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Cyclic Homology of Totally Disconnected Groups Acting on Buildings

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We prove a homological counterpart of a conjecture of P. Baum and A. Connes, concerning K-theory for convolution C^*-algebras of p-adic groups, by calculating the periodic cyclic homology for the convolution algebra of a totally disconnected group acting properly on a building.

1. INTRODUCTION

Let G be a totally disconnected group and denote by $C_c^\infty(G)$ the space of locally constant, compactly supported, complex-valued functions on G. It is an associative algebra under the convolution multiplication

$$\varphi_1 * \varphi_2(g) = \int_G \varphi_1(h)\, \varphi_2(h^{-1}g)\, dh,$$

where the integration is with respect to a left Haar measure on G. The purpose of this paper is to study the cyclic homology of $C_c^\infty(G)$ in the case where G acts properly and simplicially on an affine Bruhat–Tits building.

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If a totally disconnected group G acts properly on a simplicial complex Σ then following [3] we can define equivariant homology groups

$$H_n^G(\Sigma; \mathbb{C}) \qquad (n = 0, 1, 2, \ldots).$$

They are kinds of simplicial homology groups for the quotient $G \backslash \Sigma$, which take into account the representation theory of isotropy subgroups for the action of G on Σ.

Our main result is as follows.

(1.1) THEOREM. *If G is a totally disconnected group acting properly on an affine building Σ then*

$$PHC_j(C_c^\infty(G)) \cong \bigoplus_n H_{j+2n}^G(\Sigma; \mathbb{C}) \qquad (j = 0, 1),$$

where PHC denotes periodic cyclic homology.

For example, let G be an amalgamated free product of compact, totally disconnected groups over a common open subgroup,

$$G = K_0 *_H K_1.$$

Then G acts properly on an associated tree Σ (see [16]), and in this case the homology $H_*^G(\Sigma; \mathbb{C})$ identifies with the homology of the complex

$$0 \leftarrow C_{\mathrm{inv}}^\infty(K_0) \oplus C_{\mathrm{inv}}^\infty(K_1) \xleftarrow{I \oplus -I} C_{\mathrm{inv}}^\infty(H) \leftarrow 0 \cdots$$

where $C_{\mathrm{inv}}^\infty(\cdot)$ denotes the space of locally constant, complex-valued functions invariant under the adjoint action, and I denotes induction. In this case, and if G is discrete, then Theorem 1.1 follows from calculations in [13] (and the proof in [13] extends without difficulty to totally disconnected groups). If G is the p-adic group $SL(2, F)$, then the homology $H_*^G(\Sigma; \mathbb{C})$ is calculated in [4], whereas the periodic cyclic homology of $C_c^\infty(G)$ is calculated in [6].

Theorem 1.1 is analogous to the isomorphism

$$K_j(C_r^*(G)) \cong K_j^G(\Sigma),$$

conjectured by P. Baum and A. Connes [3], relating the K-theory of $C_r^*(G)$ to the equivariant K-homology of Σ, in the sense of Kasparov [10]. To justify this remark we recall that K-theory and periodic cyclic homology are related by a Chern character (see [12], but it should be noted that we are dealing in this paper with $C_c^\infty(G)$, not $C_r^*(G)$). On the other hand, it is shown in [2] that there is a Chern character isomorphism

$$K_j^G(\Sigma)\otimes\mathbb{C}\cong\bigoplus_n H_{2n+j}^G(\Sigma;\mathbb{C}).$$

See [3] for further information.

One might think of totally disconnected groups as lying somewhere between discrete groups and Lie groups, which are the subjects of [13] and [14]. As it happens, the very strong geometrical assumption we have imposed on G (that it acts properly on an affine building) makes available new techniques, thanks to which our arguments become more elementary than those in [13] and [14]. So we have tried to make the paper as simple and accessible as possible. In particular, although our techniques can be adapted to the case of crossed product algebras, for the sake of clarity we have not gone into that here.

We shall use two important facts about affine buildings. First, every isometry is either elliptic or hyperbolic. Second, if f is a hyperbolic isometry then its action on the set $MIN(f)$ of points x in the building which minimize $d(x, fx)$ is simply a translation. Because of this, the contribution to periodic cyclic homology of a hyperbolic conjugacy class in G can be shown to be zero, for more or less the same reason that the periodic cyclic homology for $G=\mathbb{Z}$ is localized at $0\in\mathbb{Z}$ (see, for example [13]). Once the periodic cyclic homology has been shown to be supported on the elliptic conjugacy classes of G (a result usually referred to as an "abstract Selberg principle" [6]), we can complete our calculation by analyzing a Čech complex associated to the covering of the elliptic part of G by the compact open subgroups of G.[1]

Our results have some connections with the representation theory of p-adic groups. The reader is referred to the papers [3] and [6] for a little further information.

2. PRELIMINARIES ON CYCLIC HOMOLOGY

Let us recall from [14] that a *precyclic object* in the category of complex vector spaces is a collection of vector space $\{X_n\}_{n\geqslant 0}$, together with morphisms

$$d_i\colon X_n\to X_{n-1},\qquad (i=0, ..., n),$$

$$t\colon X_n\to x_n,$$

[1] Actually, we shall proceed in reverse order, dealing with the elliptic part of G first, then proceeding to the hyperbolic part.

$$\begin{tikzcd}
{} \arrow[d, "b"'] & & {} \arrow[d, "-b'"] & & {} \arrow[d, "b"] & \\
X_2 & & X_2 \arrow[ll, "\varepsilon"'] & & X_2 \arrow[ll, "N"'] & {} \arrow[l, "\varepsilon"'] \\
X_1 \arrow[from=u, "b"'] & & X_1 \arrow[ll, "\varepsilon"'] \arrow[from=u, "-b'"] & & X_1 \arrow[ll, "N"'] \arrow[from=u, "b"] & {} \arrow[l, "\varepsilon"'] \\
X_0 \arrow[from=u, "b"'] & & X_0 \arrow[ll, "\varepsilon"'] \arrow[from=u, "-b'"] & & X_0 \arrow[ll, "N"'] \arrow[from=u, "b"] & {} \arrow[l, "\varepsilon"']
\end{tikzcd}$$

FIG. 1. The bicomplex associated to a precyclic object.

such that

C1. $d_i d_j = d_{j-1} d_i \colon X_n \to X_{n-2}$ for $i < j$,

C2. $d_i t = \begin{cases} t d_{i-1} \colon X_n \to X_{n-1} & \text{for } 1 \leqslant i \leqslant n \\ d_n \colon X_n \to X_{n-1} & \text{for } i = 0, \end{cases}$

C3. $t^{n+1} = id \colon X_n \to X_n$.

This is a weakening of Connes' notion of *cyclic object* [9], for which there are degeneracy morphisms $s_i \colon X_i \to X_{i+1}$ to go along with the face morphisms d_i.

Given a precyclic object $X = (\{X_n\}_{n \geqslant 0}, d, t)$ we define its *cyclic homology* as follows. Form the following operators on X_n:

$$b' = \sum_{i=0}^{n-1} (-1)^i d_i, \qquad b = b' + (-1)^n d_n$$

$$\varepsilon = 1 - (-1)^n t, \qquad N = \sum_{i=0}^{n} (-1)^{ni} t^i.$$

Then assemble the bicomplex shown in Fig. 1. The cyclic homology of X, denoted $HC_*(X)$, is the homology of the total complex associated to this bicomplex.

We shall be interested in precyclic objects which are *H-unital*, meaning that the b'-complex, shown as the second column in Fig. 1, is acyclic. If $s \colon X_n \to X_{n+1}$ is a contracting homotopy for the b'-complex, meaning $b's + sb' = \text{id}$, then the formula

$$B = \varepsilon s N$$

produces a *mixed complex*: a family of objects $X_0, X_1, \ldots$, along with two operators,

$$b\colon X_n \to X_{n-1} \qquad \text{and} \qquad B\colon X_n \to X_{n+1},$$

satisfying

$$B^2 = 0, \qquad b^2 = 0, \qquad \text{and} \qquad Bb + bB = 0.$$

The *cyclic homology* of a mixed complex is the homology of the total complex associated to the bicomplex shown in Fig. 2. When the mixed complex comes from a precyclic object this definition of cyclic homology agrees with the original one. See Chapter 2 of [12] for a further discussion of all this.

The *Hochschild homology* of a mixed complex, denoted $HH_*(X)$, is the homology of the complex formed from the first column of Fig. 2. This first column may be regarded as a subcomplex of the (b, B)-bicomplex, and the quotient identifies with the whole (b, B)-bicomplex, but with a shift in degree. The associated long exact sequence in homology is the *Connes–Tsygan sequence*

$$\cdots \xrightarrow{I} HC_n(X) \xrightarrow{S} HC_{n-2}(X) \xrightarrow{B}$$

$$HH_{n-1}(X) \xrightarrow{I} HC_{n-1}(X) \xrightarrow{S} \cdots.$$

Using the five lemma, it follows from the Connes–Tsygan sequence that a morphism of mixed complexes which is an isomorphism on Hochschild homology is also an isomorphism on cyclic homology. The reader is again referred to [12] for further details.

