Atiyah's proof of Bott periodicity, but using the language of asymptotic morphisms. This will set the stage for the infinite dimensional generalization in the coming sections.

4. Definition. The *Clifford operator* on E is the function $C: E \to \mathit{Cliff}(E)$ whose value at $e \in E$ is $e \in \mathit{Cliff}(E)$. It is a degree one, essentially self-adjoint, unbounded multiplier of $\mathscr{C}(E)$, with domain the compactly supported functions in $\mathscr{C}(E)$ (see Appendix A for some remarks concerning unbounded multipliers).

5. Definition. Denote by X the function of multiplication by x on $\mathbb{R}$, viewed as a degree one, essentially self-adjoint, unbounded multiplier of $\mathscr{S} = C_0(\mathbb{R})$ with domain the compactly supported functions in $\mathscr{S}$.

The operator $X \hat{\otimes} 1 + 1 \hat{\otimes} C$ is a degree one, essentially self-adjoint, unbounded multiplier of $\mathscr{S}\mathscr{C}(E) = \mathscr{S} \hat{\otimes} \mathscr{C}(E)$ (see Appendix A). Both C and X have compact resolvents (in the sense of multiplier algebra theory), and therefore so does $X \hat{\otimes} 1 + 1 \hat{\otimes} C$. So we use the functional calculus to define

$$\beta: \mathscr{S} = \mathscr{S}\mathscr{C}(0) \to \mathscr{S}\mathscr{C}(E)$$

by the formula

$$\beta: f \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} C).$$

We aim now to prove the following theorem:

6. Bott Periodicity Theorem. *The $*$-homomorphism $\beta: \mathscr{S}\mathscr{C}(0) \to \mathscr{S}\mathscr{C}(E)$ induces an isomorphism $\beta_*: K_*(\mathscr{S}\mathscr{C}(0)) \to K_*(\mathscr{S}\mathscr{C}(E))$.*

The key to the proof is the construction of an inverse to the periodicity map $\beta_*: K_*(\mathscr{S}\mathscr{C}(0)) \to K_*(\mathscr{S}\mathscr{C}(E))$ in the form of an asymptotic morphism. So we begin by recalling from [8] a few particulars from the theory of asymptotic morphisms.

If A and B are C^*-algebras then an asymptotic morphism from A to B is a family of functions $\varphi_t: A \to B$, parametrized by $t \in [1, \infty)$, such that $\varphi_t(a)$ is norm continuous in t, and

$$\lim_{t \to \infty} \begin{Bmatrix} \varphi_t(a_1)\,\varphi_t(a_2) - \varphi_t(a_1 a_2) \\ \varphi_t(a_1) + \varphi_t(a_2) - \varphi_t(a_1 + a_2) \\ \lambda\varphi_t(a_1) - \varphi_t(\lambda a_1) \\ \varphi_t(a_1^*) - \varphi_t(a_1)^* \end{Bmatrix} = 0,$$

for all $a_1, a_2 \in A$ and $\lambda \in \mathbb{C}$. An asymptotic morphism from A to B determines a $*$-homomorphism from A into the quotient C^*-algebra $\mathbf{Q}(B)$ of bounded continuous functions from $[1, \infty)$ into B, modulo the ideal of continuous functions from $[1, \infty)$ to B which vanish at infinity:

$$\varphi\colon A \to \mathbf{Q}(B) = C_b([1, \infty), B)/C_0([1, \infty), B).$$

If φ and φ' are asymptotically equivalent, that is if $\varphi_t(a) - \varphi'_t(a) \to 0$, for every $a \in A$, then φ and φ' determine the same homomorphism from A to $\mathbf{Q}(B)$. There is a one to one correspondence between $*$-homomorphisms from A to $\mathbf{Q}(B)$ and asymptotic equivalence classes of asymptotic morphisms from A to B. Every $*$-homomorphism $\varphi\colon A \to B$ may be regarded as an asymptotic morphism by setting $\varphi_t = \varphi$, for all t.

A homotopy of asymptotic morphisms is an asymptotic morphism from A to $C([0, 1], B)$. Asymptotically equivalent asymptotic morphisms are homotopy equivalent.

An asymptotic morphism $\varphi\colon A \to B$ induces a map $\varphi_*\colon K_*(A) \to K_*(B)$. The induced map depends only on the homotopy class of φ. It is obviously possible to compose asymptotic morphisms with $*$-homomorphisms (either on the left or the right) and the action on K-theory is functorial.

Finally, if $\varphi\colon A \to B$ is an asymptotic morphism and C is a third C^*-algebra then we can form the tensor product $\varphi \otimes 1\colon A \otimes C \to B \otimes C$. As a rule one must use the maximal tensor product here, but of course in the context of nuclear C^*-algebras all tensor products agree. If φ is grading preserving then there is a similar construction with the graded tensor product. Both the graded and ungraded constructions are well defined at the level of homotopy classes, and are functorial with respect to composition with $*$-homomorphisms.

Having reviewed these points, we are ready to proceed.

7. Definition. Denote by $\mathfrak{h}(E)$ the $\mathbb{Z}/2$-graded Hilbert space

$$\mathfrak{h}(E) = L^2(E, \mathit{Cliff}(E))$$

of square-integrable functions from E into $\mathit{Cliff}(E)$. The *Dirac operator* is the unbounded operator

$$D = \sum_{i=1}^{n} \widehat{e_i} \frac{\partial}{\partial x_i},$$

on $\mathfrak{h}(E)$, where $\{e_1, ..., e_n\}$ is an orthonormal basis for E, $\{x_1, ..., x_n\}$ is the dual coordinate system on E, and $\widehat{e_i}$ denotes the operator of *right* multiplication by e_i on $\mathfrak{h}(E)$, twisted by the $\mathbb{Z}/2$-grading:

$$\widehat{e_i}\,\xi = (-1)^{\deg(\xi)}\,\xi e_i.$$

The domain of D is the Schwartz subspace of $\mathfrak{h}(E)$.

The Dirac operator is a grading-degree one operator. As a constant coefficient partial differential operator, it is easily checked to be essentially self-adjoint. We can also view it as an essentially self-adjoint multiplier of the C^*-algebra $\mathfrak{K}(\mathfrak{h}(E))$ of compact operators on $\mathfrak{h}(E)$, with domain those compact operators which map $\mathfrak{h}(E)$ continuously into the Schwartz space.

8. Definition. Using the Dirac operator, we define an asymptotic morphism

$$\varphi\colon \mathscr{S}\mathscr{C}(E) \to \mathscr{S}\mathfrak{K}(\mathfrak{h}(E))$$

as follows:

$$\varphi_t\colon f \mathbin{\hat{\otimes}} h \mapsto f_t(X \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} D)(1 \mathbin{\hat{\otimes}} M_{h_t}),$$

where $f_t(x) = f(t^{-1}x)$; $h_t(e) = h(t^{-1}e)$; and M_{h_t} denotes the operator on $\mathfrak{h}(E)$ of left multiplication by h_t.

The fact that $\varphi_t(f \mathbin{\hat{\otimes}} h)$ lies in $\mathfrak{K}(\mathfrak{h}(E))$ is proved as follows. It suffices to consider the case where f is one of the generators $\exp(-x^2)$ or $x\exp(-x^2)$ of the C^*-algebra $\mathscr{S}$, and where h is compactly supported. In the first case $\varphi_t(f \mathbin{\hat{\otimes}} h)$ is the element $\exp(-t^{-2}x^2) \mathbin{\hat{\otimes}} \exp(-t^{-2}D^2)\, M_{h_t}$, while in the second it is

$$\begin{aligned} t^{-1}x\exp(-t^{-2}x^2) \mathbin{\hat{\otimes}} \exp(-t^{-2}D^2)\, M_{h_t} &+ \exp(-t^{-2}x^2) \\ &\mathbin{\hat{\otimes}} t^{-1}D\exp(-t^{-2}D^2)\, M_{h_t}. \end{aligned}$$

But $f_t(D)\, M_{h_t} \in \mathfrak{K}(\mathfrak{h}(E))$, for any $f \in C_0(\mathbb{R})$: this is an elementary consequence of the ellipticity of the Dirac operator and the Rellich lemma. Compare [16], for instance.

Granted this, our formula for φ_t defines a function from the algebraic tensor product of $\mathscr{S}$ and $\mathscr{C}(E)$ into $\mathbf{Q}(\mathscr{S}\mathfrak{K}(\mathfrak{h}(E)))$. To obtain from it an asymptotic morphism we need to note that this function is in fact a $*$-homomorphism, and hence extends to the C^*-algebra tensor product. This is an immediate consequence of the following calculation:

9. Lemma. *If $f \in C_0(\mathbb{R})$ and $h \in \mathscr{C}(E)$ then*

$$\lim_{t\to\infty} \|[f_t(X \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} D), (1 \mathbin{\hat{\otimes}} M_{h_t})]\| = 0,$$

where the brackets $[\ ,\ ]$ denote the graded commutator.

Proof. It suffices to prove this when h is smooth and compactly supported and when f is one of the generators e^{-x^2} or xe^{-x^2} for $C_0(\mathbb{R})$. The commutator above is the one of

$$e^{-t^{-2}x^2} \hat{\otimes} [e^{-t^{-2}D^2}, M_{h_t}]$$

or

$$e^{-t^{-2}x^2} \hat{\otimes} [t^{-1}De^{-t^{-2}D^2}, M_{h_t}] + t^{-1}xe^{-t^{-2}x^2} \hat{\otimes} [e^{-t^{-2}D^2}, M_{h_t}].$$

So the lemma follows from the simpler assertion that if $f \in C_0(\mathbb{R})$ then

$$\lim_{t \to \infty} \|[f_t(D), M_{h_t}]\| = 0.$$

To prove this, consider the generators $f(x) = (x \pm i)^{-1}$ and note that

$$[f_t(D), M_{h_t}] = -f_t(D)[t^{-1}D, M_{h_t}]\, \gamma^{\deg(h)}(f_t(D)),$$

where γ is the grading automorphism (see Appendix A). But

$$[t^{-1}D, M_{h_t}] = t^{-1} \sum_{i=1}^{n} \widehat{e_i} \frac{\partial h_t}{\partial x_i},$$

from which it is easy to see that the norm of the commutator is t^{-2} times the supremum of the gradient of h. ∎

10. Definition. Let $P \in \mathfrak{K}(\mathfrak{h}(E))$ be the orthogonal projection onto the one-dimensional subspace of $\mathfrak{h}(E)$ spanned by the Clifford algebra-valued function $\exp(-\frac{1}{2}\|e\|^2) \cdot 1$.

Using the stability property of K-theory [6] it is easily checked that:

11. Lemma. *If A is any graded C^*-algebra then the $*$-homomorphism $\sigma\colon A \to A \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E))$ defined by the formula $\sigma\colon a \mapsto a \hat{\otimes} P$ induces an isomorphism on K-theory.*

We can now define a left inverse to the periodicity map by the diagram

$$K_*(\mathscr{S}\mathscr{C}(E)) \xrightarrow{\varphi_*} K_*(\mathscr{S}\mathfrak{K}(\mathfrak{h}(E))) \underset{\cong}{\xrightarrow{\sigma_*^{-1}}} K_*(\mathscr{S}).$$

To prove that it really is a left inverse, we begin by considering the composition

$$\mathscr{S} \xrightarrow{\beta} \mathscr{S}\mathscr{C}(E) \xrightarrow{\varphi} \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)).$$

12. Definition. Define an unbounded symmetric operator on $\mathfrak{h}(E)$, with domain the Schwartz space, by

$$B = \sum_{i=1}^{n} \widehat{e_i} \frac{\partial}{\partial x_i} + e_i x_i.$$

Thus $B = D + C$, where D is the Dirac operator of Definition 7 and C is the Clifford operator of Definition 4.

