C^*-ALGEBRAS AND CONTROLLED TOPOLOGY

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1. Introduction

This paper is an attempt to explain some aspects of the relationship between the K-theory of C^*-algebras, on the one hand, and the categories of modules that have been developed to systematize the algebraic aspects of controlled topology, on the other. It has recently become apparent that there is a substantial conceptual overlap between the two theories, and this allows both the recognition of common techniques, and the possibility of new methods in one theory suggested by those of the other. In this first part we will concentrate on defining the C^*-algebras associated to various kinds of controlled structure and giving methods whereby their K-theory groups may be calculated in a number of cases.

From a 'revisionist' perspective[1], this study originates from an attempt to relate two approaches to the Novikov conjecture. The Novikov conjecture states that a certain assembly map is injective. The process of assembly can be thought of as the formation of a 'generalized signature' [38], and therefore to understand the connection between different approaches to the conjecture is to understand the connection between different definitions of the 'generalized signature'. Now, broadly speaking, there are two approaches to the Novikov conjecture in the literature. One approach considers the original assembly map of Wall in L-theory, and attempts to prove it to be injective by investigating homological properties of the L-theory groups (which properties themselves may be derived algebraically, or geometrically, by relating them to surgery problems). The other approach proceeds via analysis, considering the assembly map to be the formation of a generalized index of the Atiyah-Singer signature operator [3]. This approach ultimately leads to the consideration of assembly on the K-theory of C^*-algebras. Nevertheless it can be shown that the injectivity of assembly on the C^*-algebra level implies (modulo 2-torsion) the injectivity of assembly on the L-theory level. All this is explained in the paper of Rosenberg [37], to which we also refer for an extensive bibliography on the Novikov conjecture.

To proceed with the background to this paper. In the late seventies and eighties it occurred both to the topologists and independently to the analysts that a more flexible generalized signature theory might be developed if one neglected the group structure of the fundamental group π in question and considered only its "large scale" or "coarse" structure induced by some translation invariant metric; up to "large scale equivalence" the choice of such a metric is irrelevant. Some references are [26, 29, 11, 12, 14, 16, 9, 33, 34, 35, 36, 41]. Since the two theories were based on the same idea, it was inevitable that they would eventually come into interaction, but this did not happen for some while. It was the insight of Shmuel Weinberger, and especially his note [40] relating the index theory of [36] to Novikov's theorem on the topological invariance of the rational Pontrjagin classes[2], which provoked the discussions among the present authors which eventually led to the writing of this paper.

This paper is intended to be foundational, setting down some of the language in which one can talk about the relationship between analysis and controlled topology. Many of its ideas have already been worked out in the special case of bounded control in the work of the first and third authors and G. Yu [18, 20, 19]. But it seems that there is much to be gained by considering more general kinds of control, more general coefficients, "spacification" of the theory and so on, and it

[1] i.e., not what did happen but what ought to have happened.

[2] See [27] for the latest details on this argument.

has therefore become necessary to rework some of the foundations. There are a few new results here too, which will be noted later in this introduction.

Here now is an outline of the paper. Section 2 develops a general theory of *bornological spaces* (also known as *coarse spaces*) and coarse maps. This provides a framework in which one can discuss various kinds of control on a uniform footing: in particular, a metric on a space X defines one kind of coarse structure (the bounded coarse structure) on it, a compactification defines a different kind (the continuously controlled coarse structure), and there are the expected relations between them. This theory is certainly known in some form to a number of people (Carlsson in particular), and is in any event fairly natural, but we have not seen it written down anywhere. There is also a generalization to the notion of *bornological groupoid*, which is a natural framework in which to discuss various more general kinds of controlled problems, but we will not discuss this in detail here.

In Section 3, following the lead of a number of authors, we introduce the controlled module categories associated to a bornological space. This material is standard by now, and the only innovation is the introduction of certain "bounded operator" subcategories which will be better related to the C^*-algebra theory. This technique was introduced in the paper [27] on the coarse analytic signature; it seems to be a necessary intermediate in any attempt to set up an actual *mapping* from the algebraic theory to the C^*-theory.

Section 4 contains some notations and basic facts about C^*-algebras. This material will all be familiar to the C^*-algebra specialist. In section 5 we introduce the C^*-algebras associated to a bornological space, and discuss their functorial properties. When the bornological structure arises from a metric, these are the same as the C^*-algebras discussed in [36]. In general the definition of $C^*(X)$ depends on the choice of an "X-module", that is a Hilbert space over X. Two different choices give rise to K-theory groups that can be canonically identified. This slight lack of functoriality is treated in section 6 where an algebraic topological, categorical approach to these K-theory spectra is given.

In section 7 we discuss assembly. The C^*-algebras $C^*(X)$ were introduced by the third author in order to be able to do index theory on open manifolds; the assembly map is a systematization of the underlying idea here. It is a map from the locally finite K-homology of X (for which we use Kasparov's analytical model) to the K-theory of the algebra $C^*(X)$. The treatment of assembly here is based on an idea of the first author, expounded in [19] in the case of bounded control.

Section 8 discusses the special case of continuous control at infinity. It was observed by the four authors of [2] that this kind of control has especially good properties; we will reinforce this point by identifying the C^*-algebra of a continuously controlled metrizable pair with the *relative Paschke dual algebra* associated to that pair. This means that the K-theory of the continuously controlled algebra is identified with the Baum-Douglas relative K-homology of the total space relative to the interior, and therefore by excision results of Baum-Douglas and the first author with the K-homology (with dimension shifted by one) of the boundary space. This is entirely analogous to the results of [2] and [5], but the strategy of proof is quite different; instead of identifying the K-theory group as being *a priori* a homological functor of the boundary, and then appealing to some version of the Eilenberg-Steenrod uniqueness theory, we make an identification on the level of analysis with a pre-existing analytic model for the K-homology of the boundary. This calculation has not previously appeared. Furthermore, it turns out that "good compactifications" of a general bornological space, as considered for example in [5, 36], can be defined very simply in terms of the existence of a coarse map to a continuously controlled coarse structure; we therefore obtain as a corollary the splitting of the assembly map for any space admitting a good contractible compactification. A large number of results on the Novikov conjecture are included under this scheme.

In the final sections we introduce homological techniques for computing $K_*(C^*(X))$. Section 9 deals with excision and Section 10 with contractibility, that is, with Eilenberg swindles. These techniques are well known to workers in controlled algebra, and the analytical devices needed to extend them to the C^* situation were introduced in [20]; so these sections reformulate known work. Section 11 studies the notion of coarse homotopy. This was introduced by the first and third authors in [18], where an alarmingly analytic proof was given of the coarse homotopy invariance of the functor $X \mapsto K_*(C^*(X))$, in the case of bounded control. Subsequently a rather more natural analytic proof was given by Yu [41]. The proof presented here uses only excision and contractibility, and is in some sense equivalent to Yu's proof. It is an analytic version of a topological argument devised by the second author and Steve Ferry. Lastly, in section 12 we show that the continuously controlled theory defines a Steenrod homology theory. Since we have already identified continuously controlled theory with Kasparov's K-homology, one could think of this as a geometric proof of the homology properties of Kasparov's theory.

2. Bornological spaces

Let G be a locally compact Hausdorff topological groupoid. The case that will be of greatest interest to us in this paper is $G = X \times X$, where X is a topological space, and the groupoid operation is the trivial one $(x, y) \cdot (y, z) = (x, z)$, but we believe that other examples are likely to be significant, so we will give the general definition here. Our notation for groupoids is that G is the set of morphisms in a small category with every morphism invertible, G^0 is the set of objects (which we may think of as the 'diagonal' of G, that is, the set of identity morphisms), and $s, t\colon G \to G^0$ are the source and target maps.

(2.1) Definition: *A* bornology *or* coarse structure *on G is a collection of open subsets E of the set of arrows of G, called* entourages*, that have the following properties:*

- *The inverse of any entourage is contained in an entourage;*
- *The (groupoid) product of two entourages is contained in an entourage;*
- *The union of two entourages is contained in an entourage;*
- *Entourages are proper: that is, for any entourage E and any compact subset C of the objects G^0 of G, $E \cap s^{-1}(C)$ and $E \cap t^{-1}(C)$ are relatively compact;*
- *The union of all the entourages is G.*

If there is an entourage that contains the diagonal G^0, we will say that the coarse structure is unital.

We will say that two coarse structures (on the same groupoid) are *equivalent* if every entourage for one is contained in an entourage for the other and *vice versa*. We will not distinguish between equivalent coarse structures. For example, it is easy to see that if G^0 is compact then there is only one coarse structure up to equivalence, namely that consisting of all relatively compact open subsets of G. In particular this is true when G is a group, if regarded as a groupoid with one object. More interesting examples arise from groupoids whose set of objects is non-compact.

By a *coarse structure on a space* X, we will mean a coarse structure on the groupoid $X \times X$. If X is a space with a coarse structure and E is an entourage, we define, for any subset $S \subseteq X$, the *E-neighbourhood* $N_E(S)$ to be $\{x \in X : \exists s \in S,\ (x, s) \in E\}$. If $p \in X$, we write $N_E(p)$ for $N_E(\{p\})$. The E-neighbourhoods of different points of X should all be regarded as being about the same size; the coarse structure may be thought of as a means to compare the size of subsets located in different regions of X.

(2.2) Definition: *Let G and G' be bornological groupoids. A* coarse map *from G to G' is a Borel groupoid homomorphism $f\colon G \to G'$ such that*

- *f is proper, in the sense that for each compact subset C of G', the subset $f^{-1}(C)$ of G is relatively compact;*
- *For each entourage E of G, there is an entourage E' of G' such that $f(E) \subseteq E'$.*

We emphasize that coarse maps are *not* required to be continuous.

Let $\mathbf{2}$ denote the groupoid $\{0, 1\} \times \{0, 1\}$. If G is a bornological groupoid, then $G \times \mathbf{2}$ has a natural product bornology.