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
X_2 \arrow[d, "b"'] & X_1 \arrow[l, "B"'] \arrow[d, "b"] & X_0 \arrow[l, "B"'] \\
X_1 \arrow[d, "b"'] & X_0 \arrow[l, "B"'] & \\
X_0 & &
\end{tikzcd}$$

FIG. 2. The bicomplex associated to a mixed complex.

We shall be interested in the following examples.

(2.1) EXAMPLE. If A is an associative algebra over $\mathbb{C}$, and $X_n = A^{\otimes n+1}$ (the tensor product is over $\mathbb{C}$), and if we define

$$d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{for} \quad i = 0, ..., n-1 \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{for} \quad i = n, \end{cases}$$

$$t(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

then we obtain a precyclic object, which we shall write as

$$A^{\natural} = (\{A^{\otimes n+1}\}_{n \geqslant 0}, d, t).$$

If A is locally unital (i.e., a direct limit of unital algebras) then this precyclic object is H-unital. Of course we are principally interested in the algebra $A = C_c^{\infty}(G)$, where G is a totally disconnected group (that is, a locally compact Hausdorff topological group whose topology has a countable base, comprised of open-and-closed sets). This is a locally unital algebra. There are natural isomorphisms

$$C_c^{\infty}(G)^{\otimes n+1} \cong C_c^{\infty}(G^{n+1}),$$

using which the operators d_i and t are given by the formulas

$$d_i \varphi(g_0, ..., g_{n-1}) = \int_G \varphi(g_0, ..., g_{i-1}, h, h^{-1} g_i, ..., g_{n-1}) \, dh,$$

for $i = 0, ..., n-1$,

$$d_n \varphi(g_0, ..., g_{n-1}) = \int_G \varphi(h^{-1} g_0, g_1, ..., g_{n-1}, h) \, dh,$$

and

$$t\varphi(g_0, ..., g_n) = \varphi(g_1, ..., g_n, g_0).$$

If $\chi \in C_c^{\infty}(G)$ satisfies $\int_G \chi(g) \, dg = 1$ then the operator

$$s\varphi(g_0, g_1, ..., g_n) = \chi(g_0) \, \varphi(g_0 g_1, g_2, ..., g_n)$$

is an explicit contracting homotopy for the b'-complex. Compare Proposition 1.4 in [6].

(2.2) Example. Continuing the above example, suppose that U is a conjugation-invariant, open subset of G. Define

$$G^{n+1}{}_U = \{(g_0, g_1, ..., g_n) \in G^{n+1} \mid g_0 g_1 \cdots g_n \in U\}.$$

The subspaces

$$C_c^\infty(G^{n+1}{}_U) \subset C_c^\infty(G^{n+1})$$

define a subobject $C_c^\infty(G)_U^\natural$ of $C_c^\infty(G)^\natural$. (Note also that the s-operator given above restricts to $C_c^\infty(G)_U^\natural$.) Following [6] we shall call $C_c^\infty(G)_U^\natural$ the *localization* of $C_c^\infty(G)^\natural$ to U.

(2.3) Definition. Let G be a group and let Σ be a discrete set on which G acts on the left. For $n \geqslant 0$ we define the nth *Brylinski space* to be

$${}_n\hat{\Sigma} = \{(g_0, ..., g_n, \sigma) \in G^{n+1} \times \Sigma \mid g_0 \cdots g_n \sigma = \sigma\}.$$

If $n = 0$ we shall sometimes write $\hat{\Sigma}$ in place of ${}_0\hat{\Sigma}$. Note that this 0th Brylinski space is the disjoint union of the isotropy subgroups of points in Σ.

(2.4) Example. Let G be a totally disconnected group and suppose that G acts on a discrete set Σ in such a way that the isotropy subgroups of points are open subgroups of G. Denote by $C_c^\infty({}_n\hat{\Sigma})$ the locally constant, compactly supported functions on the Brylinski space ${}_n\hat{\Sigma}$. The formulas

$$d_i\varphi(g_0, ..., g_{n-1}, \sigma) = \int_G \varphi(g_0, ..., g_{i-1}, h, h^{-1}g_i, ..., g_{n-1}, \sigma)\, dh,$$

for $i = 0, ..., n-1$,

$$d_n\varphi(g_0, ..., g_{n-1}, \sigma) = \int_G \varphi(h^{-1}g_0, g_1, ..., g_{n-1}, h, h^{-1}\sigma)\, dh,$$

and

$$t\varphi(g_0, ..., g_n, \sigma) = \varphi(g_1, ..., g_n, g_0, g_0^{-1}\sigma),$$

define a precyclic object $C_c^\infty(\hat{\Sigma})^\natural$. The operator

$$s\varphi(g_0, g_1, ..., g_n, \sigma) = \chi(g_0)\, \varphi(g_0 g_1, g_2, ..., g_n, \sigma),$$

where as above $\chi \in C_c^\infty(G)$ satisfies $\int_G \chi(g)\, dg = 1$, is an explicit contracting homotopy for the b'-complex.

The construction $\Sigma \to C_c^\infty(\hat{\Sigma})^\natural$ is functorial: an equivariant map $f\colon \Sigma \to \Sigma'$ induces maps

$$f_*\colon C_c^\infty({}_n\hat{\Sigma}) \to C_c^\infty({}_n\widehat{\Sigma'})$$

by the formula

$$f_*\varphi(g_0, ..., g_n, \sigma') = \sum_{f(\sigma)=\sigma'} \varphi(g_0, ..., g_n, \sigma),$$

and f_* commutes with the maps d_i and t.

Note that if Σ is a single point then $C_c^\infty(\hat{\Sigma})^\natural = C_c^\infty(G)^\natural$.

(2.5) Example. It will be convenient to specialize the above example to the case where G is discrete. In this case we shall write $\mathbb{C}[{}_n\hat{\Sigma}]$ in place of $C_c^\infty({}_n\hat{\Sigma})$, and we shall denote by $[g_0, ..., g_n, \sigma]$ the delta function at $(g_0, ..., g_n, \sigma) \in {}_n\hat{\Sigma}$. In terms of this notation the operators d_i and t are given by

$$d_i\colon [g_0, ..., g_n, \sigma] \mapsto [g_0, ..., g_i g_{i+1}, ..., g_n, \sigma],$$

for $i = 0, ..., n-1$,

$$d_n\colon [g_0, ..., g_n, \sigma] \mapsto [g_n g_0, ..., g_{n-1}, g_n\sigma],$$

and

$$t\colon [g_0, ..., g_n, \sigma] \mapsto [g_n, g_0, ..., g_{n-1}, g_n\sigma].$$

The operator

$$s\colon [g_0, g_1, ..., g_n, \sigma] \mapsto [e, g_0, g_1, ..., g_n, \sigma]$$

is a homotopy contraction of the b'-complex.

3. EQUIVARIANT HOMOLOGY

We review the definition of the equivariant homology groups of [3].

Let G be a totally disconnected group and let Σ be a simplicial complex equipped with a simplicial action of G which is *proper*, in the sense that the stabilizer of each simplex is a compact open subgroup of G.

We shall assume that Σ is *oriented*, which means the vertices of each simplex are linearly ordered (and we regard two orderings of a simplex as the same if they differ by an even permutation). This need not be done with any regard to the group action or the inclusion relations among simplices.

Now if σ is a simplex in Σ with vertices $v_0, v_1, ..., v_p$ (written in order) and if η is a codimension one face of σ with vertices $v_0, ..., \hat{v}_i, ..., v_p$ then we define the *incidence number* $(-1)^{\langle \eta : \sigma \rangle}$ to be $+1$ if this listing of the vertices of η agrees with the given orientation of η, and we define $(-1)^{\langle \eta : \sigma \rangle}$ to be -1 otherwise.

Form the vector space

$$C_p(G; \Sigma) \stackrel{\text{def}}{=} C_c^\infty(\widehat{\Sigma^p}) \cong \bigoplus_{\dim \sigma = p} C_c^\infty(G_\sigma),$$

where Σ^p denotes the set of p-simplices in Σ, $\widehat{\Sigma^p}$ denotes the 0th Brylinski space for Σ^p, and the direct sum is over all simplices of dimension p. (We remind the reader that $\widehat{\Sigma^p}$ is the disjoint union of the isotropy subgroups of points in Σ, which explains the above isomorphism.) We shall write elements of $C_p(G; \Sigma)$ as formal sums

$$\sum_{\sigma \in \Sigma^p} \varphi_\sigma[\sigma].$$

We remark that if G is the trivial group then $C_c^\infty(\widehat{\Sigma^p})$ is just the space of simplicial p-chains in Σ (with complex coefficients).