It is not hard to show that B is essentially self-adjoint, and has compact resolvent. One way of doing this will be indicated in a moment.

13. Proposition. *The composition*

$$\mathcal{S} \xrightarrow{\beta} \mathcal{S}\mathcal{C}(E) \xrightarrow{\varphi} \mathcal{S}\mathfrak{K}(\mathfrak{h}(E)),$$

which is an asymptotic morphism from $\mathcal{S}$ *to* $\mathcal{S}\mathfrak{K}(\mathfrak{h}(E))$, *is asymptotic to the family of* $*$*-homomorphisms*

$$f \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} B).$$

We defer the proof of this, which is an application of Mehler's formula, until Appendix B.

To further analyze the composition $\varphi\beta$ we need to understand the spectral theory of B. The following is a straightforward calculation:

14. Lemma. *The square of* B *is*

$$B^2 = \sum_{i=1}^{n} -\frac{\partial^2}{\partial x_i^2} + x_i^2 + N,$$

where N *is the "number operator" on* $\mathit{Cliff}(E)$ *defined by*

$$N e_{i_1} e_{i_2} \cdots e_{i_k} = (2k - n)\, e_{i_1} e_{i_2} \cdots e_{i_k}.$$

The well known spectral theory for the harmonic oscillator $-d^2/dx^2 + x^2$ (see for instance [9]) gives us the following:

15. Corollary. (i) *B^2 admits an orthonormal eigenbasis of Schwartz-class functions, with eigenvalues* $2n$ $(n = 0, 1, ...)$, *each of finite multiplicity. Hence* B *admits an orthonormal eigenbasis of Schwartz-class functions, with eigenvalues* $\pm\sqrt{2n}$, *each of finite multiplicity.*

(ii) *The kernel of (the closure of)* B *is spanned by* $\exp(-\frac{1}{2}\|e\|^2) \cdot 1$.

Note that since B has an orthonormal eigenbasis of Schwartz class functions, it is essentially self-adjoint on the Schwartz space. In addition, it has compact resolvent, since its eigenvalue sequence converges to infinity in absolute value.

16. PROPOSITION. *The map*

$$K_*(\mathcal{S}\mathcal{C}(E)) \xrightarrow{\varphi_*} K_*(\mathcal{S}\mathfrak{K}(\mathfrak{h}(E))) \xrightarrow[\cong]{\sigma_*^{-1}} K_*(\mathcal{S})$$

is left inverse to the periodicity map $\beta_*\colon K_*(\mathcal{S}) \to K_*(\mathcal{S}\mathcal{C}(E))$.

Proof. Since the asymptotic morphism $f \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} B)$ is just a continuous family of $*$-homomorphisms, it is homotopic to the $*$-homomorphism $f \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} B)$, via the homotopy

$$f \mapsto f_{1-s+st}(X \hat{\otimes} 1 + 1 \hat{\otimes} B) \qquad s \in [0, 1].$$

Since the operator P in Definition 10 is the orthogonal projection onto the kernel of B, the $*$-homomorphism $f \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} B)$ is in turn homotopic to the $*$-homomorphism $f \mapsto f \hat{\otimes} P$, along the homotopy of $*$-homomorphisms

$$f \mapsto \begin{cases} f(X \hat{\otimes} 1 + s^{-1} \hat{\otimes} B) & s \in (0, 1] \\ f \hat{\otimes} P & s = 0. \end{cases}$$

Thus by Proposition 13 the composition $\varphi\beta$ is homotopic to σ, and hence $\sigma_*^{-1}\varphi_*\beta_* = \mathrm{id}$, as required. ∎

To complete the proof of the Bott periodicity theorem we must argue that the reverse composition of the periodicity map β_* with the homomorphism $\sigma_*^{-1}\varphi_*$ in Proposition 16 is the identity. It is given by the diagram

$$\begin{tikzcd} \mathcal{S}\mathcal{C}(E) \arrow[r, "\varphi"] & \mathcal{S}\mathfrak{K}(\mathfrak{h}(E)) & \\ & \mathcal{S} \arrow[u, "\sigma"] \arrow[r, "\beta"] & \mathcal{S}\mathcal{C}(E), \end{tikzcd}$$

which we can "complete" to get the following:

$$\begin{tikzcd} \mathcal{S}\mathcal{C}(E) \arrow[r, "\varphi"] & \mathcal{S}\mathfrak{K}(\mathfrak{h}(E)) \arrow[r, "\beta \hat{\otimes} 1"] & \mathcal{S}\mathcal{C}(E) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E)) \\ & \mathcal{S} \arrow[u, "\sigma"] \arrow[r, "\beta"] & \mathcal{S}\mathcal{C}(E). \arrow[u, "\sigma"'] \end{tikzcd}$$

Since the vertical maps induce isomorphisms on K-theory, to prove that $\sigma_*^{-1}\varphi_*$ gives a right inverse to β it suffices to prove that the composition of φ with $\beta \hat{\otimes} 1$ along the top row of the diagram induces an isomorphism in K-theory. To do this we introduce one more diagram:

$$\begin{tikzcd}
\mathscr{S}\mathscr{C}(E) \arrow[r, "\varphi"] \arrow[d, "\beta\hat{\otimes}1"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \arrow[r, "\beta\hat{\otimes}1"] & \mathscr{S}\mathscr{C}(E)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E)) \arrow[d, "\text{flip}"] \\
\mathscr{S}\mathscr{C}(E)\hat{\otimes}\mathscr{C}(E) \arrow[r, "\text{flip}"'] & \mathscr{S}\mathscr{C}(E)\hat{\otimes}\mathscr{C}(E) \arrow[r, "\tilde{\varphi}"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E))\hat{\otimes}\mathscr{C}(E).
\end{tikzcd}$$

The maps denoted "flip" are the $*$-isomorphisms $x \hat{\otimes} y \mapsto (-1)^{\deg(x)\deg(y)} y \hat{\otimes} x$ which exchange the factors of a graded tensor product and the asymptotic morphism $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}_t\colon f \hat{\otimes} h \hat{\otimes} k \mapsto \varphi_t(f \hat{\otimes} h) \hat{\otimes} k_t.$$

17. Lemma. *The above diagram asymptotically commutes.*

Proof. We shall check this on the generators $\exp(-x^2) \hat{\otimes} h$ and $x\exp(-x^2) \hat{\otimes} h$ of $\mathscr{S}\mathscr{C}(E)$. On the first generator, going around the diagram in either direction produces the element

$$\exp(-t^{-2}x^2) \hat{\otimes} \exp(-t^{-2}D^2)\, M_{h_t} \hat{\otimes} \exp(-t^{-2}C^2).$$

On the second generator we get

$$\begin{aligned}
t^{-1}x \exp(-t^{-2}x^2) &\hat{\otimes} \exp(-t^{-2}D^2)\, M_{h_t} \hat{\otimes} \exp(-t^{-2}C^2) \\
&+ \exp(-t^{-2}x^2) \hat{\otimes} t^{-1}D \exp(-t^{-2}D^2)\, M_{h_t} \hat{\otimes} \exp(-t^{-2}C^2) \\
&\pm \exp(-t^{-2}x^2) \hat{\otimes} \exp(-t^{-2}D^2)\, M_{h_t} \hat{\otimes} t^{-1}C \exp(-t^{-2}C^2),
\end{aligned}$$

where the sign varies according as h is of even or odd grading degree. In other words, the diagram is exactly commutative on generators. ∎

18. Lemma. *The flip map on $\mathscr{C}(E) \hat{\otimes} \mathscr{C}(E)$ is homotopic, through graded $*$-homomorphisms, to the map $h_1 \hat{\otimes} h_2 \mapsto h_1 \hat{\otimes} \iota h_2$, where ι is the automorphism of $\mathscr{C}(E)$ induced from the automorphism $e \mapsto -e$ of E.*

Remark. If $g\colon E_1 \to E_2$ is an isometric isomorphism then it extends to a $*$-isomorphism $g\colon \mathit{Cliff}(E_1) \to \mathit{Cliff}(E_2)$, and there is an induced $*$-isomorphism $g\colon \mathscr{C}(E_1) \to \mathscr{C}(E_2)$ defined by $(gh)(e_2) = g(h(g^{-1}e_2))$. This explains the automorphism ι in the lemma.

Proof of Lemma 18. If E_1 and E_2 are finite dimensional Euclidean spaces then there is a $*$-isomorphism

$$\mathit{Cliff}(E_1 \oplus E_2) \cong \mathit{Cliff}(E_1) \hat{\otimes} \mathit{Cliff}(E_2)$$

defined by $(e_1, e_2) \mapsto e_1 \hat{\otimes} 1 + 1 \hat{\otimes} e_2$. This in turn gives an isomorphism

$$\mathscr{C}(E_1) \hat{\otimes} \mathscr{C}(E_2) \cong \mathscr{C}(E_1 \oplus E_2)$$

defined by $h_1 \hat{\otimes} h_2(e_1, e_2) = h_1(e_1) \hat{\otimes} h_2(e_2)$, which is a $\mathit{Cliff}(E_1) \hat{\otimes} \mathit{Cliff}(E_2)$-valued function on $E_1 \oplus E_2$.

It follows from these considerations that $\mathscr{C}(E) \hat{\otimes} \mathscr{C}(E)$ is $*$-isomorphic to $\mathscr{C}(E \oplus E)$, and it is easy to check that under this identification, the flip corresponds to the automorphism of $\mathscr{C}(E \oplus E)$ induced from the flip automorphism $(e_1, e_2) \mapsto (e_2, e_1)$ on $E \oplus E$. But this flip is homotopic to the automorphism $(e_1, e_2) \mapsto (e_1, -e_2)$ through the family of rotations

$$r_s \colon (e_1, e_2) \mapsto (\sin(s)\, e_1 + \cos(s)\, e_2, \cos(s)\, e_1 - \sin(s)\, e_2),$$

where $s \in [0, \pi/2]$. ∎

Proof of Theorem 6. Returning to the last commutative diagram, and in view of Lemma 18, to prove that the top row induces an isomorphism in K-theory it suffices to show that the composition

$$\mathscr{S}\mathscr{C}(E) \xrightarrow{\beta \hat{\otimes} \iota} \mathscr{S}\mathscr{C}(E) \hat{\otimes} \mathscr{C}(E) \xrightarrow{\tilde{\varphi}} \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \hat{\otimes} \mathscr{C}(E)$$

induces an isomorphism. For this we shall follow the proof of Proposition 16. By Proposition 13 the composition is asymptotic to the family of $*$-homomorphisms

$$f \hat{\otimes} h \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} B) \hat{\otimes} \iota(h_t).$$

It is thus homotopic to the single $*$-homomorphism $f \hat{\otimes} h \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} B) \hat{\otimes} \iota(h)$ via the homotopy

$$f \hat{\otimes} h \mapsto f_{1-s+st}(X \hat{\otimes} 1 + 1 \hat{\otimes} B) \hat{\otimes} \iota(h_{1-s+st}) \qquad s \in [0, 1].$$

This in turn is homotopic to the $*$-homomorphism $\sigma \hat{\otimes} \iota \colon f \hat{\otimes} h \mapsto f \hat{\otimes} P \hat{\otimes} \iota(h)$ via the homotopy

$$f \hat{\otimes} h \mapsto \begin{cases} f(X \hat{\otimes} 1 + s^{-1} \hat{\otimes} B) \hat{\otimes} \iota(h) & s \in (0, 1] \\ f \hat{\otimes} P \hat{\otimes} \iota(h) & s = 0. \end{cases}$$

It follows from Lemma 11 that $\sigma \hat{\otimes} \iota$ induces an isomorphism in K-theory. ∎

3. THE C^*-ALGEBRA OF AN INFINITE DIMENSIONAL EUCLIDEAN SPACE

Let E be an infinite dimensional Euclidean vector space. We are going to construct a C^*-algebra $\mathscr{S}\mathscr{C}(E)$ analogous to the one constructed in the previous section for finite dimensional Euclidean spaces. It should be noted that in the infinite dimensional case $\mathscr{S}\mathscr{C}(E)$ will *not* be the suspension of a C^*-algebra $\mathscr{C}(E)$. Nevertheless our notation seems reasonably appropriate, and will (we hope) not cause any confusion.