(2.3) Definition: *Two coarse maps $f_0, f_1\colon G \to G'$ are* coarsely equivalent *if they arise by restriction from a coarse map $G \times \mathbf{2} \to G'$.*

It is equivalent to say that there is an entourage $E' \in G'$ such that, for all $\gamma \in G$, $f_0(\gamma)E' \cap E' f_1(\gamma) \neq \emptyset$.

Remark: Suppose that X is a coarse space. The objects typically studied in controlled topology are "spaces over X", by which is meant spaces M equipped with a (not necessarily continuous) proper map, called the *control map*, $c\colon M \to X$; such a space may be denoted by $\begin{pmatrix} M \\ \downarrow \\ X \end{pmatrix}$. In this regard it is relevant to note that the control map c pulls back the coarse structure on X to a coarse structure on M, and, for this coarse structure, c becomes, tautologically, a coarse map.

Example: Suppose that X is a proper metric space. ("Proper" means that closed bounded sets are compact.) A coarse structure may be defined on X by declaring that the entourages are the R-neighbourhoods of the diagonal, for real $R > 0$. We will call this the *bounded coarse structure* coming from the metric.

Example: Suppose that $(\overline{X}, Y)$ is a compact pair and that $X = \overline{X} \setminus Y$; we assume also, as the notation suggests, that X is dense in $\overline{X}$. A coarse structure on X may be defined by declaring that an entourage is a proper neighbourhood E of the diagonal in $X \times X$ such that the closure $\overline{E}$ in $\overline{X} \times \overline{X}$ intersects $Y \times Y$ only in the diagonal[3].

We will call this the *continuously controlled coarse structure* arising from the given compactification of X.

Let us relate this to the notion of continuous control at infinity studied in [2]. The following lemma shows that (at least in the case of compact metrizable pairs) the definition of that paper is the same as ours.

(2.4) Lemma: *The set E is an entourage for the continuously controlled coarse structure defined above if and only if it has the following property: for each $y \in Y$ and each open set $U \subseteq \overline{X}$ containing y, there is another open subset $V \subseteq \overline{X}$ containing y such that if $(x, x') \in E$ and one of x or x' belongs to V, the other must belong to U.*

Proof: Suppose that E satisfies this condition and that x_n is a sequence in X converging to $y \in Y$. If $(x_n, x'_n) \in E$ for all n, then given any open set $U \subseteq X$ containing y, there must be an N such that $x'_n \in U$ for all $n > N$. Thus $x'_n \to y$ also. It follows that the closure of E in $Y \times Y$ is simply the diagonal of $Y \times Y$. Conversely, suppose that the closure of E in $Y \times Y$ is the diagonal. Then, given $y \in Y$ and $U \subseteq X$ open with $y \in U$, let x belong to the (compact) set $X \setminus U$. Then (x, y) does not belong to the closure of E, and so there exist disjoint open sets U_x and V_x containing x and y respectively, such that $U_x \times V_x \cup V_x \times U_x$ does not meet the closure of E. Using compactness, take a finite cover of $X \setminus U$ by the sets U_x; the intersection of the corresponding sets V_x is the desired V. $\square$

In the application of continuous control at infinity it is often useful to know the following fact (the 'half control lemma'):

(2.5) Lemma: *In the above situation, if E satisfies the unsymmetrized control condition with the same statement as above except that we only require $(x, x') \in E, x \in V \Rightarrow x' \in U$, then E is in fact an entourage.*

For the proof, see Lemma 2.8 of [5].

In the examples above, the coarse maps may be simply characterized. In the case of maps from one boundedly controlled space to another, the coarse maps are the same as the ones defined in [36]. In the case of continuously controlled spaces, the coarse maps are the *ultimately continuous* maps: those which have continuous extensions to the given compactification Y.

Example: Generalizing the above, suppose that there is given some equivalence relation on the space Y at infinity. Then a coarse structure may be defined by requiring that the closures at infinity of the entourages are subsets of the graph of this equivalence relation. We will not study this kind of control in detail, but it includes, for example, the "control along a map" which played an important part in the paper [5].

[3] Because of the condition that the entourages are proper, this implies the apparently stronger fact that $\overline{E} \cap (\overline{X} \times Y \cup Y \times \overline{X})$ is the diagonal of $Y \times Y$.

All of the above were examples of unital coarse structures. Non-unital structures may arise from the following sort of construction. Let X be a space equipped with a proper action of some group Γ. Then a coarse structure on X may be defined by taking the entourages to be the sets

$$E = \bigcup_{\gamma\in\Gamma} \gamma C \times \gamma C$$

where C runs over the compact subsets of X. If the action is cocompact, this is a unital coarse structure, equivalent to that one would obtain as the bounded coarse structure for some Γ-invariant metric; but, in the non-cocompact case, this is an example of a non-unital structure.

As mentioned in the introduction, we will not develop our theory in detail for bornological groupoids in this paper. However, let us mention two topologically significant examples. The first is that of the fundamental groupoid of a space M, for simplicity let us say a Riemannian manifold, with the entourages defined by the homotopy classes which can be represented by paths of length $\leqslant R$. The second, somewhat related to the "tangent groupoid" of A. Connes [7], is obtained from a compact space X by forming $X \times X \times (0, 1]$ and declaring that the entourages are those proper subsets which, when closed off in the natural compactification, limit to the diagonal over the fibre at 0. This groupoid provides a way to discuss "asymptotic control" (that is, ε control for $\varepsilon \to 0$) within the framework outlined above. The resulting algebra should bear a similar relation to E-theory [8] to that which the continuously controlled algebra bears to Kasparov's model for K-homology.

3. Controlled algebra

Let X be a topological space and let R be a ring (usually, because of our analytical interests, we shall take R to be a C^*-algebra). Then the X-graded category of based free R-modules, $\mathfrak{G}(X; R)$), has as objects the (right) R-modules defined as locally finite countable direct sums of copies of R indexed by points of X, and as morphisms the *properly supported* (that is, row- and column-finite) matrices of X-module morphisms.

When R is a C^*-algebra we can define a subcategory which is better related to analysis. Notice in this case that every object of $\mathfrak{G}(X; R)$ is in a natural way a pre-Hilbert R-module[4], but that the morphisms need not be bounded relative to the norms induced by the pre-Hilbert structures. We let $\mathfrak{G}^{b.o.}(X; R)$ denote the subcategory whose morphisms are required to be bounded operators in this sense and to have bounded adjoints (compare [27]). It is a category with involution.

Of course, the category $\mathfrak{G}(X; R)$ does not contain much information about X. To encode more information, one imposes certain control restrictions. Assume that X is a bornological space in the sense of the preceding section. Then the *controlled category* $\mathfrak{C}(X; R)$ of R-modules over X has the same objects as before, but only those morphisms which are supported in some entourage (in the obvious sense). This is easily seen to be a category, with involution if R is a ring with involution; moreover, if R is a C^*-algebra, then the 'bounded operator' subcategory, $\mathfrak{C}^{b.o.}(X; R)$, is a category with involution too.

It has been shown by the work of numerous authors [2, 11, 28, 29, 32, 31] that the categories $\mathfrak{C}(X; R)$ for various kinds of control on X, and their K-theory and L-theory spaces, summarize important information relating to controlled topology over X. While we envisage that most readers will be acquainted with this theory, detailed knowledge of its results will not be assumed.

Remark: The above discussion should generalize to bornological groupoids. If G is a *discrete* bornological groupoid, we may define $\mathfrak{C}(G; R)$ as follows: the objects are locally finite direct sums of copies of R indexed by the points of G^0, and the morphisms are R-module morphisms $M \to N$ that can be represented as formal sums

$$\sum_{\gamma\in G} a_\gamma \gamma,$$

[4]This means that it has an R-valued inner product. For more detailed discussion of Hilbert modules, see the next section.

where $a_\gamma\colon M_s(\gamma) \to N_t(\gamma)$ is a morphism of (finite-dimensional free) R-modules and the support $\{\gamma : a_\gamma \neq 0\}$ is contained in an entourage; composition of such morphisms is by the natural convolution product.

Suppose that $f\colon X \to X'$ is a coarse map between coarse spaces (or, indeed, between bornological groupoids). It is routine to verify that it induces a functor $f_*\colon \mathfrak{C}(X;R) \to \mathfrak{C}(X';R)$, for any coefficients R. In detail, the image under f_* of an object $M = \oplus M_x$ of $\mathfrak{C}(X;R)$ is the object M' defined by $M'_{x'} = \oplus_{f(x)=x'} M_x$; because f is proper, this is indeed a locally finite direct sum of finite dimensional modules. Considered simply as an R-module, M' is trivially isomorphic to the original M; the effect of f_* on a morphism is simply to map it to itself under this trivial isomorphism; because f is coarse, this operation preserves the support condition on morphisms. Moreover, the procedure does not change the norm or the adjoint of a morphism, so f induces a functor on the bounded operator subcategories (when R is a C^*-algebra) as well.

4. C^*-algebras

In this section, we will review some basic facts about C^*-algebras. Let H be a Hilbert space (always assumed to be separable in this paper). The collection of all bounded linear operators on H will be denoted by $\mathfrak{B}(H)$; it is an algebra, and the collection $\mathfrak{K}(H)$ of all compact operators is a subalgebra. Both these algebras are involutive (the involution being defined by the Hilbert space adjoint) and are complete in the usual operator norm $\|T\| = \sup\{\|Tx\| : \|x\| \leqslant 1\}$.