Observe that if $\eta \subset \sigma$ then G_σ is a compact open subset of G_η. Hence

$$\eta \subset \sigma \Rightarrow C_c^\infty(G_\sigma) \subset C_c^\infty(G_\eta)$$

(we extend a function on G_σ by zero to obtain a function on G_η). In view of this we can define homomorphisms

$$\partial: C_p(G; \Sigma) \to C_{p-1}(G; \Sigma)$$

by the formula

$$\partial(\varphi_\sigma[\sigma]) = \sum_{\eta \subset \sigma} (-1)^{\langle \eta : \sigma \rangle} \varphi_\sigma[\eta],$$

where the sum is over the codimension one faces of σ. We obtain a chain complex

$$0 \leftarrow C_0(G; \Sigma) \leftarrow C_1(G; \Sigma) \leftarrow C_2(G; \Sigma) \leftarrow \cdots.$$

Suppose now, for simplicity, that the action of G on Σ is orientation-preserving. Then G acts on our complex in the following way,

$$g \sum_{\dim \sigma = p} \varphi_\sigma[\sigma] = \sum_{\dim \sigma = p} g a_\sigma[g\sigma],$$

where $g\varphi_\sigma(\gamma) = \varphi_\sigma(g^{-1}\gamma g)$ (observe that $g\varphi_\sigma \in C_c^\infty(G_{g\sigma})$, as required).

(3.1) DEFINITION. We define $H^G_*(\Sigma; \mathbb{C})$ to be the homology of the complex of coinvariants

$$0 \leftarrow C_0(G;\Sigma)_G \leftarrow C_1(G;\Sigma)_G \leftarrow C_2(G;\Sigma)_G \leftarrow \cdots.$$

(3.2) EXAMPLE. If G is the trivial group then $C_*(G;\Sigma)$ is the standard simplicial complex for Σ, and $H^G_*(\Sigma;\mathbb{C})$ is the simplicial homology of Σ with complex coefficients. If G is discrete and acts freely on Σ then $H^G_*(\Sigma;\mathbb{C})$ is the homology of the space Σ/G, with complex coefficients.

The following lemma reconciles our definition with the one in [3].

(3.3) LEMMA. (i) *Exactly the same homology groups are obtained if, in the direct sum* $\bigoplus_{\dim\sigma=p} C^\infty_c(G_\sigma)$, *the spaces* $C^\infty_c(G_\sigma)$ *are replaced by the spaces of coinvariants* $C^\infty_c(G_\sigma)_{G_\sigma}$.

(ii) *The same homology groups are also obtained if the spaces* $C^\infty_c(G_\sigma)$ *are replaced by the spaces of invariants* $C^\infty_c(G_\sigma)^{G_\sigma}$, *and the boundary map* ∂ *is replaced with*

$$\partial(\varphi_\sigma[\sigma]) = \sum_{\eta\subset\sigma} (-1)^{\langle\eta:\sigma\rangle}\operatorname{Ind}^{G_\eta}_{G_\sigma}(\varphi_\sigma)[\eta],$$

where $\operatorname{Ind}^{G_\eta}_{G_\sigma}\colon C^\infty_c(G_\sigma)^{G_\sigma} \to C^\infty_c(G_\eta)^{G_\eta}$ *is induction, defined by*

$$\operatorname{Ind}^{G_\eta}_{G_\sigma}(a)(g) = \frac{\operatorname{vol}(G_\eta)}{\operatorname{vol}(G_\sigma)}\int_{G_\sigma} a(\gamma g\gamma^{-1})\,d\gamma.$$

Proof. The first part follows from the fact that in forming $H^G_*(\Sigma;\mathbb{C})$ we are already taking coinvariants: it is easily checked that taking coinvariants first with respect to each G_σ and then with respect to G has no additional effect, in that the same complex, and hence the same homology, results.

To prove the second part of the lemma, we use the fact that the natural map from $C^\infty_c(G_\sigma)^{G_\sigma}$, a subspace of $C^\infty_c(G_\sigma)$, to the coinvariants $C^\infty_c(G_\sigma)_{G_\sigma}$, a quotient of $C^\infty_c(G_\sigma)$, is an isomorphism (an inverse is given by averaging functions with respect to Haar measure). An inclusion of groups $G_\sigma \subset G_\eta$ gives a commuting diagram

$$\begin{tikzcd} C^\infty_c(G_\sigma)_{G_\sigma} \arrow[r] \arrow[d, "\cong"'] & C^\infty_c(G_\eta)_{G_\eta} \arrow[d, "\cong"] \\ C^\infty_c(G_\sigma)^{G_\sigma} \arrow[r] & C^\infty_c(G_\eta)^{G_\eta} \end{tikzcd}$$

where the top map is induced from inclusion and the bottom is given by the induction formula. ∎

(3.4) Example. Let F be a p-adic field and let $G = SL(n, F)$. Let Σ be the affine building for $SL(n, F)$, as described in [8], for example. It is a simplicial complex of dimension $n-1$, and each closed $(n-1)$-simplex Δ in Σ is a fundamental domain for the action of G. The homology $H_*^G(\Sigma; \mathbb{C})$ identifies with the homology of the complex

$$0 \leftarrow C_0^{\mathrm{inv}}(\Delta; \mathbb{C}) \leftarrow C_1^{\mathrm{inv}}(\Delta; \mathbb{C}) \leftarrow C_2^{\mathrm{inv}}(\Delta; \mathbb{C}) \leftarrow \cdots \leftarrow C_{n-1}^{\mathrm{inv}}(\Delta; \mathbb{C}) \leftarrow 0,$$

where $C_p^{\mathrm{inv}}(\Delta; \mathbb{C})$ denotes the direct sum of the spaces of invariant functions on the isotropy subgroups of the p-dimensional faces of Δ, and the differentials are given by the formula in Lemma 3.3. See [3, 5] for a few more details. The homology groups in the case $n = 2$ were studied closely in [4].

We make the following additional assumption on fixed point sets in Σ:

F. For every compact subgroup $H \subset G$ the fixed point set Σ^H is nonempty and contractible.

As we shall note in Section 5, if Σ is an affine building then F holds. More generally, if Σ is the "universal proper G-space" $\underline{E}G$, discussed in [3], then F holds.

(3.5) Definition. An element of G is *elliptic* if it fixes some point in Σ. Denote by G_{ell} the set of all elliptic elements in G.

According to our assumptions, an element is elliptic if and only if it lies in some compact subgroup of G. The set G_{ell} is an open, conjugation invariant subset of G.

(3.6) Proposition. *Let G be a totally disconnected group acting properly on a simplicial complex Σ, in such a way that Property* F *holds. Define an augmentation*

$$C_c^\infty(G_{\mathrm{ell}}) \leftarrow C_0(\Sigma; G)$$

by summation:

$$\sum \varphi_\sigma \leftarrow \sum \varphi_\sigma[\sigma].$$

The complex

$$C_0(\Sigma; G) \xleftarrow{\partial} C_1(\Sigma; G) \xleftarrow{\partial} C_2(\Sigma; G) \xleftarrow{\partial} \cdots$$

is a resolution of the vector space $C_c^\infty(G_{\mathrm{ell}})$.

Proof. Let $\tilde{\Sigma}$ be the simplicial complex whose vertices are the vertices of Σ, and for which *every* $(p+1)$-tuple of vertices is a p-simplex. Obviously Σ is a subcomplex of $\tilde{\Sigma}$, and we orient $\tilde{\Sigma}$ in a way which extends the orientation on Σ. The group G acts on $\tilde{\Sigma}$ in the obvious way,[2] and the stabilizer of simplices are compact open subgroups of G. So we can form the complex

$$C_0(\tilde{\Sigma}; G) \xleftarrow{\partial} C_1(\tilde{\Sigma}; G) \xleftarrow{\partial} C_2(\tilde{\Sigma}; G) \xleftarrow{\partial} \cdots.$$

It is easy to see that *this* complex is a resolution of $C_c^\infty(G_{\text{ell}})$, since it is nothing more than the Čech resolution of $C_c^\infty(G_{\text{ell}})$ for the covering of G_{ell} comprised of the isotropy groups of vertices in Σ. So it suffices to exhibit the complex $C_*(\Sigma; G)$ as a direct summand of $C_*(\tilde{\Sigma};G)$; in other words it suffices to define a map of complexes

$$r_*: C_*(\tilde{\Sigma}; G) \to C_*(\Sigma; G)$$

which is left-inverse to the inclusion $j_*: C_*(\Sigma; G) \to C_*(\tilde{\Sigma}; G)$.

We shall construct the projection r by induction. To begin, we note that $C_0(\Sigma; G) = C_0(\tilde{\Sigma}; G)$, so we can take r_0 to be the identity. Suppose next that $r_p: C_p(\tilde{\Sigma}; G) \to C_p(\Sigma; G)$ has been constructed for all $p < n$ such that:

R1. $r_p \circ j_p = \mathrm{id}_{C_p(\Sigma;\, G)}$

R2. $r_p \partial = \partial r_p$

R3. If $\tilde{\eta}$ is a simplex in $\tilde{\Sigma}$ of dimension $p < n$ then there are complex coefficients λ_η indexed by the p-simplices in $\Sigma^{G_{\tilde{\eta}}}$, the fixed point set in Σ for the stabilizer of $\tilde{\eta}$, such that all but finitely many λ_η are zero and

$$r_p: \varphi_{\tilde{\eta}}[\tilde{\eta}] \to \sum_{\eta \in \Sigma_p^{G_{\tilde{\eta}}}} \lambda_\eta \varphi_{\tilde{\eta}}[\eta].$$

(We observe that if $\eta \in \Sigma^{G_{\tilde{\eta}}}$ then $G_{\tilde{\eta}}$ is a subgroup of G_η. So $\varphi_{\tilde{\eta}} \in C_c^\infty(G_\eta)$, and hence the formula makes sense.)