1. Definition. Let E_b be a finite dimensional Euclidean vector space and let E_a be a linear subspace of E_b. Define a $*$-homomorphism

$$\beta_{ba}\colon \mathscr{S}\mathscr{C}(E_a) \to \mathscr{S}\mathscr{C}(E_b)$$

as follows. Let E_{ba} be the orthogonal complement of E_a in E_b and note the isomorphism

$$\mathscr{S}\mathscr{C}(E_b) \cong \mathscr{S}\mathscr{C}(E_{ba}) \hat{\otimes} \mathscr{C}(E_a),$$

coming from the isomorphism $\mathscr{C}(E_{ba}) \hat{\otimes} \mathscr{C}(E_a) \cong \mathscr{C}(E_{ba} \oplus E_a)$ described in the proof of Lemma 2.18. Using this we define

$$\beta_{ba} = \beta \hat{\otimes} 1\colon \mathscr{S} \hat{\otimes} \mathscr{C}(E_a) \to \mathscr{S}\mathscr{C}(E_{ba}) \hat{\otimes} \mathscr{C}(E_a),$$

where $\beta\colon \mathscr{S} \to \mathscr{S}\mathscr{C}(E_{ba})$ is the Bott periodicity map from Theorem 2.6.

Remark. If we follow the $*$-homomorphism $\mathscr{S}\mathscr{C}(E_a) \to \mathscr{S}\mathscr{C}(E_b)$ with the restriction map

$$\mathscr{S}\mathscr{C}(E_b) = \mathscr{S} \hat{\otimes} C_0(E_b, \mathit{Cliff}(E_b)) \to \mathscr{S} \hat{\otimes} C_0(E_a, \mathit{Cliff}(E_b))$$

we get the map $\mathscr{S} \hat{\otimes} C_0(E_a, \mathit{Cliff}(E_a)) \hookrightarrow \mathscr{S} \hat{\otimes} C_0(E_a, \mathit{Cliff}(E_b))$ induced from the inclusion of $\mathit{Cliff}(E_a)$ into $\mathit{Cliff}(E_b)$. This shows that the $*$-homomorphism β_{ba} is injective.

2. Proposition. *Suppose that $E_a \subset E_b \subset E_c$. The composition*

$$\mathscr{S}\mathscr{C}(E_a) \xrightarrow{\beta_{ba}} \mathscr{S}\mathscr{C}(E_b) \xrightarrow{\beta_{cb}} \mathscr{S}\mathscr{C}(E_c)$$

is the $*$-homomorphism $\beta_{ca}\colon \mathscr{S}\mathscr{C}(E_a) \to \mathscr{S}\mathscr{C}(E_c)$.

Proof. The composition is as follows:

$$\begin{aligned} \mathscr{S} \hat{\otimes} \mathscr{C}(E_a) &\xrightarrow{\beta \hat{\otimes} 1} \mathscr{S}\mathscr{C}(E_{ba}) \hat{\otimes} \mathscr{C}(E_a) \\ &\xrightarrow{\beta \hat{\otimes} 1 \hat{\otimes} 1} \mathscr{S} \hat{\otimes} \mathscr{C}(E_{cb}) \hat{\otimes} \mathscr{C}(E_{ba}) \hat{\otimes} \mathscr{C}(E_a). \end{aligned}$$

Since it is a tensor product with the identity on $\mathscr{C}(E_a)$, and since β_{ca} is of the same type, we shall ignore the factor $\mathscr{C}(E_a)$ in what follows.

Let us calculate the composition on the generators $\exp(-x^2)$ and $x\exp(-x^2)$ of the C^*-algebra $\mathscr{S}$. We obtain the elements

$$\exp(-x^2)\,\hat{\otimes}\,\exp(-C_{cb}^2)\,\hat{\otimes}\,\exp(-C_{ba}^2)$$

and

$$\begin{aligned} x\exp(-x^2)\,&\hat{\otimes}\,\exp(-C_{cb}^2)\,\hat{\otimes}\,\exp(-C_{ba}^2)\\ &+\exp(-x^2)\,\hat{\otimes}\,C_{cb}\exp(-C_{cb}^2)\,\hat{\otimes}\,\exp(-C_{ba}^2)\\ &+\exp(-x^2)\,\hat{\otimes}\,\exp(-C_{cb}^2)\,\hat{\otimes}\,C_{ba}\exp(-C_{ba}^2),\end{aligned}$$

where C_{ba} denotes the Clifford operator on E_{ba}, and so on. Under the isomorphism $\mathscr{C}(E_{ca})\cong\mathscr{C}(E_{cb})\,\hat{\otimes}\,\mathscr{C}(E_{ba})$ the self-adjoint closure of C_{ca} corresponds to the self-adjoint closure of $C_{cb}\,\hat{\otimes}\,1+1\,\hat{\otimes}\,C_{ba}$. Hence the element $\exp(-C_{ca}^2)$ corresponds to $\exp(-C_{cb}^2)\,\hat{\otimes}\,\exp(-C_{ba}^2)$, while the element $C_{ca}\exp(-C_{ca})^2$ corresponds to

$$C_{cb}\exp(-C_{cb}^2)\,\hat{\otimes}\,\exp(-C_{ba}^2)+\exp(-C_{cb}^2)\,\hat{\otimes}\,C_{ba}\exp(-C_{ba}^2)$$

(see Appendix A). Therefore the composition $\beta_{cb}\beta_{ba}$ agrees with β_{ca} on generators, and hence the two maps are equal. ∎

3. Definition. Let E be an infinite dimensional Euclidean space. We define $\mathscr{SC}(E)$ to be the direct limit C^*-algebra

$$\mathscr{SC}(E)=\varinjlim_{a}\mathscr{SC}(E_a),$$

where the direct limit is over the directed set of all finite dimensional subspaces $E_a\subset E$, using the $*$-homomorphisms β_{ba} of Definition 1.

The inclusion of the zero subspace into E induces a $*$-homomorphism

$$\beta\colon \mathscr{SC}(0)\to\mathscr{SC}(E),$$

which is an "infinite dimensional" version of the Bott periodicity homomorphism considered in the last section. It follows easily from Theorem 2.6, along with the commutation of the K-functor and direct limits [6], that this infinite dimensional periodicity homomorphism induces an isomorphism on K-theory. However we are interested in an *equivariant* form of the periodicity isomorphism, which does not reduce so readily to the finite dimensional case. To formulate it, we introduce the following notation.

4. Definition. Let G be a locally compact group and let A be a C^*-algebra on which G acts continuously by C^*-algebra automorphisms. We define $K_*^G(A)$ to be the K-theory of the full crossed product C^*-algebra $A \rtimes G$:

$$K_*^G(A) = K_*(A \rtimes G).$$

See [15] for a discussion of crossed product C^*-algebras; we shall only use the theory for discrete groups in what follows. If $\varphi\colon A \to B$ is an equivariant $*$-homomorphism then there is an induced C^*-algebra homomorphism from $A \rtimes G$ to $B \rtimes G$, and hence an induced map $\varphi_*\colon K_*^G(A) \to K_*^G(B)$.

Remark. It will become clear in the next section why we choose to use the full crossed product C^*-algebra.

Suppose now that a discrete group G acts linearly and isometrically on E. For each $g \in G$ and each $E_a \subset E$ there is an induced $*$-isomorphism $g\colon \mathscr{S}\mathscr{C}(E_a) \to \mathscr{S}\mathscr{C}(gE_a)$ (see the remark following Lemma 2.18). This is functorial with respect to the $*$-homomorphisms β_{ba}, since g transforms the Clifford operator on E_a to the Clifford operator on gE_a, and so it follows that G acts on the direct limit C^*-algebra $\mathscr{S}\mathscr{C}(E)$ by $*$-automorphisms.

The inclusion $0 \subset E$ of the zero-dimensional subspace into E induces a G-equivariant inclusion $\mathscr{S}\mathscr{C}(0) \to \mathscr{S}\mathscr{C}(E)$, and the main result of this paper is as follows:

5. Theorem. *Let E be a Euclidean vector space of countable dimension and let G be a countable, discrete group which acts linearly and isometrically on E. The inclusion $0 \subset E$ induces an isomorphism in equivariant K-theory*:

$$\beta_*\colon K_*^G(\mathscr{S}\mathscr{C}(0)) \xrightarrow{\cong} K_*^G(\mathscr{S}\mathscr{C}(E)).$$

We shall divide the proof between the next two sections. In the first we shall generalize the finite dimensional argument of Section 2 to the infinite dimensional situation, and in the second we shall resolve a technical point concerning direct limits which is needed to complete the infinite dimensional argument.

4. PROOF OF THE PERIODICITY THEOREM

Let E be an infinite dimensional Euclidean space. Following the construction in Section 2, if E_a is a finite dimensional subspace of E then there is an asymptotic morphism which we shall now write as

$$\varphi_a\colon \mathscr{S}\mathscr{C}(E_a) \to \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)),$$

defined using the Dirac operator on E_a. We are going to assemble these asymptotic morphisms to obtain an asymptotic morphism

$$\varphi\colon \varinjlim_a SC(E_a) \to \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$$

(we will explain in a moment how to form the direct limit of the C^*-algebras $\mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$). Having constructed φ, we will follow the argument of Section 2 to prove the periodicity theorem.

First we must make some remarks on G-equivariant asymptotic morphisms.

Let A and B be C^*-algebras equipped with actions of a discrete group G. An equivariant asymptotic morphism from A to B is an asymptotic morphism for which the induced $*$-homomorphism $\varphi\colon A \to \mathbf{Q}(B)$ is equivariant.

It follows from the universal property of the full crossed product [15] that there is a canonical $*$-homomorphism

$$\mathbf{Q}(B) \rtimes G \to \mathbf{Q}(B \rtimes G).$$

Hence an equivariant asymptotic morphism $\varphi\colon A \to B$ determines an asymptotic morphism from $A \rtimes G$ to $B \rtimes G$, and therefore an induced K-theory map

$$\varphi_*\colon K_*^G(A) \to K_*^G(B).$$

This is functorial with respect to composition by equivariant $*$-homomorphisms.

Returning to the construction of the asymptotic morphism φ above, our first task is to assemble the C^*-algebras $\mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$ into a direct limit.