(4.1) Definition: *A* C^*-algebra of operators on H *is an involutive, complete (or equivalently, norm-closed) subalgebra of* $\mathfrak{B}(H)$*. A* C^*-algebra *is a normed involutive algebra which is faithfully representable as a C^*-algebra of operators on some H.*

One usually gives an abstract definition of C^*-algebra, and then proves it equivalent to the more concrete one given above. We say that A is *unital* if there is an element $1 \in A$ which acts as a self-adjoint unit for the algebra structure. Such an algebra need not be *unitally represented* on H, that is $1 \in A$ may not be the identity operator on H, but it will certainly be a self-adjoint projection; so by splitting off the kernel of this projection we see that the general case is the direct sum of a unital representation and the zero representation.

Let R be a C^*-algebra. A *Hilbert R-module* [22] is a right R-module M which is equipped with an R-valued inner product $\langle \cdot,\cdot \rangle$ satisfying analogues of the usual inner product axioms, and such that M is complete in the norm

$$\|x\| = \|\langle x, x\rangle\|^{1/2}.$$

On the right of this equation, the norm is that of the C^*-algebra R. An example of a Hilbert R-module is R itself, with inner product $\langle x, y\rangle = xy^*$. The tensor product (suitably completed) of R with an infinite-dimensional separable Hilbert space defines the *universal Hilbert R-module* H_R; the term 'universal' is justified by the following important result, which is known as the *stabilization theorem* and is due to Kasparov [22]:

(4.2) Proposition: *If M is any countably generated Hilbert R-module, then $M \oplus H_R$ is isomorphic as a Hilbert module to H_R.*

A *Hilbert module operator* is a linear operator between two Hilbert modules that has an adjoint in the obvious sense. A standard application of the closed graph theorem proves that such operators are norm continuous, but it can be shown that not every norm continuous linear map between Hilbert modules is a Hilbert module operator. A Hilbert module operator is of *finite rank* if it is a linear combination of the 'rank one' operators $T_{y,z}\colon x \mapsto \langle x, y\rangle z$, and it is *compact* if it is a norm limit of finite rank operators. Generalizing our previous notation in the case of Hilbert spaces (= Hilbert $\mathbb{C}$-modules) we write $\mathfrak{B}(M)$ for the set of Hilbert module operators $M \to M$ and $\mathfrak{K}(M)$ for the set of compact operators. It can be shown that these are C^*-algebras. It can also be shown that $\mathfrak{K}(H_A)$ is isomorphic to the C^*-tensor product of A with the algebra of compact operators (in the ordinary sense) on a separable Hilbert space. Moreover, $\mathfrak{B}(M)$ can be identified with the multiplier algebra [30, Section 3.12] of $\mathfrak{K}(M)$.

Now we need to discuss the K-theory of C^*-algebras. Let A be a C^*-algebra, and consider the unitary group $U = U(A \otimes \mathfrak{K}^+)$ of the unitalization of the C^*-tensor product $A \otimes \mathfrak{K}$. We consider this as a topological space, with the norm topology. It can be shown that U is an infinite loop space; in fact, it is the first term of an Ω-spectrum that satisfies Bott periodicity (in the sense that $\Omega^2 U \simeq U$).

(4.3) Definition: *We denote the infinite loop space described above (or the associated spectrum) by* $\mathbb{K}^{\mathrm{top}}(A) = \Omega^{-1} U(A \otimes \mathfrak{K}^+)$.

We remark that a unitary in A (or in a matrix algebra over A) gives rise to an element of $\pi_1 \mathbb{K}^{\mathrm{top}}(A)$, and a self-adjoint projection in $e \in A$ gives an element of $\pi_0 \mathbb{K}^{\mathrm{top}}(A)$. These considerations, together with the fact (a consequence of the functional calculus) that in a C^*-algebra the set of invertibles (resp. idempotents) is homotopy equivalent to the set of unitaries (resp. self-adjoint projections) show that there is a map from the algebraic K-theory spectrum of A to our topological K-theory spectrum $\mathbb{K}^{\mathrm{top}}(A)$; this map is usually far from being an equivalence.

Topological K-theory is covariantly functorial for C^*-algebra homomorphisms. The homological properties of topological K-theory for C^*-algebras are summarized by the next proposition.

(4.4) Proposition:

- *Suppose that* $f_0, f_1 \colon A \to B$ *are homotopic* C^**-homomorphisms; then the induced maps* $f_{0*}, f_{1*} \colon \mathbb{K}^{\mathrm{top}}(A) \to \mathbb{K}^{\mathrm{top}}(B)$ *are homotopic too.*
- *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of* C^**-algebras; then the induced sequence*
$$\mathbb{K}^{\mathrm{top}}(A) \to \mathbb{K}^{\mathrm{top}}(B) \to \mathbb{K}^{\mathrm{top}}(C)$$
is a fibration of spectra.

Proofs of these facts may be found in [15], for example.

We will now give a brief discussion of Kasparov's KK-theory [21, 23, 24] from our present perspective. We will need to use a 'spacified' version of KK-theory which is not developed in detail in the literature. Our constructions here can be related to the usual definition of KK-theory by generalizing the methods of [17].

(4.5) Definition: *Let A and B be C^*-algebras. A* Hilbert (A, B)-bimodule M *is a Hilbert B-module equipped with a representation* $\rho \colon A \to \mathfrak{B}(M)$. *In the case* $B = \mathbb{C}$ *we will say that M is* sufficiently large *if the representation ρ is nondegenerate (i.e.* $\overline{\rho(A)M} = M$*) and has the property that for every* $a \in A$ *the operator* $\rho(a)$ *is not compact. In general we will say that M is* sufficiently large *if it is of the form* $M' \otimes B$*, where M' is a sufficiently large* $(A, \mathbb{C})$*-bimodule.*

Let M be a Hilbert (A, B)-bimodule. Define C^*-algebras $\Psi_0(M)$ and $\Psi_{-1}(M)$ of operators on M as follows: $\Psi_0(M)$ consists of all operators $T \in \mathfrak{B}(M)$ such that $T\rho(a) - \rho(a)T \in \mathfrak{K}(M)$ for all $a \in A$, and $\Psi_{-1}(M)$ consists of all operators $T \in \mathfrak{B}(M)$ such that $T\rho(a)$ and $\rho(a)T$ individually belong to $\mathfrak{K}(M)$. It is easy to see that $\Psi_0(M)$ is a C^*-algebra and that $\Psi_{-1}(M)$ is an ideal in it. We will refer to the elements of $\Psi_0(M)$ as *pseudolocal* operators on M, and to the elements of $\Psi_{-1}(M)$ as *locally compact.*

(4.6) Definition: *We define the KK-theory spectrum of A and B by*
$$\mathbb{KK}^{\mathrm{top}}(A, B) = \Omega^{-1} \mathbb{K}^{\mathrm{top}}(\Psi_0(M)/\Psi_{-1}(M)),$$
for any choice of sufficiently large bimodule M.

It is implicit in this definition that $\mathbb{K}^{\mathrm{top}}(\Psi_0(M)/\Psi_{-1}(M))$ is independent (up to canonical homotopy equivalence) of the choice of sufficiently large bimodule M. This may be proved as in [17].

Remark: The functoriality up to homotopy above is sufficient for some purposes, but for others one needs a more precise functoriality. Let us indicate how this may be achieved. For simplicity we consider the case in which B is fixed, and A runs over a category $\mathfrak{W}$ of C^*-algebras having at most countably many morphisms. To begin with, for each $A \in \operatorname{ob} \mathfrak{W}$ we choose a sufficiently large (A, B)-bimodule M'_A. We then define a larger (A, B)-bimodule M_A by

$$M_A = \bigoplus f_* M'_{A'}$$

where $f \in \operatorname{mor} \mathfrak{W}$ runs over all morphisms in $\mathfrak{W}$ from A to other objects A', and $f_* M'_{A'}$ means the Hilbert (A, B)-bimodule given by $\overline{f(A) M'_{A'}}$ with A-action defined via f. Notice that, corresponding to each $g \colon A \to A'$, there is now a canonical inclusion of $M_{A'}$ as a submodule of M_A, and this induces homomorphisms $\Psi_0(M_{A'}) \to \Psi_0(M_A)$, which we may use to define the functoriality of $\mathbb{KK}^{\mathrm{top}}$-theory. The upshot of this is that we have not merely made a choice of spectra $\mathbb{KK}^{\mathrm{top}}(A, B)$ for each $A \in \operatorname{ob} \mathfrak{W}$ which define a functor up to homotopy; we have chosen (once and for all) a functor $A \mapsto \mathbb{KK}^{\mathrm{top}}(A, B)$ from $\mathfrak{W}$ to spectra. Moreover, given any two such choices of functor there is a third choice (obtained from the direct sum of the corresponding bimodules) and natural transformations from each of the original functors to the third functor which induce homotopy equivalences on objects.

One can see that functoriality in this sense also holds for B, but we will not need this.

In our applications we will mainly be interested in the case $A = C_0(X)$, X a locally compact metrizable space. In this case we have the following identification of KK-theory:

(4.7) Proposition: *If $A = C_0(X)$, then one has*

$$\mathbb{KK}^{\mathrm{top}}(A, B) \simeq \mathbb{H}^{l.f.}(X; \mathbb{K}^{\mathrm{top}}(B)),$$

the locally finite (Steenrod) homology spectrum of X with coefficients in the spectrum $\mathbb{K}^{\mathrm{top}}(B)$.

In particular, taking X to be a point, we see that $\mathbb{KK}^{\mathrm{top}}(\mathbb{C}, B) \simeq \mathbb{K}^{\mathrm{top}}(B)$.

5. C^*-algebras associated to a coarse structure

Now let X be a bornological space, and let R be a fixed C^*-algebra. We define an (X, R)-*module* to be a Hilbert R-module M that is equipped with a unital representation $B(X) \to \mathfrak{B}(M)$ of the C^*-algebra of bounded Borel functions on X. Notice that any sufficiently large $(C_0(X), R)$-bimodule is an (X, R)-module, since by the spectral theorem any action of $C_0(X)$ on a Hilbert space extends to an action of $B(X)$.