We begin the construction of r_n by defining

$$r_n(\varphi_\sigma[\sigma]) = \varphi_\sigma[\sigma],$$

for all simplices of $\tilde{\Sigma}$ which are actually simplices in Σ. In other words, we define r_n to be the identity on the subcomplex $C_n(\Sigma; G) \subset C_n(\tilde{\Sigma}; G)$. Obviously this takes care of property R1, and at the same time properties

[2] The action of G on $\tilde{\Sigma}$ may not be orientation preserving, but this will not affect us since we shall not be concerned with the action of G on $C_*(\tilde{\Sigma}; G)$.

R2 and R3 are satisfied by as much of r_n as we have yet defined. Now let $\tilde{\sigma}$ be an n-simplex in $\tilde{\Sigma}$ not in Σ. It follows from property R3 that

$$r_{n-1}(\partial([\tilde{\sigma}])) = \sum \mu_\eta[\eta],$$

for some scalars μ_η indexed by finitely many $(n-2)$-simplices in the fixed point set $\Sigma^{G_{\tilde{\sigma}}}$. It follows from property R2 that

$$\partial r_{n-1}\partial = r_{n-1}\partial\partial,$$

and so

$$\partial\left(\sum \mu_\eta[\eta]\right) = 0.$$

In other words $\sum \mu_\eta[\eta]$ is a cycle for the (ordinary) simplicial homology group $H_{n-1}(\Sigma^{G_{\tilde{\sigma}}}; \mathbb{C})$. But $\Sigma^{G_{\tilde{\sigma}}}$ is a contractible simplicial complex, and hence this simplicial cycle is a boundary:

$$\sum \mu_\eta[\eta] = \partial\left(\sum \lambda_\sigma[\sigma]\right).$$

We define

$$r_n(\varphi_{\tilde{\sigma}}) = \sum \varphi_{\tilde{\sigma}}\lambda_\sigma[\sigma].$$

It is then clear from the construction that R2 and R3 are satisfied. ∎

4. THE ELLIPTIC PART OF G

As in the previous section, let G be a totally disconnected group acting properly on a simplicial complex Σ. Assume that the Property F, described in the previous section, continues to hold.

(4.1) Definition. We shall denote by $HH_j(C_c^\infty(G))_{\mathrm{ell}}$ and $HC_j(C_c^\infty(G))_{\mathrm{ell}}$ the Hochschild and cyclic homology of $C_c^\infty(G)$, localized on the set G_{ell} of elliptic elements (see Example 2.2).

We are going to prove the following result.

(4.2) Theorem. *There are isomorphisms*

$$HH_j(C_c^\infty(G))_{\mathrm{ell}} \cong H_j^G(\Sigma; \mathbb{C}).$$

$$
\begin{tikzcd}
{} \arrow[d, "b"'] & & {} \arrow[d, "b"'] & & {} \arrow[d, "b"'] & & \\
\mathbb{C}[{}_2\widehat{\Sigma^0}] & & \mathbb{C}[{}_2\widehat{\Sigma^1}] \arrow[ll, "\partial"'] \arrow[d, "b"'] & & \mathbb{C}[{}_2\widehat{\Sigma^2}] \arrow[ll, "\partial"'] \arrow[d, "b"'] & & {} \arrow[ll, "\partial"'] \\
\mathbb{C}[{}_1\widehat{\Sigma^0}] \arrow[from=u, "b"'] \arrow[d, "b"'] & & \mathbb{C}[{}_1\widehat{\Sigma^1}] \arrow[ll, "-\partial"'] \arrow[d, "b"'] & & \mathbb{C}[{}_1\widehat{\Sigma^2}] \arrow[ll, "-\partial"'] \arrow[d, "b"'] & & {} \arrow[ll, "-\partial"'] \\
\mathbb{C}[{}_0\widehat{\Sigma^0}] & & \mathbb{C}[{}_0\widehat{\Sigma^1}] \arrow[ll, "\partial"'] & & \mathbb{C}[{}_0\widehat{\Sigma^2}] \arrow[ll, "\partial"'] & & {} \arrow[ll, "\partial"']
\end{tikzcd}
$$

FIG. 3. The double complex $\mathbb{C}[{}_*\widehat{\Sigma^*}]$.

Although the argument is essentially the same for both discrete and totally disconnected groups, the notation is a little simpler in the discrete case. We shall first assume that G is discrete, and then deal with totally disconnected groups at the end of the section.

So let G be discrete. The proof of Theorem 4.2 consists of an analysis of the double complex $\mathbb{C}[{}_*\widehat{\Sigma^*}]$ shown in Fig. 3, in which ${}_n\widehat{\Sigma^p}$ denotes the nth Brylinski space of the set Σ^p of p-simplices in Σ, and the differentials are as follows:

$$
\begin{aligned}
b\colon [g_0, ..., g_n, \sigma] \mapsto\ & [g_0 g_1, ..., g_n, \sigma] - [g_0, g_1 g_2, ..., g_n, \sigma] \\
& + \cdots \\
& + (-1)^{n-1} [g_0, ..., g_{n-1} g_n, \sigma] \\
& + (-1)^n [g_n g_0, g_1, ..., g_{n-1}, g_n \sigma], \\
\partial\colon [g_0, ..., g_n, \sigma] \mapsto\ & \sum_{\eta \in \partial\sigma} (-1)^{\langle \eta : \sigma \rangle} [g_0, ..., g_n, \eta].
\end{aligned}
$$

So each column gives the Hochschild homology for the precyclic object $\mathbb{C}[\widehat{\Sigma^p}]^\natural$, while the rows are chain complexes related to our construction of equivariant homology in the previous section.

(4.3) LEMMA. *The maps*

$$\mathbb{C}[G_{\text{ell}}^{n+1}] \leftarrow \mathbb{C}[{}_n\widehat{\Sigma^0}]$$

defined by

$$[g_0, g_1, ..., g_n] \leftarrow [g_0, g_1, ..., g_n, \sigma]$$

give a quasi-isomorphism from the double complex in Fig. 3 *to the complex*

$$\mathbb{C}[G_{\mathrm{ell}}] \xleftarrow{b} \mathbb{C}[G^2_{\mathrm{ell}}] \xleftarrow{b} \mathbb{C}[G^3_{\mathrm{ell}}] \xleftarrow{b} \cdots,$$

computing the elliptic part of Hochschild homology of $\mathbb{C}[G]$ (*which complex we regard as a double complex concentrated in the first column*).

Proof. It suffices to show that the sequence

$$0 \leftarrow \mathbb{C}[G^{n+1}_{\mathrm{ell}}] \leftarrow \mathbb{C}[{}_n\widehat{\Sigma^0}] \xleftarrow{\partial} \mathbb{C}[{}_n\widehat{\Sigma^1}] \xleftarrow{\partial} \cdots,$$

obtained by augmenting the nth row in Fig. 3, is exact. But the map

$$[g_0, g_1, ..., g_n, \sigma] \mapsto [g_0 g_1 \cdots g_n, \sigma] \otimes [g_1, ..., g_n]$$

is an isomorphism from this sequence to the sequence

$$0 \leftarrow \mathbb{C}[G_{\mathrm{ell}}] \otimes \mathbb{C}[G^n] \leftarrow \mathbb{C}[\widehat{\Sigma^0}] \otimes \mathbb{C}[G^n] \xleftarrow{\partial \otimes 1} \mathbb{C}[\widehat{\Sigma^1}] \otimes \mathbb{C}[G^n] \xleftarrow{\partial \otimes 1} \cdots,$$

which is exact by Proposition 3.6. ∎

(4.4) LEMMA. *The quotient map*

$$\begin{array}{c} \mathbb{C}[\widehat{\Sigma^p}] \\ \downarrow \\ \mathbb{C}[\widehat{\Sigma^p}]_G \end{array}$$

is a quasi-isomorphism from the double complex in Fig. 3 *to the complex*

$$\mathbb{C}[\widehat{\Sigma^0}]_G \leftarrow \mathbb{C}[\widehat{\Sigma^1}]_G \leftarrow \mathbb{C}[\widehat{\Sigma^2}]_G \leftarrow \cdots$$

computing the equivariant homology $H^G_*(\Sigma; \mathbb{C})$ (*which complex we regard as a double complex, concentrated in the bottom row*).