1. Definition. Suppose that $E_a \subset E_b$, and let E_{ba} denote the orthogonal complement of E_a in E_b. There is a natural isomorphism

$$\mathfrak{h}(E_{ba}) \hat{\otimes} \mathfrak{h}(E_a) \cong \mathfrak{h}(E_b)$$

defined by

$$\xi_{ba} \hat{\otimes} \xi_a(e_{ba} + e_a) = \xi_{ba}(e_{ba}) \hat{\otimes} \xi_a(e_a)$$

(as in Section 2 we identify the graded tensor product $\mathit{Cliff}(E_{ab}) \hat{\otimes} \mathit{Cliff}(E_a)$ with $\mathit{Cliff}(E_b)$. Using it we define a $*$-homomorphism

$$\gamma_{ba}\colon \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)) \to \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_b))$$

by the formula

$$\gamma_{ba}\colon f \hat{\otimes} T \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} B_{ba}) \hat{\otimes} T.$$

Here B_{ba} denotes the B-operator of Definition 2.12 for the finite dimensional Euclidean vector space E_{ba}. Using the argument of Proposition 3.2 it is readily verified that the composition

$$\mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \xrightarrow{\gamma_{ba}} \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_b)) \xrightarrow{\gamma_{cb}} \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_c)),$$

corresponding to a sequence of inclusions $E_a \subset E_b \subset E_c$, is the $*$-homomorphism

$$\gamma_{ca}\colon \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \to \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_c)).$$

Consequently we can form the direct limit algebra $\varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a))$ over the directed system of all finite dimensional subspaces of E.

Remark. This direct limit is *not* the suspension of a C^*-algebra of compact operators. It is in fact quite a complicated object, and we shall have to postpone until the next section a discussion of its K^G-theory.

The following proposition allows us to define the required asymptotic morphism

$$\varphi\colon \varinjlim_a \mathscr{S}\mathscr{C}(E_a) \to \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a))$$

by a simple direct limit procedure.

2. Proposition. *The diagram*

$$\begin{tikzcd}
\mathscr{S}\mathscr{C}(E_a) \arrow[r, "\varphi_a"] \arrow[d, "\beta_{ba}"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[d, "\gamma_{ba}"] \\
\mathscr{S}\mathscr{C}(E_b) \arrow[r, "\varphi_b"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_b))
\end{tikzcd}$$

is asymptotically commutative.

Proof. We shall check this on the generators $\exp(-x^2)\hat{\otimes} h$ and $x\exp(-x^2)\hat{\otimes} h$ of $\mathscr{S}\mathscr{C}(E_a)$, where $h \in \mathscr{C}(E_a)$.

To do the calculation we need to note that under the isomorphism of Hilbert spaces $\mathfrak{h}(E_b) \cong \mathfrak{h}(E_{ba}) \hat{\otimes} \mathfrak{h}(E_a)$ the Dirac operator D_b corresponds to $D_{ba} \hat{\otimes} 1 + 1 \hat{\otimes} D_a$ (to be precise, the self-adjoint closures of these essentially self-adjoint operators correspond to one another). Hence, by Appendix A, $\exp(-t^{-2}D_b^2)$ corresponds to $\exp(-t^{-2}D_{ba}^2)\hat{\otimes}\exp(-t^{-2}D_a^2)$.

Applying first φ_a, then γ_{ba}, to $\exp(-x^2)\hat{\otimes} h$ we get

$$\exp(-t^{-2}x^2)\hat{\otimes}\exp(-t^{-2}B_{ba}^2)\hat{\otimes}\exp(-t^{-2}D_a^2)\, M_{h_t},$$

while applying first β_{ba}, then φ_b, to it we get

$$\exp(-t^{-2}x^2) \hat{\otimes} \exp(-t^{-2}D_{ba}^2) \exp(-t^{-2}C_{ba}^2) \hat{\otimes} \exp(-t^{-2}D_a^2)\, M_{h_t}.$$

But by the calculations in Appendix B the two families of operators $\exp(-t^2B_{ba}^2)$ and $\exp(-t^{-2}D_{ba}^2)\exp(-t^{-2}C_{ba}^2)$ are asymptotic to one another, as $t \to \infty$.

The calculation for $x\exp(-x^2) \hat{\otimes} h$ is similar. ▮

Each φ_a defines a $*$-homomorphism $\varphi_a\colon \mathcal{S}\mathcal{C}(E_a) \to \mathbf{Q}(\mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)))$ and since $\mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$ is included in $\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$ we obtain a $*$-homomorphism

$$\varphi_a\colon \mathcal{S}\mathcal{C}(E_a) \to \mathbf{Q}(\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))).$$

By Proposition 2, the maps φ_a are compatible with the inclusion homomorphisms $\beta_{ba}\colon \mathcal{S}\mathcal{C}(E_a) \to \mathcal{S}\mathcal{C}(E_b)$, and so define a single $*$-homomorphism

$$\varphi\colon \varinjlim_a \mathcal{S}\mathcal{C}(E_a) \to \mathbf{Q}(\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))),$$

as required.

3. Definition. Define $*$-homomorphisms

$$\gamma_a\colon \mathcal{S} \to \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)),$$

along with the corresponding $*$-homomorphism

$$\mathcal{S} \xrightarrow{\gamma} \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)),$$

by $\gamma_a\colon f \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} B_{E_a})$.

We shall prove the following result in the next section:

4. Proposition. *The induced map* $\gamma_*\colon K_*^G(S) \to K_*^G(\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)))$ *is an isomorphism.*

5. Lemma. *The sequence of K-theory maps*

$$K_*^G(\mathcal{S}\mathcal{C}(E)) \xrightarrow{\varphi_*} K_*^G(\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))) \xrightarrow{\gamma_*^{-1}} K_*^G(\mathcal{S})$$

is left inverse to the periodicity map $\beta_*\colon K_*^G(\mathcal{S}) \to K_*^G(\mathcal{S}\mathcal{C}(E))$.

Proof. It follows from Proposition 2.13 that the composition

$$\mathcal{S} \xrightarrow{\beta} \mathcal{S}\mathcal{C}(E) \xrightarrow{\varphi} \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$$

is asymptotic to the family of $*$-homomorphisms $f \mapsto \gamma(f_t)$. This is in turn homotopic to the $*$-homomorphism γ via the homotopy $f \mapsto \gamma(f_{1-s+st})$. Hence $\varphi_* \beta_* = \gamma_*$, as required. ∎

To prove $\gamma_*^{-1} \varphi_*$ is also a right inverse we consider, as we did in Section 2, the diagram

$$\begin{array}{ccc}
\mathcal{S}\mathcal{C}(E) & \xrightarrow{\varphi} & \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)) & & \\
& & \uparrow{\scriptstyle \gamma} & & \\
& & \mathcal{S} & \xrightarrow{\beta} & \mathcal{S}\mathcal{C}(E).
\end{array}$$

Following the general strategy used in Section 2, we are going to “complete” it to a larger diagram, although here the details are a little more complicated.

Remark on Notation. In what follows, when we consider $*$-homomorphisms or asymptotic morphisms from one direct limit to another, constructed from a compatible family of morphisms $\psi_a \colon A_a \to B_a$, and when we consider for instance a compatible family of maps $\psi_a \hat{\otimes} 1 \colon A_a \hat{\otimes} C_a \to B_a \hat{\otimes} C_a$, we shall denote the map on the direct limit as

$$\psi \hat{\otimes} 1 \colon \varinjlim_a A_a \hat{\otimes} C_a \to \varinjlim_a B_a \hat{\otimes} C_a,$$

even if $\psi \hat{\otimes} 1$ is not actually a tensor product of maps.

Returning now to the above diagram, we complete it as follows:

$$\begin{array}{ccccc}
\varinjlim_a \mathcal{S}\mathcal{C}(E_a) & \xrightarrow{\varphi} & \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)) & \xrightarrow{\beta \hat{\otimes} 1} & \varinjlim_a \mathcal{S}\mathcal{C}(E_a) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)) \\
& & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \tilde{\gamma}} \\
& & \mathcal{S} & \xrightarrow{\beta} & \varinjlim_a \mathcal{S}\mathcal{C}(E_a).
\end{array}$$

The map $\beta \hat{\otimes} 1$ is a direct limit, as explained in the remark above. In the top right hand corner is a new direct limit C^*-algebra. The limit is indexed by the finite dimensional subspaces of E, and if $E_a \subset E_b$ then the inclusion

$*$-homomorphism from $\mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_a))$ to $\mathscr{S}\mathscr{C}(E_b)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_b))$ is the composition

$$\begin{tikzcd}
\mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\beta_{ba}\hat{\otimes}1"] & \mathscr{S}\mathscr{C}(E_b)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[d, "\text{flip}"'] & \mathscr{S}\mathscr{C}(E_b)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_b)) \\
& \mathscr{C}(E_b)\hat{\otimes}\mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "1\hat{\otimes}\gamma_{ba}"] & \mathscr{C}(E_b)\hat{\otimes}\mathscr{S}\mathfrak{K}(\mathfrak{h}(E_b)) \arrow[u, "\text{flip}"']
\end{tikzcd}$$

As in the proof of Proposition 3.2, one checks easily that the maps form a directed system. The $*$-homomorphism $\tilde{\gamma}$ is the limit of the compositions

$$\mathscr{S}\mathscr{C}(E_a)\xrightarrow{\gamma_a\hat{\otimes}1}\mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a))\hat{\otimes}\mathscr{C}(E_a)\xrightarrow{\text{flip}}\mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_a)).$$

Finally, the diagram is asymptotically commutative.

We shall postpone the proof of the following result until the next section:

6. PROPOSITION. *There are induced isomorphisms*

$$K_*^G(\varinjlim_a \mathscr{S}\mathscr{C}(E_a))\xrightarrow[\cong]{(\gamma\hat{\otimes}1)_*}K_*^G(\varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a))\hat{\otimes}\mathscr{C}(E_a))$$
$$\xrightarrow[\cong]{\text{flip}_*}K_*^G(\varinjlim_a \mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_a))).$$

Because of Proposition 6, it remains only to prove that the top row in the completed diagram induces an isomorphism in K^G-theory. For this we introduce the asymptotically commutative diagram

$$\begin{tikzcd}
\varinjlim_a \mathscr{S}\mathscr{C}(E_a) \arrow[r, "\varphi"] \arrow[d, "\beta\hat{\otimes}1"'] & \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\beta\hat{\otimes}1"] & \varinjlim_a \mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[d, "\text{flip}"] \\
\varinjlim_a \mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathscr{C}(E_a) \arrow[r, "\text{flip}"'] & \varinjlim_a \mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathscr{C}(E_a) \arrow[r, "\tilde{\varphi}"'] & \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a))\hat{\otimes}\mathscr{C}(E_a),
\end{tikzcd}$$

exactly as in Section 2 (except that here we have passed to the direct limit).