Example: For example, let $b \colon B \to X$ be a map of a countable set to X. Let M be a copy of the universal module H_R with basis indexed by B. Then M is an (X, R)-module in a natural way. For another example, suppose that μ is a Borel measure on X, and M_0 is any Hilbert R-module. There is a natural definition of a Hilbert R-module $M = L^2(X, M_0, \mu)$. This is an (X, R)-module also.

It may be helpful to think of a Hilbert (X, R)-module M as a 'bisheaf' (that is, both a sheaf and a cosheaf) of Hilbert R-modules over X. Over each open set U one has the complemented submodule $\chi_U M$ of M, where χ_U denotes the characteristic function of U; and an inclusion $U_1 \subseteq U_2$ of open sets induces both an inclusion map (the identity) and a restriction map (multiplication by χ_{U_1}) on the corresponding Hilbert modules.

Let X_1 and X_2 be coarse spaces, let M_i be an (X_i, R)-module ($i = 1, 2$), and let $S \colon M_1 \to M_2$ be a Hilbert module operator. Then there is a natural notion of the *support* of S; it is the complement of that subset of $X_1 \times X_2$ consisting of points (x_1, x_2) such that there exist open sets U_1 and U_2 containing x_1 and x_2 respectively such that $\chi U_2 S \chi_{U_1} = 0$. We denote this set $\operatorname{Supp}(S)$.

Suppose, in the above situation, that $f \colon X_1 \to X_2$ is a coarse map.

(5.1) Definition: *We say that S* covers *f if there is an entourage E for X_2 such that*

$$\operatorname{Supp}(S) \subseteq \{(x_1, x_2) : (f(x_1), x_2) \in E\}.$$

If f is the identity map, we may also say that S is a finite propagation operator*.*

Suppose that $f \colon X_1 \to X_2$ and $f' \colon X_2 \to X_3$ are coarse maps. If S covers f and S' covers g, then $S'S$ covers gf. The proof is straightforward. In particular, the composite of two finite propagation operators is a finite propagation operator.

(5.2) Definition: *An operator T on an (X,R)-module M is called* locally compact *if $\chi_{U_1} T \chi_{U_2}$ is a compact operator (in the sense of Hilbert module operators) for all relatively compact open sets U_1 and U_2.*

The locally compact finite propagation operators form an algebra, by the remarks above.

(5.3) Definition: *Let M be an (X,R)-module. By $C^*_R(X;M)$ we mean the C^*-subalgebra of $\mathfrak{B}(M)$ obtained as the norm closure of the locally compact finite propagation operators on M.*

Suppose that (X_1, M_1) and (X_2, M_2) are coarse spaces equipped with appropriate Hilbert modules, and let $f\colon X_1 \to X_2$ be a coarse map that is covered by a isometry $V\colon M_1 \to M_2$. Then one easily checks that $\mathrm{Ad}(V)\colon \mathfrak{B}(M_1) \to \mathfrak{B}(M_2)$ maps $C^*_R(X;M_1)$ to $C^*_R(X;M_2)$. Thus we have made C^*_R into a functor on the category whose objects are pairs (X,M) and whose morphisms are pairs (f,V) as described.

We will now investigate the dependence of this functor on the choices of M and V. First, a uniqueness statement.

(5.4) Lemma: *Suppose that (X_1, M_1) and (X_2, M_2) are as above, $f\colon X_1 \to X_2$ is a coarse map, and that V and V' are two isometries covering f. Then the maps $\mathrm{Ad}(V)_*$ and $Ad(V')_*$ from $\mathbb{K}^{\mathrm{top}} C^*_R(X_1, M_1)$ to $\mathbb{K}^{\mathrm{top}} C^*_R(X_2, M_2)$ are homotopic.*

Proof: This is a 'spacified' version of Lemma 3 of Section 4 of [20], and the proof is the same. □

It need not be the case that a general coarse map is covered by a isometry. However, as soon as the modules in question are sufficiently large, this will be the case. Note that, if M is a sufficiently large (X,R)-module and W is any subset of X having non-empty interior, the module $\chi_W M$ is isomorphic to H_R.

(5.5) Proposition: *Let (X_1, M_1) and (X_2, M_2) be as above, and suppose that M_2 is sufficiently large. Then any coarse map $f\colon X_1 \to X_2$ can be covered by an isometry.*

Proof: This is essentially lemma 2 of section 4 of [20], whose proof is unfortunately somewhat garbled. Choose an entourage E for X_2 and partition X_2 into a countable union of Borel components W_j having nonempty interior and such that $\bigcup_j W_j \times W_j \subseteq E$. Then write

$$M_1 = \bigoplus_j \chi_{f^{-1}(W_j)} M_1, \qquad M_2 = \bigoplus_j \chi_{W_j} M_2.$$

By the Stabilization Theorem there exist isometries $V_j\colon \chi_{f^{-1}(W_j)} M_1 \to \chi_{W_j} M_2$; define

$$V = \bigoplus_j V_j.$$

By construction, $(x_1, x_2) \in \mathrm{Supp}(V)$ only if $(f(x_1), x_2) \in \bigcup_j W_j \times W_j$, so V covers f. □

From the results above it follows that one obtains a well-defined functor from the category of coarse spaces[5] to the homotopy category of spectra by selecting once and for all a sufficiently large (X,R)-module M_X for each coarse space X, and then sending the space X to the spectrum $\mathbb{K}^{\mathrm{top}}(C^*_R(X;M_X))$. We shall sometimes allow ourselves to suppress mention of the choice of module, and just to write $C^*_R(X)$ for $C^*_R(X;M_X)$. Of course, in a particular application there may be special reasons (for example, equivariant considerations, compare [5, 6]) for making a particular choice of module.

Remark: Once again there is an element of choice here. However, just as in the remark after 4.6, we can arrange to select our modules in such a way as to have a functor from some specified subcategory $\mathfrak{X}$ of the category of coarse spaces, having at most countably many morphisms, to the category of C^*-algebras; and thereby obtain a choice of functor from $\mathfrak{X}$ to spectra by passing to K-theory.

[5] Or at least from any small subcategory.

Now let us consider the relation between the spaces $\mathbb{K}^{\mathrm{top}}(C^*_R(X))$ and the algebraic K or L theory of the categories $\mathfrak{C}^{b.o.}_R(X)$ discussed previously.

(5.6) Proposition: *There are natural maps*

$$\mathbb{K}(\mathfrak{C}^{b.o.}_R(X)) \to \mathbb{K}^{\mathrm{top}}(C^*_R(X)), \quad \pi_*\mathbb{L}(\mathfrak{C}^{b.o.}_R(X)) \to \pi_*\mathbb{K}^{\mathrm{top}}(C^*_R(X)).$$

Proof: Let us note that an object M of $\mathfrak{C}^{b.o.}_R(X)$ is in a natural way an (X,R)-module. Moreover, since the basis of M is locally finite, any endomorphism of M is locally compact, and it is by definition a Hilbert module operator supported in an entourage, by definition. Thus we get a map

$$\mathrm{End}(M) \to C^*(X; M).$$

From the definition of the K-theory of a category this gives us a map

$$\mathbb{K}(\mathfrak{C}^{b.o.}_R(X)) \to \mathbb{K}^{\mathrm{top}}(C^*_R(X)).$$

where we have also made use of the natural map from algebraic to topological K-theory. On the level of L-theory we may similarly construct

$$\mathbb{L}(\mathfrak{C}^{b.o.}_R(X)) \to \mathbb{L}^p(C^*_R(X)),$$

and we then follow this by Mischenko's natural map from L-theory to topological K-theory for C^*-algebras [25, 37]. $\square$

Remark: Note that it is not asserted that the map from L-theory to K-theory is a map of spectra.

6. The categorical approach

Given a bornological space X, we have constructed a C^*-algebra $C^*(X; R)$ and corresponding spectrum $\mathbb{K}^{\mathrm{top}}C^*(X; R)$. In this section we present a categorical approach to the construction of such spectra which is closer to the traditional way of thinking in algebraic topology.

(6.1) Definition: *A C^*-category is a Banach category with involution such that $\|TT^*\| = \|T\|^2$ for all morphisms T (see* [13] *for details).*

(Note that this implies that the endomorphisms of an object in a C^*-category form a C^*-algebra.)

Given a bornological space X we have defined a category $\mathfrak{C}^{b.o.}(X; R)$ for any C^*-algebra R. Given objects A and B in this category, any element $\varphi \in \mathrm{Hom}(A, B)$ determines a bounded operator from the pre-Hilbert R-module given by A to the pre-Hilbert R-module given by B, thus presenting $\mathrm{Hom}(A, B)$ as a subset of the bounded operators from A to B. We define $\overline{\mathrm{Hom}}(A, B)$ to be the closure of $\mathrm{Hom}(A, B)$ in the space of all bounded operators.

(6.2) Definition: $\overline{\mathfrak{C}}^{b.o.}(X; R)$ *is the category with the same objects as* $\mathfrak{C}^{b.o.}(X; R)$ *but with* Hom*-sets* $\overline{\mathrm{Hom}}(A, B)$. *We shall call this category the* analytical category with labels in X and coefficients in R *and denote it equivalently as*

$$\mathfrak{A}(X; R) = \overline{\mathfrak{C}}^{b.o.}(X; R)$$

(6.3) Lemma: $\mathfrak{A}(X; R)$ *is a C^*-category.*

Proof: Clear from the definition. $\square$

Restricting the morphism to the isomorphisms we may thus think of $\mathfrak{A}(X;R)$ as a topological symmetric monoidal category.