Proof. It suffices to show that the sequence

$$0 \leftarrow \mathbb{C}[\widehat{\Sigma^p}]_G \leftarrow \mathbb{C}[{}_0\widehat{\Sigma^p}] \xleftarrow{b} \mathbb{C}[{}_1\widehat{\Sigma^p}] \xleftarrow{b} \mathbb{C}[{}_2\widehat{\Sigma^p}] \xleftarrow{b} \cdots,$$

which is obtained by augmenting the pth column in Fig. 3, is exact. But the map

$$[g_0, ..., g_n, \sigma] \mapsto [g_1, ..., g_n] \otimes [g_0 g_1 \cdots g_n, \sigma]$$

gives an isomorphism from this to the sequence

$$0 \leftarrow \mathbb{C}[\widehat{\Sigma^p}]_G \leftarrow \mathbb{C} \otimes \mathbb{C}[\widehat{\Sigma^p}] \xleftarrow{d} \mathbb{C}[G] \otimes \mathbb{C}[\widehat{\Sigma^p}] \xleftarrow{d} \mathbb{C}[G^2] \otimes \mathbb{C}[\widehat{\Sigma^p}] \xleftarrow{d} \cdots,$$

where the differential is given by the formula

$$\begin{aligned} d\colon [g_1, ..., g_n] \otimes [g, \sigma] \mapsto {} & [g_2, ..., g_n] \otimes [g, \sigma] \\ & - [g_1 g_2, ..., g_n] \otimes [g, \sigma] \\ & + \cdots \\ & + (-1)^{n-1} [g_1, ..., g_{n-1} g_n] \otimes [g, \sigma] \\ & + (-1)^n [g_1, ..., g_{n-1}] \otimes [g_n g g_n^{-1}, g_n \sigma]. \end{aligned}$$

Now the homology of the complex $(\mathbb{C}[G^*] \otimes \mathbb{C}[\widehat{\Sigma^p}], d)$ is the group homology of G with coefficients in the left G-module $\mathbb{C}[\widehat{\Sigma^p}]$ (see [7]). If $\sigma_1, \sigma_2, \ldots$ is a list of representatives from among the G-orbits of p-simplices in Σ, then the G-module $\mathbb{C}[\widehat{\Sigma^p}]$ identifies with a direct sum of modules induced from the G_{σ_j}-modules $\mathbb{C}[G_{\sigma_j}]$ (the G_{σ_j}-action is given by conjugation). So, since finite groups have vanishing homology in positive degree, it follows from Shapiro's lemma that the complex $(\mathbb{C}[\widehat{\Sigma^p}] \otimes \mathbb{C}[G^*], d)$ gives a resolution of $\mathbb{C}[\widehat{\Sigma^p}]_G$, as required. ∎

Proof of Theorem 4.2. This follows immediately from the preceding two lemmas. ∎

Here is the analogous result for cyclic homology.

(4.5) THEOREM. *If a totally disconnected group G acts on a simplicial complex Σ in such a way that property* F *holds then*:

$$HC_j(C_c^\infty(G))_{\mathrm{ell}} \cong H_j^G(\Sigma; \mathbb{C}) \oplus H_{j-2}^G(\Sigma; \mathbb{C}) \oplus \cdots \oplus H_0^G(\Sigma; \mathbb{C}),$$

if j is even, and

$$HC_j(C_c^\infty(G))_{\mathrm{ell}} \cong H_j^G(\Sigma; \mathbb{C}) \oplus H_{j-2}^G(\Sigma; \mathbb{C}) \oplus \cdots \oplus H_1^G(\Sigma; \mathbb{C}),$$

if j is odd.

Proof. Once again, let us consider first the case where G is discrete. From the precyclic structure described in Example 2.5 we obtain maps

$$B\colon \mathbb{C}[{}_n\widehat{\Sigma^p}] \to \mathbb{C}[{}_{n+1}\widehat{\Sigma^p}]$$

which anticommute with b and commute with ∂. Totalizing, we obtain a mixed complex

$$(\mathrm{Tot}(\mathbb{C}[{}_*\widehat{\Sigma^*}]), b \pm \partial, B).$$

The map in Lemma 4.3 gives a morphism from this mixed complex to the mixed complex $(\mathbb{C}[G^{*+1}_{\mathrm{ell}}], b, B)$ associated to the precyclic object $C_c^\infty(G)^\natural_{G_{\mathrm{ell}}}$. Lemma 4.3 asserts that this map is an isomorphism on Hochschild homology, and so it is also an isomorphism on cyclic homology. Hence, it suffices to calculate to cyclic homology of the mixed complex $(\mathrm{Tot}(\mathbb{C}[{}_*\widehat{\Sigma^*}]), b \pm \partial, B)$. But the map in Lemma 4.4 is a morphism from $(\mathrm{Tot}(\mathbb{C}[{}_*\widehat{\Sigma^*}]), b \pm \partial, B)$ to the mixed complex $(\mathbb{C}[\widehat{\Sigma^*}], \partial, 0)$, in which the B-operator is zero. Lemma 4.4 asserts that the map is an isomorphism on Hochschild homology. Hence it is an isomorphism on cyclic homology. But since the B-operator in this last mixed complex is zero, the cyclic homology reduces to a direct sum of Hochschild homology groups, as required. ∎

We turn now to the case of a general totally disconnected group. We shall use some basic facts about differentiable group homology for totally disconnected groups worked out by Blanc and Brylinski [6].

(4.6) DEFINITION. A *differentiable G-module* is a complex vector space equipped with an action of G such that the stabilizer of every vector is an open subgroup of G. If V is a differentiable G-module then its *differentiable group homology* $H_*(G, V)$ is the homology of the complex

$$C_c^\infty(G^0, V) \xleftarrow{d} C_c^\infty(G^1, V) \xleftarrow{d} C_c^\infty(G^2, V) \xleftarrow{d} \cdots,$$

where $C_c^\infty(G^n, V)$ denotes the locally constant, compactly supported functions from G^n to V, and the differential is given by the formula[3]

$$\begin{aligned} d(\varphi)(g_1, ..., g_{n-1}) = &\int_G \varphi(h, g_1, ..., g_{n-1})\, dh \\ &+ \sum_{i=1}^{n-1} (-1)^i \int_G \varphi(g_1, ..., g_{i-1}, h, h^{-1}g_i, ..., g_{n-1})\, dh \\ &+ (-1)^n \int_G h \cdot \varphi(g_1, ..., g_{n-1}, h)\, dh. \end{aligned}$$

[3] Our complex maps isomorphically to the one considered in [6] by the change of variables $\check{\varphi}(g_1, ..., g_n) = \varphi(g_n^{-1}, ..., g_1^{-1})\, \Delta(g_1 \cdots g_n)^{-1}$, where Δ is the modular function of G.

Of course we are interested in the modules $V = C_c^\infty(\widehat{\Sigma^p})$. Using the natural isomorphism

$$C_c^\infty(G^n, C_c^\infty(\widehat{\Sigma^p})) \cong C_c^\infty(G^n \times \widehat{\Sigma^p})$$

the differential d becomes

$$\begin{aligned} d(\varphi)(g_1, ..., g_{n-1}, g, \sigma) &= \int_G \varphi(h, g_1, ..., g_{n-1}, g, \sigma)\, dh \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \int_G \varphi(g_1, ..., g_{i-1}, h, h^{-1}g_i, ..., g_{n-1}, g, \sigma)\, dh \\ &\quad + (-1)^n \int_G \varphi(g_1, ..., g_{n-1}, h, h^{-1}gh, h^{-1}\sigma)\, dh. \end{aligned}$$

Note that if G is discrete then this is the same as the formula in the proof of Lemma 4.4. Now the formula

$$\tilde{\varphi}(g_0, ..., g_n, \sigma) = \varphi(g_1, ..., g_n, g_0 g_1 \cdots g_n, \sigma)$$

gives an isomorphism

$$\varphi\colon C_c^\infty(G^n, C_c^\infty(\widehat{\Sigma^p})) \to C_c^\infty({}_n\widehat{\Sigma^p}),$$

with respect to which the group homology differential corresponds to the differential

$$b\colon C_c^\infty({}_n\widehat{\Sigma^p}) \to C_c^\infty({}_{n-1}\widehat{\Sigma^p})$$

$$\begin{aligned} b\varphi(g_0, ..., g_{n-1}, \sigma) &= \sum_{i=0}^{n-1} (-1)^i \int_G \varphi(g_0, ..., g_{i-1}, h, h^{-1}g_i, ..., g_{p-1}, \sigma)\, dh \\ &\quad + (-1)^n \int_G \varphi(h^{-1}g_0, g_1, ..., g_{p-1}, h, h^{-1}\sigma)\, dh. \end{aligned}$$

This is the b-operator associated to the precyclic object $C_c^\infty(\widehat{\Sigma^p})^\natural$ defined in Example 2.4. If we define also

$$\partial\colon C_c^\infty({}_n\widehat{\Sigma^p}) \to C_c^\infty({}_n\widehat{\Sigma^{p-1}})$$

$$\partial\varphi(g_0, ..., g_n, \eta) = \sum_{\eta \in \partial\sigma} (-1)^{\langle \eta, \sigma \rangle} \varphi(g_0, ..., g_q, \sigma),$$

then we can form a bicomplex $(C_c^\infty(\widehat{{}_*\Sigma^*}), b, \pm\partial)$ analogous to the one in Fig. 3. Using the change of variables $\varphi \leftrightarrow \tilde{\varphi}$ given above, we obtain the analogues of Lemmas 4.3 and 4.4 (the proof of Shapiro's Lemma in differentiable group homology is given in [6]). This proves Theorem 4.2 in the totally disconnected case, and then Theorem 4.5 may be proved in the totally disconnected case using the same argument as in the discrete case.

5. AFFINE BUILDINGS

In this section we shall review the definition of an affine building, and establish a few simple facts about their isometries. A convenient reference on buildings is the book of K. Brown [8], which we shall follow closely.