As in Section 2, the flip map on $\mathscr{C}(E_a)\hat{\otimes}\mathscr{C}(E_a)$ is homotopic through a rotation to the map $\iota_a\hat{\otimes}1$, where $\iota_a\colon\mathscr{C}(E_a)\to\mathscr{C}(E_a)$ is induced from the inversion $e\mapsto -e$ on E_a. This homotopy is compatible with direct limits, and so to prove that the top row induces an isomorphism in K^G-theory it suffices to show that the composition

$$\varinjlim_a \mathscr{S}\mathscr{C}(E_a)\xrightarrow{\beta\hat{\otimes}\iota}\varinjlim_a \mathscr{S}\mathscr{C}(E_a)\hat{\otimes}\mathscr{C}(E_a)\xrightarrow{\tilde{\varphi}}\varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a))\hat{\otimes}\mathscr{C}(E_a)$$

induces an isomorphism. The composition is asymptotic to the family of $*$-homomorphisms

$$f \hat{\otimes} h \mapsto \gamma(f_t) \hat{\otimes} \iota(h_t)$$

and this is homotopic to $f \hat{\otimes} h \mapsto \gamma(f) \hat{\otimes} \iota(h)$ via the homotopy

$$f \hat{\otimes} h \mapsto \gamma(f_{1-s+st}) \hat{\otimes} \iota(h_{1-s+st}) \qquad s \in [0, 1].$$

It follows from Proposition 6 the map $f \hat{\otimes} h \mapsto \gamma(f) \hat{\otimes} \iota(h)$ induces an isomorphism in K^G-theory.

5. CONCLUSION OF THE PROOF

In this section we will assume that E is a countably infinite dimensional Euclidean vector space on which a countable discrete group G acts by linear isometries. We shall complete our proof of the periodicity theorem in this case by dealing with the following technical point from the last section:

1. PROPOSITION. *The $*$-homomorphisms*

$$\text{(i)} \quad \gamma\colon \mathscr{S} \to \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a))$$

and

$$\text{(ii)} \quad \gamma \hat{\otimes} 1\colon \varinjlim_a \mathscr{S}\mathscr{C}(E_a) \to \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathscr{C}(E_a),$$

which were defined in Section 4, *induce isomorphisms in K^G-theory.*

We shall prove the Proposition by constructing suitable asymptotic morphisms

$$\psi\colon \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \to \mathscr{S} \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E))$$

and

$$\psi \hat{\otimes} 1\colon \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathscr{C}(E_a) \to \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \hat{\otimes} \mathscr{C}(E_a).$$

Here $\mathfrak{h}(E)$ is a Hilbert space constructed from the infinite dimensional Euclidean space E as follows:

2. DEFINITION. Let E_b be a finite dimensional Euclidean vector space and let E_a be a subspace of E_b. We define an isometry $T_{ba}\colon \mathfrak{h}(E_a)\to\mathfrak{h}(E_b)$ by

$$(T_{ba}\xi)(e_{ba}+e_a)=\pi^{-n/4}\exp(-\tfrac{1}{2}\,\|e_{ba}\|^2)\,\xi(e_a),$$

where $e_a\in E_a$, $e_{ba}\in E_{ba}$, and n is the difference of the dimensions of E_b and E_a.

Note that the function $\pi^{-n/4}\exp(-\frac{1}{2}\,\|e_{ba}\|^2)$ has norm one in $\mathfrak{h}(E_{ba})$, which ensures that the map T_{ba} is indeed isometric.

If $E_a\subset E_b\subset E_c$ then the composition of isometries

$$\mathfrak{h}(E_a)\to\mathfrak{h}(E_b)\to\mathfrak{h}(E_c)$$

is equal to the isometry $\mathfrak{h}(E_a)\to\mathfrak{h}(E_c)$ associated to the inclusion $E_a\subset E_b$, and because of this we can make the following definition.

3. DEFINITION. We define $\mathfrak{h}(E)$ to be the Hilbert space direct limit of the spaces $\mathfrak{h}(E_a)$, where the limit is taken over the directed system of finite dimensional subspaces of H, using the isometric inclusions $T_{ba}\colon \mathfrak{h}(E_a)\to\mathfrak{h}(E_b)$.

Before defining the asymptotic morphisms ψ and $\tilde{\psi}$ we must develop an infinite dimensional generalization of the operator B of Definition 2.12. This is a little technical, and will occupy our attention for the next several pages.

Let V be a countably infinite dimensional Euclidean space and, as above, define $\mathfrak{h}(V)$ to be the direct limit of Hilbert spaces $\mathfrak{h}(V_a)$, indexed by the finite dimensional subspaces of V.

4. DEFINITION. If V_a is a finite dimensional subspace of V then denote by $\mathfrak{s}(V_a)$ the Schwartz subspace of the L^2-space $\mathfrak{h}(V_a)$. Denote by $\mathfrak{s}(V)$ the algebraic direct limit of the vector spaces $\mathfrak{s}(V_a)$, under the inclusions $T_{ba}\colon \mathfrak{s}(V_a)\to\mathfrak{s}(V_b)$.

5. DEFINITION. Let W be a finite dimensional subspace of V. As in Definition 2.12 we form the operator

$$B_W=\sum_{i=1}^{n}\widehat{w_i}\,\frac{\partial}{\partial x_i}+w_i x_i,$$

where $\{w_1, ..., w_n\}$ is an orthonormal basis for W and $\{x_1, ..., x_n\}$ are coordinates in W dual to this basis. We view it now as an operator

$$B_W \colon \mathfrak{s}(V) \to \mathfrak{s}(V)$$

by noting that it is a well defined operator on $\mathfrak{s}(V_a)$, for any V_a containing W, and that the diagram

$$\begin{tikzcd} \mathfrak{s}(V_a) \arrow[r, "T_{ba}"] \arrow[d, "B_W"'] & \mathfrak{s}(V_b) \arrow[d, "B_W"] \\ \mathfrak{s}(V_a) \arrow[r, "T_{ba}"'] & \mathfrak{s}(V_b) \end{tikzcd}$$

commutes, for every inclusion $V_a \subset V_b$.

The following very simple observation will be crucial for us.

6. Lemma. *If W is orthogonal to V_a then* $\mathfrak{s}(V_a) \subset \mathrm{Kernel}(B_W)$.

Proof. The embedding of $\mathfrak{s}(V_a)$ into $\mathfrak{s}(W \oplus V_a)$ maps $\xi \in \mathfrak{s}(V_a)$ to the function

$$T\xi(w + e_a) = \pi^{-n/4} \exp(-\|w\|^2/2)\, \xi(e_a).$$

The function $\exp(-\frac{1}{2}\|w\|^2)$ lies in the kernel of B_W^2 (see Corollary 2.15) and hence in the kernel of B_W. Thus $T\xi$ lies in the kernel of B_W.

Suppose now that V is written as an (algebraic) orthogonal direct sum of finite dimensional subspaces:

$$V = W_0 \oplus W_1 \oplus W_2 \oplus \cdots.$$

Otherwise put, suppose that V is written as an increasing union of finite dimensional subspaces

$$V_n = W_0 \oplus \cdots \oplus W_n.$$

Using this extra structure on V we define an operator on $\mathfrak{s}(V)$ (actually a family of operators, depending on a parameter $t \in [1, \infty)$) as follows.

7. Definition. Let $t \in [1, \infty)$. Define an unbounded operator B_t on $\mathfrak{h}(V)$, with domain $\mathfrak{s}(V)$, by the formula

$$B_t = t_0 B_{W_0} + t_1 B_{W_1} + t_2 B_{W_2} + \cdots,$$

where $t_n = 1 + t^{-1} n$.

The infinite sum is well defined because, by Lemma 6, when B_t is applied to any vector in $\mathfrak{s}(V)=\bigcup_n \mathfrak{s}(V_n)$ the resulting infinite series in $\mathfrak{s}(V)$ has only finitely many non-zero terms.

8. LEMMA. *The operator B_t is essentially self-adjoint on $\mathfrak{h}(V)$ and has compact resolvent.*

Proof. Squaring B_t (considered as an operator on $\mathfrak{s}(V)$) we get

$$B_t^2 = t_0^2 B_{W_0}^2 + t_1^2 B_{W_1}^2 + t_2^2 B_{W_2}^2 + \cdots.$$

If $\xi_j \in \mathfrak{s}(W_j)$ is an eigenfunction for $B_{W_j}^2$, with eigenvalue λ_j, then the function $\xi \in \mathfrak{s}(V_n)$ defined by

$$\xi(w_0 + w_1 + \cdots + w_n) = \xi_0(w_0) \mathbin{\hat{\otimes}} \xi_0(w_0) \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \xi_n(w_n)$$

is an eigenfunction for B_t^2 with eigenvalue

$$\lambda = t_0^2 \lambda_0 + t_1^2 \lambda_1 + \cdots + t_n^2 \lambda_n.$$

It follows that B_t^2, and hence B_t has an orthonormal eigenbasis within $\mathfrak{s}(V)$, which proves that it is essentially self-adjoint. Furthermore the eigenvalues of B_t^2 are the scalars λ as above. Since the only accumulation point of the eigenvalue sequence is infinity, the operator B_t^2, and hence also the operator B_t, has compact resolvent. ∎

We now investigate what happens to the operator B_t when the direct sum decomposition

$$V = W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

is altered a little.

9. LEMMA. *Suppose that V is provided with a second decomposition,*

$$V = \tilde{W}_0 \oplus \tilde{W}_1 \oplus \tilde{W}_2 \oplus \cdots$$

and let $\tilde{V}_n = \tilde{W}_0 \oplus \cdots \oplus \tilde{W}_n$. Let $\tilde{B}_t$ be the corresponding essentially self-adjoint operator on $\mathfrak{h}(V)$, and let $f \in C_0(\mathbb{R})$. If either

$$\text{(i)} \quad \tilde{V}_j = V_{j+1}, \qquad \textit{for} \quad j = 0, 1, 2, \ldots$$

or

$$\text{(ii)} \quad V_0 \subset \tilde{V}_0 \subset V_1 \subset \tilde{V}_1 \subset \cdots$$

then $\lim_{t\to\infty} \|f(\tilde{B}_t) - f(B_t)\| = 0$.

Proof. By the Stone–Weierstrass theorem, it suffices to prove the lemma for the function $f(x)=(x+i)^{-1}$. In case (i) note that

$$B_t-\tilde{B}_t=t^{-1}B_{W_1}+t^{-1}B_{W_2}+\cdots,$$

and so if $\xi\in\mathfrak{s}(V)$ then

$$\|(B_t-\tilde{B}_t)\,\xi\|^2=t^{-2}\,\|B_{W_1}\xi\|^2+t^{-2}\,\|B_{W_2}\xi\|^2+\cdots.$$

On the other hand

$$\|(B_t+i)\,\xi\|^2=\|\xi\|^2+t_0^2\,\|B_{W_0}\xi\|^2+t_1^2\,\|B_{W_1}\xi\|^2+\cdots,$$

and since $t^{-2}\leqslant t^{-2}t_n^2$ we get that

$$\|(\tilde{B}_t-B_t)\,\xi\|^2\leqslant t^{-2}\,\|(B_t+i)\,\xi\|^2.$$

It follows that $\|(\tilde{B}_t-B_t)(B_t+i)^{-1}\|\leqslant t^{-1}$. From the formula

$$(\tilde{B}_t+i)^{-1}-(B_t+i)^{-1}=(\tilde{B}_t+i)^{-1}\,(B_t-\tilde{B}_t)(B_t+i)^{-1}$$

(valid on the dense set of those $\eta\in\mathfrak{h}(V)$ for which $(B_t+i)^{-1}\eta\in\mathfrak{s}(V)$) and the above inequality we get $\|(\tilde{B}_t+i)^{-1}-(B_t+i)^{-1}\|\leqslant t^{-1}$.