Notice there is a directed system of the objects of $\mathfrak{A}(X;R)$ given by $A \leqslant B$ if the basis for A is contained in the basis for B. Clearly

$$C^*(X;R) = \lim_{\to}(\mathrm{End}(A))$$

Since K-theory commutes with direct limits we thus get

$$\mathbb{K}^{\mathrm{top}}C^*(X;R) = \lim_{\to}(\mathbb{K}^{\mathrm{top}}(\mathrm{End}(A)))$$

We are now in a position to use the May-Segal machinery to produce a spectrum from the category $\mathfrak{A}(X;R)$ namely

$$K\mathfrak{A}(X;R) = \Omega B \coprod B(\mathrm{Aut}(A))$$

(6.4) Theorem: *The spectrum $K(\mathfrak{A}(X;R))$ is a connective cover of $\mathbb{K}^{\mathrm{top}}(C^*(X;R))$. We recover the spectrum $\mathbb{K}^{\mathrm{top}}(C^*(X;R))$ from the categorical approach as the homotopy colimit*

$$\mathbb{K}^{\mathrm{top}}\mathfrak{C}(X;R) = \mathrm{hocolim}\,\Omega^i K(\mathfrak{A}(X\times\mathbb{R}^i;R))$$

Proof: (Sketch) We shall use [1] as a standard reference on infinite loop spaces. According to [1, section 3.2] we find that the 0'th space of $K(\mathfrak{A}(X;R))$ is the product of $K_0(\mathfrak{A}(X;R))$ and the plus construction applied to the limit of $B(\mathrm{Aut}(A))$. But since π_1 is commutative the plus construction has no effect so the space is the classifying space of of units in the limit i.e. the units in $C^*(X;R)$. Since every object in $\mathfrak{A}(X;R)$ is a projective $C^*(X;R)$-module we get a monomorphism $K_0(\mathfrak{A}(X;R)) \to K_0(C^*(X;R))$. This proves the first statement. As for the second statement notice that

$$K(\mathfrak{A}(X\times\mathbb{R}^i;R)) \to K(\mathfrak{A}(X\times\mathbb{R}^{i+1};R))$$

is null homotopic in two ways by Eilenberg swindles hence give a map

$$K(\mathfrak{A}(X\times\mathbb{R}^i;R)) \to \Omega K(\mathfrak{A}(X\times\mathbb{R}^{i+1};R))$$

which is an isomorphism on homotopy groups in positive dimensions. The similarly defined map

$$\mathbb{K}^{\mathrm{top}}C^*(X\times\mathbb{R}^i;R) \to \Omega\mathbb{K}^{\mathrm{top}}C^*(X\times\mathbb{R}^{i+1};R)$$

is a homotopy equivalence of spectra i.e. isomorphism in homotopy groups in all dimensions. The result follows from this since each increase in i means getting isomorphism in one more negative homotopy group. □

7. The assembly map

In this section we will define the *assembly map* for a coarse space X. This is a map

$$A\colon \mathbb{KK}^{\mathrm{top}}(C_0(X),R) \to \mathbb{K}^{\mathrm{top}}(C^*_R(X))$$

defined for any coefficient C^*-algebra R. Here the left-hand side is Kasparov's KK-theory [21, 23, 24], made into a spectrum according to the procedure already described. In the case that $R=\mathbb{C}$, the complex numbers, this KK-theory spectrum is a model for the locally finite K-homology spectrum $\mathbb{H}^{l.f.}(X;\mathbb{K}^{\mathrm{top}}\mathbb{C})$, and the assembly map becomes a C^*-version (see [6]) of the assembly map of [5]. The technique that we use to construct the assembly map is borrowed from [19]. We begin by defining another C^*-algebra associated to a coarse space, whose K-theory spectrum will be a kind of "analytic structure set" for the space.

(7.1) Definition: *Let X be a coarse space, and let M be an (X,R)-module. We define $D^*_R(X;M)$ to be the C^*-algebra generated by all pseudolocal, finite propagation operators on M.*

Recall that an operator $T\in\mathfrak{B}(M)$ is *pseudolocal* if $\varphi T - T\varphi$ is compact for all $\varphi\in C_0(M)$. In the case of bimodules for which the left-hand algebra is commutative, as here, one has an alternative characterization of pseudolocality which is extremely useful.

(7.2) Lemma: (Kasparov [21]) *An operator T is pseudolocal if and only if $\varphi T\psi$ is compact whenever φ and ψ are continuous functions with disjoint supports.*

Clearly, $C^*_R(X;M)$ is an ideal in the unital algebra $D^*_R(X;M)$. Notice that $D^*_R(X;M)$ does depend on the local topological structure of the space X, as well as on its coarse structure; for this reason, its K-theory will be functorial only under *continuous* coarse maps. As before, we will suppress mention of the module M if it is assumed that a standard choice has been made.

There is a natural induced map $i_X\colon \mathbb{K}^{\mathrm{top}}C^*_R(X) \to \mathbb{K}^{\mathrm{top}}D^*_R(X)$. The existence and properties of the assembly map follow easily from

(7.3) Proposition: *The homotopy fibre of the map i_X is $\mathbb{KK}^{\mathrm{top}}(C_0(X),R)$.*

Proof: Recall that $\mathbb{KK}^{\mathrm{top}}(C_0(X),R) = \Omega^{-1}\mathbb{K}^{\mathrm{top}}(\Psi_0(M)/\Psi_{-1}(M))$, where M is a sufficiently large Hilbert $(C_0(X),R)$-bimodule. Using the long exact sequence of topological K-theory (4.4), we see that it suffices to prove that

$$D^*_R(X;M)/C^*_R(X;M) = \Psi_0(M)/\Psi_{-1}(M).$$

This is an analogue of lemma 6.2 of [19]. The proof is the same, except that we have to replace the *strong* topology (used in the Hilbert space case in [19]) by the *strict* topology. We will carry out the details.

Recall that $\mathfrak{B}(M)$ can be regarded as the multiplier algebra of $\mathfrak{K}(M)$, and thus acquires a *strict topology*, defined by the seminorms $T \mapsto \|TA\| + \|AT\|$, $A \in \mathfrak{K}(M)$. Let E be an entourage for X, and choose a partition of unity ψ_j^2 subordinate to a locally finite open cover U_j of X such that $\bigcup_j U_j \times U_j \subseteq E$. Let T be a pseudolocal operator. We claim that the series

$$T' = \sum_j \psi_j T \psi_j \tag{$*$}$$

converges in the strict topology. Indeed, the partial sums are bounded in norm, and, if A belongs to the total subset of $\mathfrak{K}(M)$ consisting of rank one operators $u \mapsto \langle u,v\rangle w$, with v and w compactly supported[6], then the series $\sum_j \psi_j T\psi_j A$ and $\sum_j A\psi_j T\psi_j$ converge in norm — in fact, they become finite sums. By a standard result on the strict topology (see, for example, [39], lemma 2.3.6), this ensures that the series $(*)$ converges strictly.

Clearly, now, T' is an operator of finite propagation. But, if φ is a function of compact support, then

$$(T - T')\varphi = \sum_j [\psi_j, T]\psi_j\varphi$$

is a finite sum of compact operators, hence is compact; and, similarly, $\varphi(T-T')$ is compact. Thus $T - T'$ is locally compact. We have written T as the sum of a finite propagation operator and a locally compact operator, so we have shown that

$$\Psi_0(M) = D^*_R(X;M) + \Psi_{-1}(M).$$

But it is clear that

$$C^*_R(X;M) = D^*_R(X;M) \cap \Psi_{-1}(M),$$

and the desired result follows immediately. $\square$

Recall that $\mathbb{KK}^{\mathrm{top}}(C_0(X),R) \simeq \mathbb{H}^{\mathrm{l.f.}}(X,\mathbb{K}^{\mathrm{top}}R)$. Thus we have a fibration of spectra

$$\mathbb{H}^{\mathrm{l.f.}}(X,\mathbb{K}^{\mathrm{top}}R) \to \mathbb{K}^{\mathrm{top}}(C^*_R(X)) \to \mathbb{K}^{\mathrm{top}}(D^*_R(X)), \tag{$\dagger$}$$

which it is appropriate to think of as an analytic analogue of the (spacified) bounded surgery exact sequence. In particular, a splitting of the first arrow gives a Novikov conjecture type statement. For example, suppose that Γ is a finitely generated group having a compact classifying space $B\Gamma$ with universal cover $E\Gamma$. Then an appropriate splitting of the analytic assembly map in the sequence $(\dagger)$ implies the usual Novikov conjecture for Γ. (This is the 'principle of descent' for

[6] That is, there is a compactly supported function χ on M with $v = \chi v$ and $w = \chi w$.

which see [5, 12].) In the next section we will develop some analytic techniques for splitting this map.

8. The special case of continuous control at infinity

Let X be a locally compact, metrizable space. Let $\overline{X}$ be a compactification of X, with $Y = \overline{X}\backslash X$ the 'space at infinity' of the compactification, and X dense in $\overline{X}$. Recall that this data allows us to define the *continuously controlled coarse structure* on X, by decreeing that entourages should 'close off to the diagonal at infinity'. In this section we will compute the K-theory of $C^*_R(X)$ when X is endowed with this coarse structure. It will usually be necessary to assume that the compactification $\overline{X}$ is metrizable.

Suppose that M is a sufficiently large Hilbert $(C_0(X), R)$-bimodule. We may also regard it as a Hilbert $(C(\overline{X}), R)$-bimodule, since a continuous function on $\overline{X}$ is uniquely determined by its restriction to a (bounded) function on X. We will use the notation $\overline{M}$ for M thought of as a bimodule in this latter sense.

(8.1) Proposition: *Let X be endowed with the continuously controlled coarse structure coming from a metrizable compactification $\overline{X}$. Then*

$$D^*_R(X; M) = \Psi_0(\overline{M}).$$

Proof: We prove the two inclusions separately.