An *affine Coxeter complex*, is the simplicial complex underlying a tessellation of Euclidean space associated to an affine Weyl group (the term "simplicial complex" refers in this paper to the geometric realization, not the abstract combinatorial object, but all maps between geometric realizations will be taken to be simplicial). Examples are the line, divided into a chain of intervals of unit length, and the plane, tessellated in the usual fashion by equilateral triangles. See Section 2 in Chapter 6 of [8] for a precise definition. Each affine Coxeter complex possesses a canonical metric (i.e. distance function), invariant under simplicial automorphisms, for which the diameter of each chamber (a simplex of maximal dimension) is 1.

(5.1) Definition. Let Σ be a simplicial complex. A *system of apartments* for Σ is collection of subcomplexes, called apartments, with the following properties.

B1. Each apartment is isomorphic to an affine Coxeter complex

B2. Any two simplices of Σ are contained in some common apartment

B3. If two apartments both contain two given simplices of Σ then there is an isomorphism of one apartment onto the other which fixes the two simplices.

An *affine building* is a simplicial complex for which there exists a system of apartments.

It turns out that the union of all systems of apartments for an affine building Σ is itself a system of apartments (see Section IV.4 in [8]); so Σ has a canonical, maximal system of apartments.

According to Axiom B2, any two points in an affine building Σ are contained within some single apartment A, and if we define the distance between them as the distance inside A, we obtain a well defined metric on Σ (see Section VI.3 of [8]). It has the following properties.

M1. If x_0 and x_1 are any two points in Σ then for all $0 \leqslant t \leqslant 1$ there is a unique point $x_t \in \Sigma$ such that

$$d(x_0, x_t) = td(x_0, x_1) \qquad \text{and} \qquad d(x_t, x_1) = (1-t)\, d(x_0, x_1).$$

M2. If $x_0, x_1, y \in \Sigma$ then

$$d(y, x_t)^2 \leqslant td(y, x_0)^2 + (1-t)\, d(y, x_1)^2 - t(1-t)\, d(x_0, x_1)^2,$$

where x_t is as in M1.

Let us refer to the path $\{x_t\}_{t \in [0,1]}$ in M1 as a *segment*, and denote it by $[x_0, x_1]$.

The following proposition, which is valid for any metric space satisfying the above two axioms, summarizes much of the geometry of Σ that we shall need. It is part of a more general theorem of Reshetnyak [15]. For a proof of this special case see Theorem 2.1.2 in [11].

(5.2) PROPOSITION. *Let $[x_0, x_1]$ and $[y_0, y_1]$ be segments in Σ. There are segments $[\tilde{x}_0, \tilde{x}_1]$ and $[\tilde{y}_0, \tilde{y}_1]$ in the Euclidean plane such that*

$$d(x_0, x_1) = d(\tilde{x}_0, \tilde{x}_1), \qquad d(y_0, y_1) = d(\tilde{y}_0, \tilde{y}_1),$$
$$d(y_0, x_0) = d(\tilde{y}_0, \tilde{x}_0), \qquad d(y_1, x_1) = d(\tilde{y}_1, \tilde{x}_1)$$

and

$$d(x_t, y_s) \leqslant d(\tilde{x}_t, \tilde{y}_s) \qquad \textit{for all} \quad 0 \leqslant t, s \leqslant 1.$$

(5.3) COROLLARY. *If $[x_0, x_1]$ and $[y_0, y_1]$ are segments in Σ then $d(x_t, y_t)$ is a convex function of t.*

Proof. By Proposition 5.2 this follows from the corresponding fact about segments in the place. ∎

(5.4) COROLLARY (Pythagoras' Theorem). *Let $[x_0, x_1]$ and $[y_0, y_1]$ be segments in Σ such that* (i) *$d(x_t, y_t)$ is constant*; *and* (ii) *for each t, x_t is the nearest point in $[x_0, x_1]$ to y_t. Then*

$$d(x_0, y_1)^2 = d(x_0, x_1)^2 + d(x_1, y_1)^2.$$

Proof. Let $[\tilde{x}_0, \tilde{x}_1]$ and $[\tilde{y}_0, \tilde{y}_1]$ be corresponding segments in the plane, as in Proposition 5.2. Our hypotheses imply that $d(\tilde{x}_t, \tilde{y}_t)$ is constant and that for each t the point $\tilde{x}_t$ is the nearest point in $[\tilde{x}_0, \tilde{x}_1]$ to $\tilde{y}_t$. So $[\tilde{x}_0, \tilde{x}_1]$ and $[\tilde{y}_0, \tilde{y}_1]$ are opposite sides of a rectangle, and hence

$$\begin{aligned} d(x_0, y_1)^2 &\leqslant d(\tilde{x}_0, \tilde{y}_1)^2 \\ &= d(\tilde{x}_0, \tilde{x}_1)^S + d(\tilde{x}_1, \tilde{y}_1)^2 \\ &= d(x_0, x_1)^2 + d(x_1, y_1)^2. \end{aligned}$$

To get the reverse inequality apply Proposition 5.2 to $[x_0, x_1]$ and the degenerate segment $[y_1, y_1]$. Our hypotheses imply that the Euclidean triangle $\tilde{x}_0\tilde{x}_1\tilde{y}_1$ has an obtuse angle at $\tilde{x}_1$, which gives

$$\begin{aligned} d(x_0, y_1)^2 &= d(\tilde{x}_0, \tilde{y}_1)^2 \\ &\geqslant d(\tilde{x}_0, \tilde{x}_1)^2 + d(\tilde{x}_1, \tilde{y}_1)^2 \\ &= d(x_0, x_1)^2 + d(x_1, y_1)^2. \quad \blacksquare \end{aligned}$$

We turn now to isometries.

(5.5) DEFINITION. Let f be an automorphism of Σ (it is automatically isometric). We say that f is *elliptic* if it has a fixed point in Σ. It is *hyperbolic* if it has no fixed point but there is an $x_0 \in \Sigma$ such that

$$d(x_0, fx_0) = \min\{d(x, fx) : x \in \Sigma\}.$$

We have the following important result.

(5.6) THEOREM. *Every automorphism of an affine building is either elliptic or hyperbolic.*

The theorem is proved by reducing to the study of isometries of apartments, where we have the following result.

(5.7) LEMMA. *Let $A_1, A_2, \ldots$ be affine Coxeter complexes, all isomorphic to some single complex A. Let $h_n \colon A_n \to A_n$ be isomorphisms and let $x_n \in A_n$ be such that $\lim_{n \to \infty} d(x_n, h_n x_n)$ exists, say*

$$\lim_{n \to \infty} d(x_n, h_n x_n) = a.$$

Then for some n there is a point $x_0 \in A_n$, lying in the smallest closed simplex of A_n containing x_n, such that

$$d(x_0, h_n x_0) = a.$$

Proof. Fix isomorphisms $A_n \cong A$, which map the points x_n to points $\bar{x}_n$ all lying within a single closed chamber C in A. Using these isomorphisms, identify h_n with an isometry $\bar{h}_n$ of A. The $\bar{h}_n$ map C a uniformly bounded distance away from itself, and since there are only finitely many automorphisms of A with this property, by passing to a subsequence, we get

$$\bar{h}_{n_1} = \bar{h}_{n_2} = \cdots.$$

In addition, since all the $\bar{x}_n$ lie within a compact set, by passing to a further subsequence if necessary, we may assume that all the $\bar{x}_{n_j}$ have the same smallest closed simplex and that $\bar{x}_{n_j}$ converges to some $\bar{x}_0$. Take x_0 to be the inverse image of $\bar{x}_0$ under the isomorphism $A_{n_1} \cong A$. ▌

Proof of Theorem 5.6. It suffices to show that there is some $x_0 \in \Sigma$ such that

$$d(x_0, fx_0) = \min\{d(x, fx) : x \in \Sigma\}.$$

Let $x_1, x_2, \ldots$ be a sequence in Σ such that

$$\lim_{n \to \infty} d(x_n, fx_n) = a = \inf\{d(x, fx) : x \in \Sigma\}.$$

For each n let A_n be an apartment in Σ containing both x_n and fx_n (it exists by axiom B2 for a building) and identify A_n and fA_n by an isomorphism $\iota_n : fA_n \to A_n$ which fixes the smallest closed simplex containing fx_n. Let

$$h_n = \iota_n \circ f : A_n \to A_n.$$

Then $h_n x_n = fx_n$ and so $\lim_{n \to \infty} d(x_n, h_n x_n) = a$. Hence by Lemma 5.7 there is a point y in some A_n such that $d(y, h_n y) = a$ and such that y lies in the smallest closed simplex containing x_n. It follows that $h_n y = fy$, and so $d(y, fy) = a$. ▌

(5.8) Definition. Let f be an automorphism of an affine building Σ. We define

$$d_f = \min\{d(x, fx) : x \in \Sigma\},$$

and

$$MIN(f) = \{x \in \Sigma \mid d(x, fx) = d_f\}$$

(if f is elliptic then this is of course the fixed point set of f).

Now let G be a totally disconnected group acting properly (by simplicial automorphisms) on a locally finite affine building Σ.