The proof of the second part is quite similar. Let $X_n=V_n\ominus\tilde{V}_{n-1}$ and $Y_n=\tilde{V}_n\ominus V_n$. Then

$$B_t-\tilde{B}_t=t^{-1}B_{Y_0}+t^{-1}B_{Y_1}+t^{-1}B_{Y_2}+\cdots,$$

whereas

$$B_t=t_0B_{X_0}+t_1B_{Y_0}+t_1B_{X_1}+t_2B_{Y_1}+t_2B_{X_2}+\cdots.$$

We see again that

$$\|(B_t-\tilde{B}_t)\,\xi\|^2\leqslant t^{-2}\,\|(B_t+i)\|^2,$$

and hence that

$$\|(\tilde{B}_t+i)^{-1}-(B_t+i)^{-1}\|\leqslant t^{-1}. \quad \blacksquare$$

Let us return now to the countably infinite dimensional Euclidean vector space E, on which a countable discrete group G is acting by isometries.

10. Definition. Fix (for the rest of the section) a sequence

$$E_0\subset E_1\subset E_2\subset\cdots\subset E$$

of finite dimensional linear subspaces of E such that $\bigcup_n E_n = E$ and such that if $g \in G$ then $gE_n \subset E_{n+1}$, for all large enough n. (Note that we are assuming E is countably infinite dimensional, so it is possible to arrange for the union $\bigcup_n E_n$ to be all of E, not just dense in E.) Let W_n be the orthogonal complement of E_{n-1} in E_n (we set $W_0 = E_0$), so that

$$E = W_0 \oplus W_1 \oplus W_2 \oplus \cdots.$$

This is an algebraic direct sum.

Now let E_a be a finite dimensional subspace of E. Since E_a is contained in E_n, for some n, we may write the orthogonal complement of E_a in E as an algebraic direct sum

$$E_a^\perp = W_a \oplus W_{n+1} \oplus W_{n+2} \oplus \cdots,$$

where W_a is the orthogonal complement of E_a in E_n. It is convenient *not* to necessarily choose the smallest n such that $E_a \subset E_n$, and because of this our decomposition of $E_a^\perp$ is not canonical. However we have the following stability result:

11. Lemma. *If B_t and $\tilde{B}_t$ denote the operators associated to different decompositions of $E_a^\perp$ (using different values of n) then* $\lim_{t\to\infty} \|f(B_t) - f(\tilde{B}_t)\| = 0$.

Proof. This follows immediately from part (i) of Lemma 9. ∎

Remark. In what follows we shall need a slight strengthening of Lemmas 9 and 11: with the same hypotheses, we have in fact that

$$\lim_{t\to\infty} \|f(s^{-1}\tilde{B}_t) - f(s^{-1}B_t)\| = 0,$$

uniformly in $s \in (0, 1]$. To see this, just repeat the proofs with sB_t in place of B_t.

12. Lemma. *Let g be an isometric isomorphism of E onto itself and let U_g be the corresponding unitary isomorphism of $\mathfrak{h}(E_a^\perp)$ onto $\mathfrak{h}(gE_a^\perp)$. Let B_t^a and B_t^{ga} be the operators associated to the decompositions*

$$E_a^\perp = W_a \oplus W_m \oplus W_{m+1} \oplus \cdots$$

and

$$gE_a^\perp = W_{ga} \oplus W_n \oplus W_{n+1} \oplus \cdots.$$

If $f \in \mathcal{S}$ then $\lim_{t\to\infty} \|f(B_t^a) - U_g^* f(B_t^{ga})\, U_g\| = 0$.

Proof. By Lemma 11, we may assume that $m=n$, and that

$$E_a,\ gE_a,\ g^{-1}E_a \subset E_n$$

and

$$gE_{n+k},\ g^{-1}E_{n+k} \subset E_{n+k+1} \qquad k=0, 1, 2, \ldots$$

(compare Definition 10). Let $\tilde{E}_{n+k}$ be the subspace spanned by E_{n+k} and $g^{-1}E_{n+k}$ and define $\tilde{E}_a$ similarly. Observe that there are inclusions

$$\begin{tikzcd}
E_a \arrow[dr] & & E_n \arrow[dr] & & E_{n+1} & \\
& \tilde{E}_a \arrow[ur] \arrow[dr] & & \tilde{E}_n \arrow[ur] \arrow[dr] & & \cdots. \\
g^{-1}E_a \arrow[ur] & & g^{-1}E_n \arrow[ur] & & g^{-1}E_{n+1} &
\end{tikzcd}$$

The operator B_t^a is constructed from the top row of ascending subspaces, while the operator $U_g^* B_t^{ga} U_g$ is constructed from the bottom row. By considering the operator $\tilde{B}_t$ constructed from the middle row (that is, from the spaces $\tilde{E}_{n+k}$), and by using twice the second part of Lemma 9, we get the required result. ∎

Remark. At one point below we shall need to know that

$$\lim_{t\to\infty} \| f(s^{-1}B_t^a) - U_g^* f(s^{-1}B_t^{ga})\, U_g \| = 0,$$

uniformly in $s \in (0, 1]$. This follows from the above argument and the remark following Lemma 11.

13. Definition. Let E_a be a finite dimensional subspace of E, and let B_t, for $t \in [1, \infty)$, be the operator on $\mathfrak{h}(E_a^\perp)$ associated to a decomposition

$$E_a^\perp = W_a \oplus W_{n+1} \oplus W_{n+2} \oplus \cdots.$$

Define an asymptotic morphism

$$\psi_a \colon \mathscr{S}\mathfrak{R}(\mathfrak{h}(E_a)) \to \mathscr{S}\mathfrak{R}(\mathfrak{h}(E))$$

by the formula

$$\psi_{a,t} \colon f \hat{\otimes} T \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} B_t) \hat{\otimes} T.$$

We are using here the natural isomorphism of Hilbert spaces

$$\mathfrak{h}(E) \cong \mathfrak{h}(E_a^\perp) \mathbin{\hat{\otimes}} \mathfrak{h}(E_a),$$

coming from the isomorphisms $\mathfrak{h}(E_b) \cong \mathfrak{h}(E_{ba}) \mathbin{\hat{\otimes}} \mathfrak{h}(E_a)$ already noted in Definition 4.1.

Since each $\psi_{a,t}$ is in fact a $*$-homomorphism, to show that ψ_a is indeed an asymptotic morphism we need only note that $\psi_{a,t}(f \mathbin{\hat{\otimes}} T)$ is norm continuous in t. This in turn follows from the simple fact that $f(B_t)$ is norm continuous in t, for all $f \in \mathscr{S}$.

By Lemma 11, the asymptotic morphism ψ_a is, up to asymptotic equivalence, independent of the choice of decomposition of $E_a^\perp$ (in other words it is independent of the choice of n such that $E_a \subset E_n$).

14. Lemma. *If $E_a \subset E_b$ then the diagram*

$$\begin{tikzcd} \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\psi_a"] \arrow[d, "\gamma_{ba}"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \arrow[d, "="] \\ \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_b)) \arrow[r, "\psi_b"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \end{tikzcd}$$

is asymptotically commutative.

Proof. A calculation with the generators $\exp(-x^2) \mathbin{\hat{\otimes}} T$ and $x\exp(-x^2) \mathbin{\hat{\otimes}} T$ shows that if we use the decompositions

$$E_a^\perp = W_a \oplus W_{n+1} \oplus W_{n+2} \oplus \cdots,$$

and

$$E_b^\perp = W_b \oplus W_{n+1} \oplus W_{n+2} \oplus \cdots,$$

(where the same n is used in both cases) to define the B-operators on $E_a^\perp$ and $E_b^\perp$ then the diagram is exactly commutative. ∎

It follows from the lemma, and from the argument used in the last section, that the asymptotic morphisms ψ_a combine to form a single asymptotic morphism

$$\psi\colon \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \to \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)).$$

Since G acts on E by linear isometries, it acts unitarily on $\mathfrak{h}(E)$. From Lemma 12 we get that:

15. LEMMA. *The asymptotic morphism* ψ: $\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)) \to \mathcal{S}\mathfrak{K}(\mathfrak{h}(E))$ *is G-equivariant.*

Following the general strategy employed in Sections 2 and 4, we should like to obtain from ψ an inverse, at the level of K-theory, to γ.

We note first the a simple equivariant version of Lemma 2.11. Let $P \in \mathfrak{K}(\mathfrak{h}(E))$ be the orthogonal projection onto the kernel of B_t (which is spanned by the single function $\exp(-\frac{1}{2}\|e\|^2)$). Note that P is the projection onto a one-dimensional subspace of vectors in $\mathfrak{h}(E)$ which are fixed by the action of G.

16. LEMMA. *Let A be a graded C^*-algebra, equipped with an action of G. The map σ: $a \mapsto a \mathbin{\hat{\otimes}} P$ induces an isomorphism σ: $K_*^G(A) \to K_*^G(A \mathbin{\hat{\otimes}} \mathfrak{K}(\mathfrak{h}(E)))$.*

17. PROPOSITION. *The composition*

$$K_*^G(\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))) \xrightarrow{\psi_*} K_*^G(\mathcal{S}\mathfrak{K}(\mathfrak{h}(E))) \underset{\cong}{\xrightarrow{\sigma_*^{-1}}} K_*^g(\mathcal{S})$$

is left inverse to the K-theory map γ_*: $K_*^G(\mathcal{S}) \to K_*^G(\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)))$.

Proof. We will prove that the composition

$$\mathcal{S} \xrightarrow{\gamma} \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)) \xrightarrow{\psi} \mathcal{S} \mathbin{\hat{\otimes}} \mathfrak{K}(\mathfrak{h}(E))$$

is homotopic to σ: $\mathcal{S} \to \mathcal{S}\mathfrak{K}(\mathfrak{h}(E))$. The composition is given by

$$f \mapsto f(X \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} B_t),$$

where B_t is the operator associated to the decomposition

$$E = W_0 \oplus W_1 \oplus W_2 \oplus \cdots.$$

the desired homotopy is

$$f \mapsto \begin{cases} f(X \mathbin{\hat{\otimes}} 1 + s^{-1} \mathbin{\hat{\otimes}} B_t) & s \in (0, 1] \\ f \mathbin{\hat{\otimes}} P & s = 0. \end{cases}$$

Note that $f(s^{-1}B_t)$ is asymptotically G-invariant, as $t \to \infty$, uniformly in s, so the homotopy is a homotopy of G-equivariant asymptotic morphisms. ∎

Proof of Proposition 1, *Part* (i). It remains to show that the composition $\sigma_*^{-1}\psi_*$ is also a right inverse to γ_*. We employ the same rotation argument already used in Sections 2 and 4, beginning with the diagram

$$
\begin{tikzcd}
\varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\psi"] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\gamma\hat{\otimes}1"] & \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)) \\
& \mathscr{S} \arrow[u, "\sigma"] \arrow[r, "\gamma"] & \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)). \arrow[u, "\sigma"']
\end{tikzcd}
$$