Proof of $\subseteq$: To prove that $D^*_R(X; M) \subseteq \Psi_0(\overline{M})$, we make use of Kasparov's lemma (7.2). It suffices to show that if φ and ψ are continuous functions on $\overline{X}$ with disjoint supports, and T is a pseudolocal operator supported in an entourage E, then $\varphi T\psi$ is compact. Notice first that $\varphi T\psi$ is certainly *locally* compact, since by pseudolocality of T it differs by a locally compact operator from $T\varphi\psi = 0$. Thus we need only prove that $\varphi T\psi$ is compactly supported. But this is clear: since $\operatorname{Supp}\varphi \times \operatorname{Supp}\psi$ does not meet the diagonal in $\overline{X} \times \overline{X}$, it follows from the definition of the continuously controlled coarse structure that $E \cap \operatorname{Supp}\varphi \times \operatorname{Supp}\psi$ is a compact subset of $X \times X$. Thus $\varphi T\psi$ is compactly supported and locally compact, hence it is compact.

Proof of $\supseteq$: By lemmas 2.4 and 2.5, it is enough to show that any $T \in \Psi_0(\overline{M})$ can be approximated (in norm) by pseudolocal operators that are half controlled in the sense that given any compact $K \subseteq Y$ and open $V \subseteq X$ with $K \subseteq V$, there exists an open U with $K \subseteq U \subseteq V$ and such that T' does not propagate from outside V to inside U (in other words, $fT'g = 0$ whenever g is supported outside V and f is supported within U).

I claim that it is in fact enough to show the following: given any $\varepsilon > 0$ and $T \in \Psi_0(\overline{M})$, and any V and K as above, there exists a U as above and an operator $T' \in \Psi_0(\overline{M})$, such that $\operatorname{Supp}(T') \subseteq \operatorname{Supp}(T)$, $\|T - T'\| < \varepsilon$, and $fT'g = 0$ whenever g is supported outside V and f is supported within U. The reason that this weaker claim suffices is the following. Because (X, Y) is a compact metric pair, one can find a sequence of pairs $V_n \supseteq K_n$ of open subsets of X and compact subsets of Y, such that for any pair $V \supseteq K$ there is an n such that $V \supseteq V_n \supseteq K_n \supseteq K$. Using the claim, construct inductively a sequence of operators T_n such that $\|T_n - T_{n-1}\| < \varepsilon 2^{-n}$, $\operatorname{Supp}(T_n) \subseteq \operatorname{Supp}(T_{n-1})$, and T_n does not propagate from outside V_n to inside U_n, where U_n is some open set containing K_n. The sequence T_n is Cauchy, so tends to a limit operator T which does not propagate from outside any V_n to inside any U_n; in particular, therefore, it does not propagate from outside V to inside some neighbourhood of K.

Now let us prove the claim. Choose a bump function φ on X equal to 1 off V and equal to zero on a neighbourhood W of K. Write $T = T\varphi + T(1 - \varphi)$. Then $\chi_W T\varphi$ is a compact operator. Let now W_n, with $W_0 = W$, be a decreasing sequence of neighbourhoods of K, with intersection K. The multiplication operators $\chi_{W_n} \in \mathfrak{B}(M)$ then tend strictly to zero, so $\chi_{W_n} T\varphi = \chi W_n \chi_W T\varphi$ tend in norm to zero. Thus one can find an n such that $\|\chi_{W_n} T\varphi\| < \varepsilon$. Take $U = W_n$ and take $T' = (1 - \chi_U)T\varphi + T(1 - \varphi)$. It is easy to check that $fT'g = 0$ if f is supported within U and g is supported outside V. $\square$

We may now identify the K-theory of $D^*_R(X;M) = \Psi_0(\overline{M})$. There is a quotient map $\Psi_0(\overline{M}) \to \Psi_0(\overline{M})/\Psi_{-1}(\overline{M})$, which induces a map

$$\mathbb{K}^{\text{top}}(D^*_R(X)) \to \Omega\mathbb{KK}^{\text{top}}(C(\overline{X}), R) \simeq \Omega\mathbb{H}(X; \mathbb{K}^{\text{top}}(R)).$$

The homotopy fibre of this map is $\mathbb{K}^{\text{top}}(\Psi_{-1}(\overline{M})$, which since $\overline{X}$ is compact is simply a copy of $\mathbb{K}^{\text{top}}(R)$. In fact it is not hard to see that this gives us an equivalence

$$\mathbb{K}^{\text{top}}(D^*_R(X)) \simeq \Omega\widetilde{\mathbb{H}}(X; \mathbb{K}^{\text{top}}(R)),$$

where $\widetilde{\mathbb{H}}$ denotes *reduced* Steenrod homology. Thus we obtain the following definitive result on the isomorphism properties of the analytic assembly map for continuous control at infinity:

(8.2) PROPOSITION: *Let X be a locally compact, metrizable space equipped with a continuously controlled coarse structure coming from a metrizable compactification $\overline{X}$. Then the bounded analytic assembly map for X (the first map in the sequence* (†) *above) is an equivalence if and only if the inclusion of a point in $\overline{X}$ induces an isomorphism on (Steenrod) homology with $\mathbb{K}^{\text{top}}(R)$ coefficients. In particular, if $\overline{X}$ is contractible, then assembly is an isomorphism.*

REMARK: One can identify the entire exact sequence (†) in this case with the reduced homology exact sequence of the pair $(\overline{X}, Y)$. One notes that, as a consequence of 8.1, one can identify $C^*_R(X;M)$ with $\Psi_0(\overline{M}) \cap \Psi_{-1}(M)$. Now an appropriate generalization of the 'relative Paschke duality theory' of [17] shows that $\mathbb{K}^{\text{top}}(C^*_R(X)) \simeq \Omega\widetilde{\mathbb{H}}(Y; \mathbb{K}^{\text{top}}(R))$, and that the maps are those of the relative homology sequence.

This gives a result on the splitting of assembly for more general coarse structures which admit appropriately good compactifications. The relevance of such compactification conditions to the Novikov conjecture was first made explicit by Farrell and Hsiang [10] (see also [5]).

(8.3) DEFINITION: *Let X be a coarse space and let $\overline{X}$ be a compactification of X; by this we mean that there is given a compact pair $(\overline{X}, Y)$ and a homeomorphism $h\colon X \to \overline{X} \setminus Y$. We say that $\overline{X}$ is a* coarse compactification *(or that Y is a* coarse corona*) of X if the homeomorphism h, considered as a map from X with its ambient coarse structure to $\overline{X} \setminus Y$ with the continuously controlled structure induced by the compactification, is a coarse map.*

It is equivalent (provided everything is metrizable) to say that whenever x_n and x'_n are two sequences in X such that (x_n, x'_n) lies in some entourage for all n, then x_n and x'_n must both converge to the same point of Y if one of them converges there at all. In particular, if a group Γ acts cocompactly on X and the coarse structure is Γ-invariant, then a coarse compactification of X is precisely one in which 'compact sets become small at infinity under translation' in the sense of [10, 5].

(8.4) PROPOSITION: *Let X be a coarse space. If X admits a contractible coarse compactification, then the bounded analytic assembly map for X is a split monomorphism.*

PROOF: Let X^* denote X with the continuously controlled coarse structure. By the functoriality of the exact sequence (†) one has a commutative diagram of 'bounded analytic surgery sequences',

$$\begin{tikzcd}
\mathbb{H}^{\text{l.f.}}(X, \mathbb{K}^{\text{top}}R) \arrow[r, "\alpha"'] \arrow[d, "="'] & \mathbb{K}^{\text{top}}(C^*_R(X)) \arrow[r] \arrow[d] & \mathbb{K}^{\text{top}}(D^*_R(X)) \arrow[d] \\
\mathbb{H}^{\text{l.f.}}(X^*, \mathbb{K}^{\text{top}}R) \arrow[r, "\alpha^*"'] & \mathbb{K}^{\text{top}}(C^*_R(X^*)) \arrow[r] & \mathbb{K}^{\text{top}}(D^*_R(X))
\end{tikzcd}$$

By the previous result α^* is an equivalence. Hence α is a split monomorphism, as required. □

9. Excision

We will now discuss excision in the general context of coarse spaces for the C^*-algebras that we have introduced. For metric coarse structures this is essentially carried out in [20].

(9.1) Definition: *An* ideal *in a coarse space X is a collection J of subsets of X with the properties that*

- *if $j \in J$, then any subset of j is in J;*
- *if $j \in J$ and E is an entourage, then $N_E(j)$ is in J.*

The set of all subsets of X is an ideal (which by abuse of notation we will sometimes write as X also). The empty set is an ideal. If L is any subset of X, the set

$$\langle L\rangle = \{j \subseteq N_E(L) \text{ for some entourage } E\}$$

is an ideal, called the *ideal generated by* L. If $(\overline{X}, Y)$ is a coarse compactification of X and W is an open subset of Y, the set

$$W^\circ = \{j \subseteq X : \overline{j} \cap W = \emptyset\}$$

is an ideal, called the *annihilator* of W. If I and J are ideals, the sets

$$I \vee J = \{i \cup j : i \in I, j \in J\}, \quad I \wedge J = \{k \subseteq i \cap j : i \in I, j \in J\}$$

are ideals, called the *join* and *meet* of I and J.

Let M be an (X, R)-module. If I is an ideal in X, denote by $C^*_R(I, X; M)$ the norm closure of the set of all locally compact finite propagation operators on M whose support is contained in $i \times i$ for some $i \in I$. (We will refer to such operators simply as 'supported in I'. As usual, we will omit mention of M if it is unimportant or obvious from the context.) It is easy to see that $C^*_R(I, X)$ is a closed two-sided ideal in $C^*_R(X)$.