(5.9) LEMMA. *The function $g \mapsto d_g$ in Definition 5.5 is locally constant.*

Proof. Fix $g \in G$ and let $x_0 \in MIN(g)$. Let V be a bounded open neighborhood of $[x_0, gx_0]$ in Σ, and let U be the set of all $g' \in G$ whose action on V agrees with that of g. The local finiteness of Σ implies that U is open. We shall show that if $g' \in U$ then $x_0 \in MIN(g')$, which is enough to prove the lemma. Suppose, for the sake of a contradiction, that $g' \in U$ and $d(x_1, g'x_1) < d(x_0, g'x_0)$, for some $x_1 \in \Sigma$, and denote the points on the segment $[x_0, x_1]$ by x_t. Then it follows from Corollary 5.3, applied to the segments $[x_0, x_1]$ and $[g'x_0, g'x_1]$, that $d(x_t, g'x_t) < d(x_0, g'x_0)$ for all $t > 0$. But for sufficiently small $t > 0$ we have $g'x_t = gx_t$, by definition of the open set U, so that $d(x_t, gx_t) < d(x_0, gx_0)$ for small enough t. Contradiction. ∎

Now let us write

$$G_{\text{ell}} = \{\text{elliptic elements in } G\},$$

and

$$G_{\text{hyp}} = \{\text{hyperbolic elements in } G\}.$$

As a consequence of the above results we have:

(5.10) THEOREM. *The sets G_{ell} and G_{hyp} are both open and closed.*

We remark that G_{ell} is easily characterized as the set of elements of G which are contained in some compact open subgroup. Indeed if $g \in G$ is elliptic then it is contained in the isotropy group of some point. By properness of the action of G, this isotropy group is compact and open. On the other hand, if K is a compact open subgroup then a well known fixed point theorem (see Section 4.4 of [8]) asserts that K has a fixed point in Σ.

6. THE HYPERBOLIC PART OF G

Let G be a totally disconnected group which acts properly on a locally finite affine building.

(6.1) DEFINITION. Denote by $HC_*(C_c^\infty(G))_{\text{hyp}}$ the cyclic homology of $C_c(G)$, localized on the set of elliptic elements (in the sense of Example 2.2).

In view of Theorem 5.10 there is a direct sum decomposition

$$HC_*(C_c^\infty(G)) = HC_*(C_c^\infty(G))_{\text{ell}} \oplus HC_*(C_c^\infty(G))_{\text{hyp}}.$$

Here is the main result of this section.

(6.2) THEOREM. *Let G be a totally disconnected group acting properly on an affine building. The periodicity map*

$$S\colon HC_{n+2}(C_c^\infty(G))_{\text{hyp}} \to HC_n(C_c^\infty(G))_{\text{hyp}}$$

in the Connes–Tsygan sequence is zero.

Our proof makes use of the following notion.

(6.3) DEFINITION. (See [14].) A *quasicyclic object* in the category of vector spaces is a family spaces $\{X_n\}_{n\geqslant 0}$, together with morphisms

$$d_i\colon X_n \to X_{n-1} \quad (i=0, ..., n) \qquad \text{and} \qquad T\colon X_n \to X_n$$

satisfying the relations C1 and C2 given in Section 2, but not necessarily C3.

We observe that if X is a quasicyclic object then C1 implies

$$d_j T^{n+1} = T^n d_j\colon X_n \to X_{n-1}.$$

Because of this, the quotient

$$X_T = (\{X_n/(1-T^{n+1})\, X_n\}_{n\geqslant 0}, d, T)$$

is, in a natural way, a precyclic object. The following result is proved in [14].

(6.4) PROPOSITION. *Let X be a quasicyclic object such that $1-T^{n+1}$: $X_n \to X_n$ is injective. Then $S=0$ on $HC_*(X_T)$.*

In the remainder of this section we shall use the geometry of affine buildings to construct a quasicyclic object X so that X_T identifies with $C_c^\infty(G)^\natural_{\text{hyp}}$.

(6.5) DEFINITION. Let Σ be an affine building. A doubly infinite path $c(t)$ in Σ is called a *geodesic* if

$$d(c(t_1), c(t_2)) = t_1 - t_2 \qquad \text{for all} \quad t_1 \geqslant t_2 \in \mathbb{R}$$

(in other words each finite part of $c(t)$ is a segment). Two geodesics $c_1(t)$ and $c_2(t)$ are *parallel* if

$$\sup\{d(c_1(t), c_2(t))\colon t \in \mathbb{R}\} < \infty.$$

Two parallel geodesics are *synchronous* if

$$d(c_1(t), c_2(t)) = \min\{d(c_1(t), c_2(s)) : s \in \mathbb{R}\},$$

for every t.

(6.6) LEMMA. (i) *If $c_1(t)$ and $c_2(t)$ are parallel then $d(c_1(t), c_2(t))$ is constant.*

(ii) *If $c_1(t)$ and $c_2(t)$ are synchronous then*

$$\begin{aligned} d(c_1(t), c_2(t)) &= \min\{d(c_1(t), c_2(s)) : s \in \mathbb{R}\} \\ &= \min\{d(c_1(s), c_2(t)) : s \in \mathbb{R}\} \end{aligned}$$

for all t.

(iii) *If any two geodesic are parallel then either one of them may be reparametrized in a unique way so as to be synchronous with the other.*

Proof. Part (i) follows from Corollary 5.3 and the fact that a bounded convex function on $\mathbb{R}$ is constant. Parts (ii) and (iii) follow easily from part (i). ∎

(6.7) LEMMA. *Let $c_1(t)$, $c_2(t)$, and $c_3(t)$ be the three parallel geodesics. If $c_1(t)$ and $c_2(t)$ are synchronous, and $c_2(t)$ and $c_3(t)$ are synchronous, then $c_1(t)$ and $c_3(t)$ are synchronous.*

Proof. Let $c_3(t_0)$ be the point on c_3 closest to $c_1(0)$. Then $c_1(t)$ and $c_3(t+t_0)$ are synchronous, and we must show that $t_0 = 0$. It follows from Pythagoras' Theorem (5.4) that

$$\begin{aligned} d(c_3(T), c_1(0))^2 &= d(c_3(T), c_3(t_0))^2 + d(c_3(t_0), c_1(0))^2 \\ &= (T - t_0)^2 + d(c_3(t_0), c_1(0))^2, \end{aligned}$$

so that

$$\lim_{T \to \infty} (T - t_0) - d(c_3(T), c_1(0)) = 0.$$

Two similar applications of Pythagoras' Theorem give

$$\lim_{T \to \infty} T/2 - d(c_3(T), c_2(T/2)) = 0$$

and

$$\lim_{T \to \infty} T/2 - d(c_2(T/2), c_1(0)) = 0.$$

Combining these three limits using the triangle inequality gives $T - t_0 \leqslant T$, and hence $t_0 \geqslant 0$. Repeating the argument, but taking limits now as $T \to -\infty$, gives the reverse inequality. ▮

(6.8) DEFINITION. If f is an isometry of Σ and $c(t)$ is a geodesic in Σ then we shall say that $c(t)$ is *translated* by f if $f(c(t)) = c(t + d_f)$.

It is not hard to show that if $f(c(t)) = c(t+d)$, for some $d \geqslant 0$ then $d = d_f$.

Observe that for f to translate $c(t)$ we require that f shift $c(t)$ by a "positive amount."

(6.9) EXAMPLE. If Σ is a tree and $f\colon \Sigma \to \Sigma$ is hyperbolic then $MIN(f)$ is a line in Σ. This line (appropriately oriented) is the unique geodesic translated into itself by f.

(6.10) LEMMA. *If f is a hyperbolic automorphism of Σ and $x \in MIN(f)$ then there is a unique geodesic $c(t)$ such that $c(0) = x$ and such that $c(t)$ is translated by f.*

Proof. (Compare part (1) of Lemma 6.5 in [1].) It is easily verified that if $c(t)$ is path such that: (i) for all n the finite path $\{c(t): n \leqslant t \leqslant n+1\}$ is a segment; and (ii) for all n the point $c(n+1)$ is the midpoint of the segment $[c(n), c(n+2)]$; then $c(t)$ is a geodesic. So it suffices to prove that for every $x \in MIN(f)$ the point $f(x)$ is the midpoint of the segment $[x, f^2x]$, for then given $x \in MIN(f)$ the piecewise linear path $c(t)$ with $c(n) = f^n x$ for all $n \in \mathbb{Z}$ will be a geodesic translated by f.

Now the point fx is the midpoint of $[x, f^2x]$ if and only if

$$d(x, f^2x) = d(x, fx) + d(fx, f^2x) = 2d_f.$$

Let y be the point midway between x and fx. Applying Corollary 5.3 to the segments $[x, fx]$ and $[f^2x, fx]$ we see that if $d(x, f^2x) < 2d_f$ then $d(y, fy) < d_f$. But the inequality $d(y, fy) < d_f$ contradicts the definition of d_f as a minimum. ▮

(6.11) DEFINITION. Let g be a hyperbolic element of G. If $x_1, x_2 \in MIN(g)$ then let us write

$$x_1 \sim x_2 \iff \begin{array}{l}\text{there are synchronous geodesics } c_1(t) \text{ and} \\ c_2(t), \text{ both translated by } g, \text{ such that} \\ x_1 = c_1(0) \text{ and } x_2 = c_2(0).\end{array}$$

Observe that by Lemma 6.7 this is an equivalence relation on $MIN(g)$. We define E_g to be the set of equivalence classes:

$$E_g = MIN(g)/\sim.$$

We shall denote the equivalence class of $x \in MIN(g)$ by $[x] \in E_g$.