It suffices to prove that the composition along the top induces an isomorphism in K^G-theory. We then consider the diagram

$$
\begin{tikzcd}
\mathscr{S} \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\psi"] \arrow[d, "\gamma\hat{\otimes}1"'] & \mathscr{S} \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E)) \arrow[r, "\gamma\hat{\otimes}1"] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)) \arrow[d, "\text{flip}"] \\
\mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\text{flip}"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)) \arrow[r, "\psi\hat{\otimes}1"'] & \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)),
\end{tikzcd}
$$

and use the same type of rotation argument as in Sections 2 and 4: the flip map on $\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a))$ is homotopic through a rotation to the map $\iota_a \hat{\otimes} 1$, where ι_a: $\mathfrak{K}(\mathfrak{h}(E_a)) \to \mathfrak{K}(\mathfrak{h}(E_a))$ is induced from the inversion $e \mapsto -e$ on E_a. The homotopy is compatible with direct limits and so it suffices to show that the composition

$$
\varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \xrightarrow{\gamma\hat{\otimes}\iota} \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a))
$$

$$
\xrightarrow{\psi\hat{\otimes}1} \varinjlim_a \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a))
$$

induces an isomorphism in K^G-theory (after first passing to direct limits). The composition is

$$
\mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \xrightarrow{\psi\gamma\hat{\otimes}\iota} \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \hat{\otimes} \mathfrak{K}(\mathfrak{h}(E_a)),
$$

which is homotopic to the map $\sigma \hat{\otimes} \iota$: $f \hat{\otimes} T \mapsto f \hat{\otimes} P \hat{\otimes} T$ via the homotopy

$$
f \hat{\otimes} T \mapsto \begin{cases} f(x \hat{\otimes} 1 + s^{-1} \hat{\otimes} B_t) \hat{\otimes} T & s \in (0, 1] \\ f \hat{\otimes} P \hat{\otimes} T & s = 0. \end{cases}
$$

It follows from Lemma 16 that $\sigma \hat{\otimes} \iota$ induces an isomorphism on K^G-theory. ∎

The proof of the second part of Proposition 1 is essentially the same. We assemble the asymptotic morphisms

$$
\psi_a \hat{\otimes} 1\colon \mathscr{S}\mathfrak{K}(\mathfrak{h}(E_a)) \hat{\otimes} \mathscr{C}(E_a) \to \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)) \hat{\otimes} \mathscr{C}(E_a)
$$

to get an asymptotic morphism

$$\psi \mathbin{\hat{\otimes}} 1\colon \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)) \mathbin{\hat{\otimes}} \mathcal{C}(E_a) \to \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E)) \mathbin{\hat{\otimes}} \mathcal{C}(E_a).$$

By incorporating an extra factor of $\mathcal{C}(E_a)$ into the argument for ψ, we see that $\psi \mathbin{\hat{\otimes}} 1$ determines an inverse to $\gamma \mathbin{\hat{\otimes}} 1$, at the level of K^G-theory, as required.

APPENDIX A

Graded C-Algebras, Tensor Products and Multipliers*

A.1. *Graded Commutators*

Let A be a $\mathbb{Z}/2$-graded algebra with grading automorphism $\gamma(a) = (-1)^{\deg(a)}\, a$. The graded commutator of two elements in A is

$$[a_1, a_2] = a_1 a_2 - (-1)^{\deg(a_1)\deg(a_2)}\, a_2 a_1$$

(this formula is valid for homogeneous elements, and is extended by linearity to all of A). A useful identity is

$$[x^{-1}, y] = -x^{-1}[x, y]\, \gamma^{\deg(y)}(x^{-1}).$$

The formula is written for homogeneous y, and extends by linearity to all y. Similarly

$$[x, y^{-1}] = -\gamma^{\deg(x)}(y^{-1})[x, y]\, y^{-1}.$$

A.2. *Tensor Products*

If A and B are $\mathbb{Z}/2$-graded C^*-algebras then the graded tensor product of A and B, as $\mathbb{Z}/2$-graded algebras, may in general be completed in more than one way to a C^*-algebra.

The *minimal*, or *spatial* C^*-algebra completion of $A \mathbin{\hat{\otimes}}_{\mathbb{C}} B$ is obtained by representing A and B faithfully on $\mathbb{Z}/2$-graded Hilbert spaces $\mathfrak{h}_A$ and $\mathfrak{h}_B$; then representing $A \mathbin{\hat{\otimes}}_{\mathbb{C}} B$ via the formula

$$\pi_{A,B}(a \mathbin{\hat{\otimes}} b)(v \otimes w) = (-1)^{\deg(b)\deg(v)}\, \pi_A(a)\, v \otimes \pi_B(b)\, w;$$

then taking the completion in the operator norm topology. This does not depend on the choice of representations.

The *maximal* C^*-algebra completion of $A \hat{\otimes}_{\mathbb{C}} B$ has the characteristic property that a pair of $*$-homomorphisms

$$A \to C \qquad \text{and} \qquad B \to C,$$

into a $\mathbb{Z}/2$-graded C^*-algebra whose images graded commute with one another induces a morphism

$$A \hat{\otimes}_{\max} B \to C.$$

If one of A or B is a nuclear C^*-algebra (forgetting the grading) then the maximal norm is equal to the minimal one. Since this will be the case throughout the paper we shall just write $A \hat{\otimes} B$ for the C^*-algebraic graded tensor product.

A.3. *Multipliers*

Let A be a $\mathbb{Z}/2$-graded C^*-algebra. An *unbounded selfadjoint multiplier* of A is an A-linear map D from a dense, $\mathbb{Z}/2$-graded, right A-submodule, $\mathscr{A} \subset A$, into A such that

(i) $\langle Dx, y \rangle = \langle x, Dy \rangle$, for all $x, y \in \mathscr{A}$, where the angle brackets denote the pairing ($\langle x, y \rangle = x^* y$; and

(ii) The operators $D \pm iI$ are isomorphisms from $\mathscr{A}$ onto A.

(iii) $\deg(Dx) \equiv \deg(x) + 1$, mod 2, for all $x \in \mathscr{A}$.

Compare [7, 3]. Note that we have built into our definition that D have grading degree one (because all our examples will be this way).

If $\mathscr{A}$ is a dense, $\mathbb{Z}/2$-graded, right A-submodule of A and if $D: \mathscr{A} \to A$ satisfies

(i) $\langle Dx, y \rangle = \langle x, Dy \rangle$, for all $x, y \in \mathscr{A}$;

(ii) $\deg(Dx) \equiv \deg(x) + 1$, mod 2, for all $x \in \mathscr{A}$;

(iii) the operators $D \pm iI$ have dense range;

then the closure of D (whose graph is the norm closure in $A \oplus A$ of the graph of D) is a self-adjoint multiplier. We shall call D an *essentially self-adjoint* multiplier of A. To avoid cluttering our notation we use the same symbol for both D and its closure.

If D is an unbounded self-adjoint multiplier of A then the resolvent operators $(D \pm iI)^{-1}$ (viewed as maps of A into itself) are bounded multipliers of A, in the ordinary sense of C^*-algebra theory [15]. If $C_b(\mathbb{R})$ denotes the C^*-algebra of bounded, continuous complex-valued functions on $\mathbb{R}$, and $M(A)$ denotes the multiplier C^*-algebra of A, then there is a unique *functional calculus* homomorphism

$$C_b(\mathbb{R}) \to M(A)$$
$$f \mapsto f(D),$$

mapping the resolvent functions $(x \pm i)^{-1}$ to $(D \pm iI)^{-1}$. Compare [7]. It has the property that if $xf(x) = g(x)$ then $g(D) = Df(D)$. The functional calculus homomorphism is grading preserving, if we grade $C_b(\mathbb{R})$ by even and odd functions.

A.4. *Tensor products and multipliers*

If D is a self-adjoint (or essentially self-adjoint) unbounded multiplier of A, and if B is a second $\mathbb{Z}/2$-graded C^*-algebra, then the operator $D \hat{\otimes} 1$, with domain $\mathscr{A} \hat{\otimes}_{alg} B$ is an essentially self-adjoint multiplier of $A \hat{\otimes} B$. The functional calculus homomorphism for $D \hat{\otimes} 1$ is $f(D \hat{\otimes} 1) = f(D) \hat{\otimes} 1$.

Lemma [3]. *If C and D are essentially self-adjoint multipliers of A and B then $C \hat{\otimes} 1 + 1 \hat{\otimes} D$ is an essentially self-adjoint multiplier of $A \hat{\otimes} B$, with domain the algebraic tensor product of the domains of C and D.*

Lemma. $\exp(-(C \hat{\otimes} 1 + 1 \hat{\otimes} D)^2) = \exp(-C^2) \hat{\otimes} \exp(-D^2)$.

Remarks. The notation means that we apply the function e^{-x^2} to the essentially self-adjoint operators C, D and $C \hat{\otimes} 1 + 1 \hat{\otimes} D$. We shall say that C has *compact resolvent* if $f(C) \in A$, for every $f \in C_0(\mathbb{R})$. From the lemma it is clear that if C and D have compact resolvents then so does $C \hat{\otimes} 1 + 1 \hat{\otimes} D$.

Proof. The formula is certainly correct if C and D are bounded operators, for then we can expand both sides as power series. But if $\varphi \in C_0(\mathbb{R})$ is compactly supported then C and D define bounded multipliers on the C^*-algebra closures of $\varphi(C)\, A\varphi(C)$ and $\varphi(D)\, B\varphi(D)$. The result follows from the fact that as φ ranges over all compactly supported functions, the union of $\varphi(C)\, A\varphi(C) \hat{\otimes} \varphi(D)\, B\varphi(D)$ is dense in $A \hat{\otimes} B$. ∎

APPENDIX B

Mehler's Formula

Our aim is to prove the following result from Section 2.

Proposition. *The composition*

$$\mathscr{S}\mathscr{C}(0) \xrightarrow{\beta} \mathscr{S}\mathscr{C}(E) \xrightarrow{\varphi} \mathscr{S}\mathfrak{K}(\mathfrak{h}(E)),$$

which is an asymptotic morphism from $\mathcal{S}$ to $\mathcal{S}\mathfrak{K}(\mathfrak{h}(E))$, is asymptotic to the family of $*$-homomorphisms $f \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} B)$.

The main tool will be Mehler's formula from quantum theory:

MEHLER'S FORMULA. *If $d = -id/dx$ and if $\alpha > 0$ then*

$$\exp(-\alpha(d^2 + x^2)) = \exp(-\beta x^2)\exp(-\gamma d^2)\exp(-\beta x^2),$$

as bounded operators on $L^2(\mathbb{R})$, where $\beta = (\cosh(2\alpha) - 1)/2\sinh(2\alpha)$ and $\gamma = \sinh(2\alpha)/2$. In addition,

$$\exp(-\alpha(d^2 + x^2)) = \exp(-\beta d^2)\exp(-\gamma x^2)\exp(-\beta d^2),$$

for the same β and γ.

See for example [9]. Note that the second identity follows from the first upon taking the Fourier transform on $L^2(\mathbb{R})$, which interchanges the operators d and x.

In view of Mehler's formula and the formula for B^2 (Lemma 2.14), we get

$$\exp(-t^{-2}B^2) = \exp(-uC^2)\exp(-vD^2)\exp(-uC^2)\exp(-t^{-2}N),$$

where

$$u = (\cosh(2t^{-2}) - 1)/\sinh(2t^{-2}), \qquad v = \sinh(2t^{-2})/2.$$

In addition,

$$\exp(-t^{-2}B^2) = \exp(-uD^2)\exp(-vC^2)\exp(-uD^2)\exp(-t^{-2}N).$$

LEMMA. *If X is any unbounded self-adjoint operator then there are asymptotic equivalences*

$$\exp(-uX^2) \sim \exp(-\tfrac{1}{2}t^{-2}X^2), \qquad \exp(-vX^2) \sim \exp(-t^{-2}X^2)$$

and

$$t^{-1}X\exp(-uX^2) \sim t^{-1}X\exp(-\tfrac{1}{2}t^{-2}X^2),$$

where u and v are the above defined functions of t.

Remark. By "asymptotic equivalence" we mean here that the differences between the left and right hand sides in the above relations all converge to zero, in the operator norm, as t tends to infinity.