(9.2) Proposition: *Let I and J be ideals in a coarse space X. Then*

$$C^*_R(I \vee J; X) = C^*_R(I; X) + C^*_R(J; X), \quad C^*_R(I \wedge J; X) = C^*_R(I; X) \cap C^*_R(J; X).$$

Proof: (Compare [20], Lemma 5.2) A simple partition of unity argument shows that any locally compact finite propagation operator supported in $I \vee J$ can be decomposed into the sum of two such operators, one supported in I and one supported in J; therefore, $C^*_R(I; X) + C^*_R(J; X)$ is dense in $C^*_R(I \vee J; X)$. The first result now follows from Lemma 1 in Section 3 of [20], applied in the C^*-algebra $C^*_R(I \vee J; X)$. The second follows from the observations that

$$C^*_R(I \wedge J; X) \subseteq C^*_R(I; X) \cap C^*_R(J; X), \quad C^*_R(I; X) \cdot C^*_R(J; X) \subseteq C^*_R(I \wedge J; X),$$

together with the fact (a consequence of the functional calculus) that in a C^*-algebra the intersection of two closed ideals is equal to their product. $\square$

(9.3) Corollary: *Let I and J be above. There is a homotopy Cartesian square*

$$\begin{array}{ccc} \mathbb{K}^{\text{top}} C^*_R(I \wedge J; X) & \longrightarrow & \mathbb{K}^{\text{top}} C^*_R(I; X) \\ \downarrow & & \downarrow \\ \mathbb{K} C^*_R(J; X) & \longrightarrow & \mathbb{K}^{\text{top}} C^*_R(I \vee J; X) \end{array}$$

Proof: Consider the commutative diagram of short exact sequences of C^*_R- algebras

$$\begin{array}{ccccc} C^*_R(I \wedge J; X) & \longrightarrow & C^*_R(I; X) & \longrightarrow & C^*_R(I; X)/C^*_R(I \wedge J; X) \\ \downarrow & & \downarrow & & \downarrow \\ C^*_R(J; X) & \longrightarrow & C^*_R(I \vee J; X) & \longrightarrow & C^*_R(I \vee J; X)/C^*_R(J; X) \end{array}$$

The right-hand vertical arrow is an isomorphism of C^*-algebras, by the proposition above. However, a short exact sequence of C^*-algebras gives rise to a homotopy fibration on (topological) K-theory (Proposition 4.4) so the result follows. $\square$

To apply this result one needs to understand the meet and join of two ideals, in the cases where the ideals arise from the geometric constructions that we mentioned earlier.

Consider first the case of generated ideals. Here one has the following observation. Let $L \subseteq X$ be an inclusion of coarse spaces. To define the functorially induced map on the K-theory of the corresponding C^*-algebras we use the homomorphism $\mathrm{Ad}(V)\colon C^*_R(L) \to C^*_R(X)$, where V is any isometry covering the inclusion map. It is easy to see that the range of this homomorphism (for any choice of V) is contained in $C^*(\langle L\rangle, X)$, and that two choices of V give rise to homotopic maps $\mathbb{K}^{\mathrm{top}}C^*_R(L) \to \mathbb{K}^{\mathrm{top}}C^*_R(\langle L\rangle, X)$. In fact

(9.4) Proposition: *Let $L \subseteq X$ be an inclusion of coarse spaces. Then the induced map $\mathbb{K}^{\mathrm{top}}C^*_R(L) \to \mathbb{K}^{\mathrm{top}}C^*_R(X)$ is a homotopy equivalence onto $\mathbb{K}^{\mathrm{top}}C^*_R(\langle L\rangle; X)$.*

The proof is exactly as in [20].

If L and M are subsets of X, it is always true that $\langle L\rangle \vee \langle M\rangle = \langle L \cup M\rangle$. However, as pointed out by Carlsson in the metric context [4], it is not always the case that $\langle L\rangle \wedge \langle M\rangle = \langle L \cap M\rangle$. When this is so, we say that L, M form a *coarsely excisive* pair (in the metric case, this coincides with the definition of 'ω-excisive' in [20]).

Then we have

(9.5) Corollary: *If $X = L \cup M$ is a coarsely excisive decomposition of a coarse space X, then there is a homotopy Cartesian square*

$$\begin{array}{ccc} \mathbb{K}C^*_R(L\cap M) & \longrightarrow & \mathbb{K}^{\mathrm{top}}C^*_R(L) \\ \downarrow & & \downarrow \\ \mathbb{K}^{\mathrm{top}}C^*_R(M) & \longrightarrow & \mathbb{K}C^*_R(X) \end{array}$$

giving rise to a Mayer-Vietoris sequence of K-theory groups.

It is particularly appropriate to consider generated ideals, and their associated excision properties, in the boundedly controlled case. By contrast, annihilator ideals seem particularly appropriate to the case of continuous control at infinity, and give an analytic counterpart to the excision results in section 2 of [5]. Indeed, let X be continuously controlled with boundary Y, and let $W \subseteq Y$ be open. Then the C^*-algebra $C^*(W^\circ, X)$ corresponds to the category $\mathcal{B}(X,Y)_{Y\setminus W}$ of [5], and the quotient algebra $C^*(X)/C^*(W^\circ, X)$ corresponds to the 'germ category' $\mathcal{B}(X,Y)^W$. The short exact sequence of C^*-algebras

$$0 \to C^*(W^\circ, X) \to C^*(X) \to C^*(X)/C^*(W^\circ, X) \to 0$$

gives rise to a fibration of $\mathbb{K}^{\mathrm{top}}$-spectra corresponding to the fibration of algebraic K-theory spectra in Corollary 2.30 of [5].

For future reference we remark that the analogue of the coarsely excisive condition is always true for annihilator ideals; it is always the case that $(U \cup V)^\circ = U^\circ \wedge V^\circ$ and $(U \cap V)^\circ = U^\circ \vee V^\circ$. This corresponds to the fact that the homology theory of the space at infinity defined by taking the K-theory of the continuously controlled category satisfies the strong excision axiom [5, Theorem 2.36]. See section 12.

10. Contractibility

This section is about Eilenberg swindles. Let X be a coarse space. We will say that X if *flasque* if there is a coarse map $t\colon X \to X$ such that

- For any compact $K \subseteq X$, there is an $n_0 \in \mathbb{N}$ such that $\mathrm{Im}(t^n) \cap K = \emptyset$ for all $n \geqslant n_0$. (In other words, the action of t eventually leaves any compact set.)
- The powers of t are *uniformly* coarse: given an entourage E, there is another entourage E' such that t^n maps E into E' for all n.
- The map t is coarsely equivalent to the identity map.

(10.1) Proposition: *If X is flasque then $\mathbb{K}^{\mathrm{top}}C^*_R(X)$ is contractible. The same applies to $\mathbb{K}\mathfrak{C}^{b.o.}_R(X)$ and $\mathbb{L}\mathfrak{C}^{b.o.}_R(X)$.*

The usual proof applies. To check that it works in an analytic context, we need only to see that if $T \in C^*_R(X; M)$, then the infinite sum

$$T \oplus tTt^{-1} \oplus t^2Tt^{-2} \oplus \cdots$$

is a *bounded* operator on $M \oplus M \oplus \cdots$. But this is true, because conjugation by t is an isometry, and the norm of an orthogonal direct sum of this kind is the supremum of the norms of its constituent summands. For the details (in the case of bounded control), see [20, Proposition 7.1].

11. Lipschitz homotopy invariance

In [18], the notion of *coarse homotopy* was introduced as an equivalence relation between coarse maps, and it was proved that coarsely homotopic coarse maps induce the same homomorphism on the C^*-algebra K-theory groups. In [42], the same result was proved for the notion of *Lipschitz homotopy* due to Gromov [14]. The precise relation between the notions of Lipschitz homotopy and coarse homotopy is rather unclear, but what is clear is that they perform similar functions, and that all of the applications of which we are aware could be based on either concept. Thus, for example, the radial contraction maps on an open cone, the exponential and logarithm maps of a Hadamard manifold, and so on, are both coarse homotopy equivalences and Lipschitz homotopy equivalences. In this paper we will work with Lipschitz homotopy as it seems more suitable for the kind of argument that we want to give.

(11.1) Definition: *Let X and Y be coarse spaces. A* Lipschitz homotopy *is a coarse map $H\colon X \times \mathbb{R}_+ \to Y \times \mathbb{R}_+$ of the form $H(x,t) = (H_t(x), t)$ which has the property that for each compact $K \subseteq X$ there is $t_K \in \mathbb{R}_+$ such that $H_t(x)$ is constant in t for $x \in K$ and $t \geqslant t_K$.*

Intuitively, a Lipschitz homotopy is one that runs at unit speed, but perhaps for a longer and longer time before finishing. In these circumstances we say that $f_0(x) = H(0,x)$ and $f_\infty(x) = \lim_{t\to\infty} H(t,x)$ are *(elementary) Lipschitz homotopic.*

The relation of elementary Lipschitz homotopy is obviously reflexive, but it is not clear that it is either symmetric or transitive. We let *Lipschitz homotopy* be the equivalence relation (on coarse maps) generated by elementary Lipschitz homotopy, and we define the notion of *Lipschitz homotopy equivalence* (between coarse spaces) in the obvious way.

(11.2) Theorem: *Lipschitz homotopic coarse maps induce homotopic maps on the $\mathbb{K}^{\mathrm{top}}$-theory of the C^*-algebras.*

Proof: It is enough to prove this for an elementary Lipschitz homotopy. Given such a homotopy H we extend it (constantly) over $\mathbb{R}^-$, so that H becomes a coarse map from $W = X \times \mathbb{R}$ to Y. By definition of coarse homotopy, $H(x,t)$ is constant in x outside some region Z of the form

$$Z = \{(x,t) : 0 \leqslant t \leqslant \varphi(x)\},$$

for a suitable (perhaps rapidly increasing) function $\varphi\colon X \to \mathbb{R}_+$. Let $X_0 \subset W$ be the space $\{(x,0) : x \in X\}$, which is simply a copy of X in W. Let $X_1 \subset W$ be the space $\{(x,\varphi(x)) : x \in X\}$; there is an obvious coarse map $q\colon X_1 \to X$.