We shall now glue the lines E_g together so as to obtain a principal $\mathbb{R}$-bundle

$$E = \bigcup_g E_g$$

over the space G_{hyp}. Given $x \in X$ let

$$U_x = \{g \in G_{\text{hyp}} \mid x \in MIN(g)\}.$$

By Lemma 5.9 this is an open set. Of course it might be empty, but the family of all the U_x covers G_{hyp}. If $g \in U_x$ then identify E_g with $\mathbb{R}$ by mapping $[c_{g,x}(t)]$ to t/d_g, where $c_{g,x}(t)$ is the geodesic translated by g for which $c_{g,x}(0) = x$. In this way we identify $\bigcup_{g \in U_x} E_g$ with $U_x \times \mathbb{R}$.

(6.12) Lemma. *If $x, y \in \Sigma$ then the transition function*

$$(U_x \cap U_y) \times \mathbb{R} \to (U_x \cap U_y) \times \mathbb{R}$$

is given by a map $(g, t) \mapsto (g, t + t_g)$, where t_g is a locally constant function of g.

Proof. First note that if $g \in U_x \cap U_y$ then the transition function on $\{g\} \times \mathbb{R}$ is given by $(g, t) \mapsto (g, t + t_{g,x,y}/d_g)$, where $c_{g,x}(t_{g,x,y})$ is the nearest point on the geodesic $c_{g,x}(t)$ to y. In view of Lemma 5.9 it suffices to show that $t_{g,x,y}$ is a locally constant function of g. So fix $g \in U_x \cap U_y$ and choose $N > 1$ large enough so that the point $c_{g,x}(t_{g,x,y})$ lies on the segment in Σ between $g^{-N}x$ and $g^N x$. Let $U \subset U_x \cap U_y$ be the open neighborhood comprised of all g' whose action on the segment $[g^{-2N}x, g^{2N}x]$ is the same as the action of g. Then the geodesics $c_{g,x}(t)$ and $c_{g',x}(t)$ both contain the segment $[g^{-2N}x, g^{2N}x]$. Since $d(c_{g',x}(t), y)$ has a local minimum in the interior of this segment (namely at $c_{g,x}(t_{g,x,y})$), it follows from Corollary 5.3 that this is a global minimum, so that $c_{g',x}(t_{g,x,y})$ is the closest point on $c_{g',x}(t)$ to y, and hence $t_{g,x,y} = t_{g',x,y}$. ∎

We shall write the $\mathbb{R}$-action on E as

$$s: e \mapsto e + s \qquad (e \in E).$$

If $g \in G_{\mathrm{hyp}}$ and $c(t)$ is a geodesic translated by g then $[c(t)] + s = c(t + d_g s)$ for all t and s. Note that

$$[c(t)] + 1 = [c(t + d_g)] = [gc(t)].$$

So if we define an action of G on E by the formula

$$g \cdot [x] = [gx],$$

then the G-action and $\mathbb{R}$-action on E commute with one another, and

$$g[x] = [x] + 1, \qquad \text{if} \quad [x] \in E_g.$$

(6.13) Definition. Denote by E_{n+1} the pull-back of the bundle E to the set $G_{\mathrm{hyp}}^{n+1} = \{(g_0, g_1, ..., g_n) \mid g_0 g_1 \cdots g_n \in G_{\mathrm{hyp}}\}$ along the multiplication map

$$(g_0, g_1, ..., g_n) \mapsto g_0 g_1 \cdots g_n.$$

So a typical point of E_{n+1} is an $(n+2)$-tuple

$$(g_0, g_1, ..., g_n, [x]), \qquad \text{where} \quad [x] \in E_{g_0 g_1 \cdots g_n}.$$

(6.14) Definition. A compactly supported, complex-valued function φ on E_{n+1} is *smooth* of for each point $(g_0, g_1, ..., g_n, [x]) \in E_{n+1}$, there is a coordinate neighborhood $V \cong U \times \mathbb{R}$ such that $\varphi((g_0, ..., g_n), t) = \psi_g(t)$, where $g \mapsto \psi_g$ is a locally constant function from U to the smooth, compactly supported functions on $\mathbb{R}$.

We shall not really use smoothness in the $\mathbb{R}$ direction—what is important is that smooth functions on E are locally constant in the base direction.

Form the space $C_c^\infty(E_{n+1})$ of smooth, compactly supported functions on E_{n+1} and define operators

$$d_i\colon \tilde{X}_n \to \tilde{X}_{n-1} \qquad (i = 0, ..., n)$$

$$T\colon \tilde{X}_n \to \tilde{X}_n$$

by the formulas

$$d_i \varphi(g_0, ..., g_{n-1}, [x]) = \int_G \varphi(g_0, ..., g_{i-1}, h, h^{-1} g_i, ..., g_{n-1}, [x])\, dh$$

for $i=0, \ldots, n-1$,

$$d_n\varphi(g_0, \ldots, g_{n-1}, [x]) = \int_G \varphi(h^{-1}g_0, g_1, \ldots, g_{n-1}, h, [h^{-1}x])\, dx,$$

and

$$T\varphi(g_0, \ldots, g_n, [x]) = \varphi(g_1, \ldots, g_n, g_0, [g_0^{-1}x]).$$

(6.15) LEMMA. *With these operators we obtain a quasicyclic object*

$$C_c^\infty(E)^\natural = (\{C_n^\infty(E_{n+1})\}_{n\geqslant 0}, d, T).$$

The operator $T^{n+1}: X_n \to X_n$ *is given by the formula*

$$T^{n+1}\varphi(g_0, \ldots, g_n, [x]) = \varphi(g_0, \ldots, g_n, [(g_0, \cdots g_n)^{-1}x])$$

and is injective.

Proof. It is a simple calculation to show that $C_c^\infty(E)^\natural$ is a quasicyclic object, and to verify the formula for T^{n+1}. It follows from this formula that $1-T^{n+1}: \tilde{X}_n \to \tilde{X}_n$ is injective. Indeed the map

$$(g_0, \ldots, g_n, [x]) \mapsto (g_0, \ldots, g_n, [(g_0 \cdots g_n)^{-1}x])$$

is given by translation by $-1 \in \mathbb{R}$ and so leaves no compact set in the principal $\mathbb{R}$-bundle E_{n+1} invariant. Hence it leaves no compactly supported function invariant. ▮

(6.16) DEFINITION. Denote by $D_c^\infty(E)^\natural$ the subobject of $C_c^\infty(E)^\natural$ comprised of functions $\varphi \in C_c^\infty(E_{n+1})$ such that the function $\tilde{\varphi}(e) = \sum_{k\in\mathbb{Z}} \varphi(e+k)$ is constant on each fibre of E_{n+1}.

Note that the infinite sum make sense, thanks to the fact that φ is compactly supported and E_{n+1} is a principal $\mathbb{R}$-space.

(6.17) LEMMA. *The precyclic object* $D_c^\infty(E)_T^\natural$, *obtained from the quasicyclic object* $D_c^\infty(E)^\natural$ *by dividing by the action of* $1-T^{n+1}$, *is isomorphic to the localized precyclic object* $C_c^\infty(G)_{G_{\mathrm{hyp}}}^\natural$.

Proof. The map $\varphi \mapsto \tilde{\varphi}$, where $\tilde{\varphi}$ is as in Definition 6.16, induces an isomorphism from $D_c^\infty(E)_T^\natural$ to the functions on E_{n+1} which are constant in the $\mathbb{R}$-direction and locally constant in the base, G_{hyp}^{n+1}. Obviously this space identifies with $C_c^\infty(G_{\mathrm{hyp}}^{n+1})$, and this identification gives the desired isomorphism of quasicyclic objects. ▮

7. PERIODIC CYCLIC HOMOLOGY

The theorem on periodic cyclic homology stated in the introduction follows easily from Theorems 4.5 and 6.2. Indeed for any precyclic object X there is a $\lim^1$ sequence relating HC_* to PHC_*,

$$0 \to \lim_r{}^1 HC_{j+2r+1}(X) \to PHC_j(X) \to \lim_r HC_{n+2r}(X) \to 0 \qquad (j=0, 1),$$

where the limits are over the inverse system

$$HC_j(X) \xleftarrow{S} HC_{j+2}(X) \xleftarrow{S} HC_{j+4}(X) \xleftarrow{S} \cdots$$

(compare, for example, Chapter 5 of [12]). If we decompose $C_c^\infty(G)^\natural$ into its elliptic and hyperbolic parts, then for the elliptic part the maps S are eventually isomorphisms (by a slight extension of the argument in the proof of Theorem 4.5), while for the hyperbolic part we have just shown that $S=0$.

Note added in proof. After we submitted this paper we learned that P. Schneider has obtained a very similar result.

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