Proof of the Lemma. By the spectral theorem it suffices to consider the same problem with the self-adjoint operator X replaced by a real variable

x and the operator norm replaced by the supremum norm on $C_0(\mathbb{R})$. The lemma is then a simple calculus exercise, based on the Taylor series expansions $u = \frac{1}{2}t^{-2} + o(t^{-2})$ and $v = t^{-2} + o(t^{-2})$. ▮

LEMMA. *If $f, g \in \mathcal{S} = C_0(\mathbb{R})$ then $[f(t^{-1}C), g(t^{-1}D)] \sim 0$.*

Proof. For any fixed f, the set of g for which the lemma holds is a C^*-subalgebra of $C_0(\mathbb{R})$. So by the Stone–Weierstrass theorem it suffices to prove the lemma when g is one of the resolvent functions $(x \pm i)^{-1}$. In this case, for a C^∞-function f we have

$$\|[f(t^{-1}C), g(t^{-1}D)]\| \leqslant \|[f(t^{-1}C), t^{-1}D]\|$$

by the commutator identity for resolvents. But as $f(t^{-1}c)$ is a smooth, Clifford algebra-valued function on E we have that

$$\begin{aligned}\|[f(t^{-1}C), t^{-1}D]\| &= t^{-1}\,\|[f(t^{-1}C), D]\| \\ &= t^{-1}\,\|\operatorname{grad}(f(t^{-1}C))\| = t^{-2}\,\|\operatorname{grad}(f(C))\|.\end{aligned}$$

This proves the lemma. ▮

Proof of the Proposition. The lemmas imply that there are asymptotic equivalences:

$$\exp(-t^{-2}B^2) \sim \exp(-t^{-2}D^2)\exp(-t^{-2}C^2)$$

and

$$\begin{aligned}t^{-1}B\exp(-t^{-2}B^2) \sim{}& \exp(-t^{-2}D^2)\,t^{-1}C\exp(-t^{-2}C^2) \\ &+ t^{-1}D\exp(-t^{-2}D^2)\exp(-t^{-2}C^2).\end{aligned}$$

Hence there are asymptotic equivalences

$$\begin{aligned}&\exp(-t^{-2}X^2)\,\hat{\otimes}\,\exp(-t^{-2}B^2) \\ &\quad\sim \exp(-t^{-2}X^2)\,\hat{\otimes}\,\exp(-t^{-2}D^2)\exp(-t^{-2}C^2)\end{aligned}$$

and

$$\begin{aligned}&t^{-1}X\exp(-t^{-2}X^2)\,\hat{\otimes}\,\exp(-t^{-2}B^2) \\ &\qquad+ \exp(-t^{-2}X^2)\,\hat{\otimes}\,t^{-1}B\exp(-t^{-2}B^2) \\ &\quad\sim t^{-1}X\exp(-t^{-2}X^2)\,\hat{\otimes}\,\exp(-t^{-2}D^2)\exp(-t^{-2}C^2) \\ &\qquad+ \exp(-t^{-2}X^2)\,\hat{\otimes}\,\exp(-t^{-2}D^2)\,t^{-1}C\exp(-t^{-2}C^2) \\ &\qquad+ \exp(-t^{-2}X^2)\,\hat{\otimes}\,t^{-1}D\exp(-t^{-2}D^2)\exp(-t^{-2}C^2).\end{aligned}$$

But on the left hand sides are the results of the morphism $f \mapsto f_t(X \hat{\otimes} 1 + 1 \hat{\otimes} B)$, applied to the generators $\exp(-x^2)$ and $x \exp(-x^2)$ of $\mathcal{S}$, while on the right hand sides are the results of the composition $S \xrightarrow{\beta} \mathcal{S}\mathcal{C}(E) \xrightarrow{\varphi} \mathcal{S}\mathfrak{K}(\mathfrak{h}(E))$, applied to the same generators. ∎

APPENDIX C

Continuous Groups

Suppose that E is a Euclidean space which is not necessarily of countably infinite dimensions. If E is at least separable in the norm topology then there is an increasing sequence of finite dimensional subspaces

$$E_1 \subset E_2 \subset \cdots \subset E$$

whose union is dense in E. It is easy to check that:

1. Lemma. *The natural inclusion*

$$\varinjlim_n \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_n)) \to \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$$

is an isomorphism of C^*-algebras.

Because of this, our arguments in the previous sections generalize immediately to the case of a countable discrete group acting on a separable Euclidean space. In particular our Bott periodicity theorem holds for separable Hilbert space.

Suppose now that G is a second countable, locally compact Hausdorff topological group, and that G acts isometrically and *continuously* on E, in the sense that the map $G \times E \to E$ is continuous when E is given its norm topology. We also leave to the reader the following calculation:

2. Lemma. *The natural actions of G on $\mathcal{S}\mathcal{C}(E)$ and $\varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$ are continuous.*

Let us consider how the arguments of the previous sections carry over to the case of a non-discrete group. We define the notion of G-equivariant asymptotic morphism just as we did in Section 4 for discrete groups. A G-equivariant asymptotic morphism induces an asymptotic morphism of full crossed product C^*-algebras and the entire argument of Section 4 carries over verbatim to the case of a non-discrete group G. A little more complicated is the argument of Section 5. The problem is to show that for a suitable increasing sequence of finite dimensional subspaces

$$E_1 \subset E_2 \subset \cdots \subset E,$$

whose union is dense in E, the asymptotic morphism ψ of Definition 5.13 is G-equivariant. Once this is done the remainder of Section 5 carries over to the non-discrete case.

The subspaces E_n are constructed as follows:

3. Lemma. *There is an increasing sequence of finite dimensional subspaces E_n as above with the property that for every $\varepsilon > 0$, every $g \in G$ can be written as a product of isometric isomorphisms $g = g_1 g_2$, where*

(i) *g_1 and g_1^{-1} both map E_n into E_{n+1}, for sufficiently large n; and*

(ii) $\|1 - g_2\| < \varepsilon$.

Remark. The isometries g_1 and g_2 do not necessarily belong to the group G. They are simply isometries of E.

Proof of the Lemma. Let $\{K_n\}$ be an increasing sequence of compact subsets of G whose union is all of G. It suffices to inductively choose E_n large enough that for every $g \in K_n$, every vector in gE_n or $g^{-1}E_n$ is within 2^{-n} of E_{n+1} (and of course such that the union of the E_n is dense in E). It is then a simple matter to construct small norm perturbations g_1 of any g with the required properties. ∎

Fix a sequence $\{E_n\}$ as in the lemma. Using Lemma 1 and the formula in Definition 5.13 we construct an asymptotic morphism

$$\psi\colon \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a)) = \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_n)) \to \mathcal{S}\mathfrak{K}(\mathfrak{h}(E)).$$

We are going to prove:

4. Proposition. *The asymptotic morphism ψ is G-equivariant.*

Form the unbounded, essentially self-adjoint operator

$$B = B_{W_1} + B_{W_2} + B_{W_3} + \cdots$$

on $\mathfrak{h}(E)$. This is a fully equivariant version of the operator considered in Section 5. Note however that B does not have compact resolvent; the eigenvalues of B^2 are 0, 2, 4, ..., and each except 0 has infinite multiplicity.

If g is an isometric isomorphism of E onto itself then let $\pi(g)$ be the corresponding unitary operator on $\mathfrak{h}(E)$.

5. LEMMA. *Let* $\mathfrak{h}_{2n}(E) \subset \mathfrak{h}(E)$ *be the* $2n$*-eigenspace of the operator* B^2. *For every* $\varepsilon > 0$ *there is a* $\delta > 0$ (*depending on* n) *such that if* $\|1 - g\| < \delta$ *then* $\|1 - \pi(g)\|_{\mathfrak{h}(E)_{2n}} < \varepsilon$.

Proof. Let E_a be a finite dimensional subspace of E, and note that $\mathfrak{h}(E_a) \cong L^2(E_a) \otimes \Lambda^*(E)$, where Λ^* denotes the (complexified) exterior algebra. Let $\mathfrak{h}_{j,k}(E_a) \subset \mathfrak{h}(E_a)$ denote the span of the elements $p(e)\exp(-\frac{1}{2}\|e\|^2) \otimes \omega$, where $p(x)$ is a homogeneous polynomial of degree j and ω is a form of degree k. Let $\mathfrak{h}_{j,k}(E) \subset \mathfrak{h}(E)$ be the direct limit of the spaces $\mathfrak{h}_{j,k}(E_a)$ under the inclusions described in Definition 5.2. From Lemma 2.14 and the spectral theory of the harmonic oscillator it follows that the $2n$-eigenspace of B_a^2 lies within $\bigoplus_{j+k \leqslant n} \mathfrak{h}_{j,k}(E)$, so to prove the lemma it suffices to prove a similar statement for each $\mathfrak{h}_{j,k}(E)$. But $\mathfrak{h}_{j,k}(E) \cong S^j(E) \otimes \Lambda^k(E)$, where $S^j(E)$ is the jth symmetric power of (the completion of) E and $\Lambda^k(E)$ is the kth antisymmetric power, and the statement of the lemma is clear for $S^j(E) \otimes \Lambda^k(E)$. ∎

Proof of Proposition 4. We must show that if $x \in \varinjlim_a \mathcal{S}\mathfrak{K}(\mathfrak{h}(E_a))$ and $g \in G$ then there is an asymptotic equivalence $g(\psi_t(x)) \sim \psi_t(g(x))$. In fact it suffices to prove this for all g and all x in a dense subalgebra of the direct limit, and we shall consider those x lying within the union of the subalgebras

$$C_0(-K, K) \hat{\otimes} \mathfrak{K}(\mathfrak{h}_{[N]}(E_a)),$$

where $\mathfrak{h}_{[N]}(E_a) \subset \mathfrak{h}(E_a)$ denotes the direct sum of all eigenspaces of B_a^2 corresponding to eigenvalues less than or equal to N. By so restricting x we ensure that for all t, $\psi_t(x)$ is supported on the subspace $\mathfrak{h}_{[K^2+N]}(E) \subset \mathfrak{h}(E)$ comprised of the eigenspaces of B^2 corresponding to eigenvalues less than $K^2 + N$.

Let us use the notation

$$X_t \underset{\alpha}{\sim} Y_t \Leftrightarrow \limsup_t \|X_t - Y_t\| \leqslant \alpha.$$

We will show that for every $\varepsilon > 0$,

$$\psi_t(g(x)) \underset{4\varepsilon}{\sim} g(\psi_t(x)).$$

As in Lemma 3, write $g = g_1 g_2$ in such a way that $g_1 E_n$, $g_1^{-1} E_n \subset E_{n+1}$ for large enough n and $\|1 - g_2\| < \delta$, where δ is small enough that $\pi(g_2)$ is within ε of the identity on the Hilbert space $\mathfrak{h}_{[K^2+N]}(E)$. By the argument in Section 5, $\psi_t(g_1(x)) \sim g_1(\psi_t(x))$. Furthermore $\|g_2(x) - x\| < 2\varepsilon$ and $\|g_2(\psi_t(x)) - \psi_t(x)\| < 2\varepsilon$, and so

$$\begin{aligned}\psi_t(g(x)) = \psi_t(g_1 g_2(x)) &\underset{2\varepsilon}{\sim} \psi_t(g_1(x)) \\ &\sim g_1(\psi_t(x)) \\ &\underset{2\varepsilon}{\sim} g_1 g_2(\psi_t(x)) = g(\psi_t(x)). \quad \blacksquare\end{aligned}$$

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