Consider the commuting diagram

$$\begin{tikzcd} X_0 \arrow[r, "i_0"] \arrow[d, "id"] & Z \arrow[d, "H"] & X_1 \arrow[l, "i_1"'] \arrow[d, "q"] \\ X \arrow[r, "f_0"] & Y & X \arrow[l, "f_\infty"'] \end{tikzcd} .$$

We will prove the homotopy invariance by showing that the maps i_0, i_1, and q each induce homotopy equivalences on $\mathbb{K}^{\mathrm{top}}$. In fact, it will be enough to show that i_0 and i_1 induce equivalences, and that the obvious projection $p\colon Z \to X$ induces the homotopy inverse of the map induced by i_0; this is because $q = pi_1$. We will do this by comparing the Mayer-Vietoris sequences corresponding to different ω-excisive decompositions of W. Let $W^- = \{(x,t) : t \leqslant 0\} \subset W$ and let $W^+ = \{(x,t) : t \geqslant \varphi(x)\}$. To prove that i_0 induces an equivalence whose inverse is induced by p, we compare the decomposition $W = A \cup B$ where

$$A = W^-, \quad B = Z \cup W^+$$

with the decomposition $W = A' \cup B'$ where

$$A' = W^- \cup Z, \quad B' = Z \cup W^+.$$

Notice that $A \cap B = X_0$ and $A' \cap B' = Z$. From these decompositions and corollary 9.5 we get a cube diagram

$$\begin{tikzcd}
& \mathbb{K}^{\mathrm{top}}C^*(A' \cap B') \arrow[rr] \arrow[dd] & & \mathbb{K}^{\mathrm{top}}C^*(A') \arrow[dd] \\
\mathbb{K}^{\mathrm{top}}C^*(A \cap B) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{K}^{\mathrm{top}}C^*(A) \arrow[ur] \arrow[dd] & \\
& \mathbb{K}^{\mathrm{top}}C^*(B') \arrow[rr] & & \mathbb{K}^{\mathrm{top}}C^*(W) \\
\mathbb{K}^{\mathrm{top}}C^*(B) \arrow[ur] \arrow[rr] & & \mathbb{K}^{\mathrm{top}}C^*(W) \arrow[ur, equal] &
\end{tikzcd}$$

in which the front and back faces are homotopy Cartesian. We want to prove that the top left-hand arrow is a homotopy equivalence, and to do this it will be enough to prove that $\mathbb{K}^{\mathrm{top}}C^*(A)$, $\mathbb{K}^{\mathrm{top}}C^*(B)$, $\mathbb{K}^{\mathrm{top}}C^*(A')$, and $\mathbb{K}^{\mathrm{top}}C^*(B')$ are trivial. But A, B, A', and B' are all flasque by appropriate right or left shifts, so this follows from the result of the previous section. A similar argument works for i_1, completing the proof. $\square$

The same result as above holds for algebraic K-theory and for L-theory. Since this is not recorded in the literature we state

(11.3) Theorem: *Lipschitz homotopic coarse maps $X \to Y$, induce homotopic maps on the algebraic K- and L-theory of the categories $\mathfrak{C}(X;R) \to \mathfrak{C}(Y;R)$ for any ring R (with involution in the case of L-theory).*

Proof: The proof in algebraic K-theory is exactly the same, replacing the C^*-algebras by the categories and $\mathbb{K}^{\mathrm{top}}$ by $\mathbb{K}^{-\infty}$. In L-theory we need to worry about decorations. From [32] or [5] we get the following diagram with notation as in the theorem above

$$\begin{tikzcd}
& \mathbb{L}^p\mathfrak{C}(A' \cap B';R) \arrow[rr] \arrow[dd] & & \mathbb{L}^h\mathfrak{C}(A';R) \arrow[dd] \\
\mathbb{L}^p\mathfrak{C}(A \cap B;R) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{L}^h\mathfrak{C}(A;R) \arrow[ur] \arrow[dd] & \\
& \mathbb{L}^h\mathfrak{C}(B';R) \arrow[rr] & & \mathbb{L}^h\mathfrak{C}(W;R) \\
\mathbb{L}^h\mathfrak{C}(B;R) \arrow[ur] \arrow[rr] & & \mathbb{L}^h\mathfrak{C}(W;R) \arrow[ur, equal] &
\end{tikzcd}$$

The p-decorations are due to the flasqueness of the categories $\mathfrak{C}(A';R)$, $\mathfrak{C}(B';R)$, $\mathfrak{C}(A;R)$ and $\mathfrak{C}(B;R)$. This flasqueness makes L-theory trivial, so we get an equivalence $\mathbb{L}^p\mathfrak{C}(A \cap B;R) \to \mathbb{L}^p\mathfrak{C}(A' \cap B';R)$. To get isomorphism in $\mathbb{L}^h$ from the isomorphism in $\mathbb{L}^p$, we use the Rothenberg-Ranicki exact sequence and the fact that the map induces isomorphism in algebraic K-theory. $\square$

REMARK: Notice that we get Lipschitz homotopy invariance in L-theory with any decoration by the argument above, once we have invariance for one decoration. This follows from the K-theory statement and Rothenberg-Ranicki exact sequences.

12. Continuous control and Steenrod homology

For a compact metrizable space X, let cX denote the (closed) cone on X. Let $\mathcal{O}X$ denote the open cone $cX \setminus X$ with the continuously controlled coarse structure coming from its compactification to cX. The object of this section is to prove

(12.1) PROPOSITION: *For a fixed C^*-algebra R, the functor $X \mapsto \mathbb{K}^{\mathrm{top}} C^*_R(\mathcal{O}X)$ is a reduced Steenrod homology functor.*

REMARK: Notice that the results of section 8 already give us an identification $\mathbb{K}^{\mathrm{top}} C^*_R(\mathcal{O}X) \simeq \Omega\widetilde{\mathbb{KK}}^{\mathrm{top}}(C(X), R)$ (reduced Kasparov K-homology). Thus our proposition follows from the identification of KK-theory in which the first algebra is commutative with Steenrod homology. However, in this section we will proceed directly; by reversing the argument we may use the proposition as a proof of the homological properties of KK-theory.

PROOF: According to [5], to show that $k(X) = \mathbb{K}^{\mathrm{top}} C^*_R(\mathcal{O}X)$ is a Steenrod functor we have to prove three things.

(i) $k(cX)$ is contractible for every X;

(ii) For a closed subset $A \subseteq X$ there is a fibration
$$k(A) \to k(X) \to k(X/A);$$

(iii) for a 'strong wedge' of countably many metric spaces, $X = \bigvee_i X_i$, the projection maps induce a weak equivalence
$$k\left(\bigvee X_i\right) \simeq \prod k(X_i).$$

We should check first that we do have a functor! Indeed, a continuous map $X \to Y$ naturally induces a coarse map $\mathcal{O}X \to \mathcal{O}Y$, hence a map on K-theory.

To verify property (i) we follow the argument of [2]. As a topological space, $Z =_{df} \mathcal{O}cX = \mathcal{O}X \times \mathbb{R}^+$. It is not known whether or not this space is flasque (in the continuously controlled structure); however, it clearly is flasque in the product coarse structure, and it is shown in [2] that there are sufficiently many coarse maps from Z with the product structure to Z with the continuously controlled structure that their images generate the whole of K-theory. The result follows.

We check property (ii). Let $A \subseteq X$ and let I be the annihilator ideal of the open subset $X \setminus A$ of X. Because we are working with continuous control, this is the same as the ideal generated by $\mathcal{O}A \subseteq \mathcal{O}X$. Thus the K-theory of the C^*-algebra $C^*_R(I; \mathcal{O}X)$ is equal to $k(A)$. On the other hand there is an isomorphism of C^*-algebras
$$C^*_R(\mathcal{O}X)/C^*_R(I; \mathcal{O}X) \equiv C^*_R(\mathcal{O}(X/A))/C^*R(\langle\mathcal{O}(*)\rangle; \mathcal{O}(X/A)).$$
Since $\mathbb{K}^{\mathrm{top}} C^*R(\langle\mathcal{O}(*)\rangle; \mathcal{O}(X/A))$ is contractible by (i), Proposition 4.4 shows that the K-theory of this quotient C^*-algebra is homotopy equivalent to $k(X/A)$, and hence that there is a fibration
$$k(A) \to k(X) \to k(X/A)$$
as required.

Finally we check (iii). Let $*$ be the wedge point. We use the result, which was just shown, that it makes no difference up to homotopy if we consider $k'(X_i) = \mathbb{K}^{\mathrm{top}} C^*_R(\mathcal{O}(X_i))/C^*R(\langle\mathcal{O}(*)\rangle; \mathcal{O}(X_i))$ in place of $k(X_i)$. Let us do this. Now all elements in these quotient C^*-algebras can be represented as operators supported near infinity and away from the wedge point. The continuous control condition therefore gives
$$\mathbb{K}^{\mathrm{top}} C^*_R(\mathcal{O}(\bigvee X_i))/C^*R(\langle\mathcal{O}(*)\rangle; \mathcal{O}(\bigvee X_i)) = \bigoplus \mathbb{K}^{\mathrm{top}} C^*_R(\mathcal{O}(X_i))/C^*R(\langle\mathcal{O}(*)\rangle; \mathcal{O}(X_i)).$$

From this and the continuity of topological K-theory with respect to direct limits we get the desired result. □

Remark: It is also possible to prove the analogous theorem when $\mathcal{O}X$ refers to the open cone equipped with *bounded* (metric) control, rather than continuous control. The wedge axiom is rather harder to prove in this case: one must rescale the $\mathcal{O}X_i$ using Lipschitz homotopy invariance, so as to obtain representatives for homology classes which have a common propagation bound, which can then be glued together. See [11, Theorem 16.7] for an algebraic version of this.

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