ON THE COARSE BAUM-CONNES CONJECTURE

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1. Introduction

The *Baum-Connes conjecture* [2, 3] concerns the K-theory of the reduced group C^*-algebra $C^*_r(G)$ for a locally compact group G. One can define a map from the equivariant K-homology of the universal proper G-space $\underline{E}G$ to $K_*(C^*_r(G))$: each K-homology class defines an index problem, and the map associates to each such problem its analytic index. The conjecture is that this map is an isomorphism. The injectivity of the map has geometric and topological consequences, implying the Novikov conjecture for example; the surjectivity has consequences for C^*-algebra theory and is related to problems in harmonic analysis.

In geometric topology it has proved to be very useful to move from studying classical surgery problems on a compact manifold M to studying bounded surgery problems over its universal cover (see [8, 24] for example). In terms of L-theory, one replaces the classical L-theory of $\mathbf{Z}\pi$ by the L-theory, bounded over $|\pi|$, of $\mathbf{Z}$ (here $|\pi|$ denotes π considered as a metric space, with a word length metric). Now the authors, motivated by considerations of index theory on open manifolds, have studied a C^*-algebra $C^*(X)$ associated to any proper metric space X, and it has recently become quite clear that the passage from $C^*_r(\pi)$ to $C^*(|\pi|)$ is an analytic version of the same geometric idea. Moreover, various descent arguments have been given [4, 9, 5, 17, 27], both in the topological and analytic contexts, which show that a 'sufficiently canonical' proof of an analogue of the Baum-Connes conjecture in the bounded category will imply the classical version of the Novikov conjecture.

The purpose of this paper is to give a precise formulation of the Baum-Connes conjecture for the C^*-algebras $C^*(X)$ (filling in the details of the hints in the last section of [26]) and to prove the conjecture for spaces which are non-positively curved in some sense, including affine buildings and hyperbolic metric spaces in the sense of Gromov. Notice that while the classical Novikov conjecture has been established for the analogous class of groups, the Baum-Connes conjecture has not. Unfortunately there does not seem to be any descent principle for the surjectivity side of the Baum-Connes conjecture as there is for the injectivity side.

The main tool that we will use in this paper is the invariance of $K_*(C^*(X))$ under *coarse homotopy*, established by the authors in [15]. Coarse homotopy is a rather weak equivalence relation on metric spaces, weak enough that (for example) many spaces are coarse homotopy equivalent to open cones $\mathcal{O}Y$ on compact spaces Y.

The idea of 'reduction to a cone on an ideal boundary' is also used in some topological approaches to the Novikov conjecture, but the notion of coarse homotopy

Date: May 15, 1997.
Nigel Higson is an Alfred P. Sloan Foundation Research Fellow.

appears at present[1] to be peculiar to our analytic set-up. Certainly the proofs in [15] make heavy use of C^*-algebraic machinery.

A different and very interesting approach to the coarse Baum-Connes conjecture has been proposed by G. Yu [31].

2. Coarse homology theories

The *coarse category* UBB was defined in [26] to be the category whose objects are proper metric spaces (that is, metric spaces in which closed bounded sets are compact) and whose maps are proper Borel maps f satisfying the growth condition

$$\forall R > 0 \quad \exists S > 0 \quad \text{such that} \quad d(x_0, x_1) < R \quad \Rightarrow \quad d(f(x_0), f(x_1)) < S.$$

Two morphisms f and g are called *bornotopic* if there is a constant C such that $d(f(x), g(x)) < C$ for all x.

A substantial part of the paper [26] was devoted to the explicit construction of a bornotopy-invariant cohomology theory on the coarse category. Now let M_* be any generalized homology theory on the category of locally compact spaces and proper maps. We will show how it is possible to 'coarsen' M to a 'coarse homology theory' MX_*, functorial on the coarse category and invariant under bornotopy.

To do this recall from [26], Definition 3.13, that an *anti-Čech system* for a proper metric space X is a sequence $\mathcal{U}_1, \mathcal{U}_2$ of successively coarser open covers of X, with the property that the diameter of each set in $\mathcal{U}_n$ is bounded by a constant R_n which is less than the Lebesgue number of $\mathcal{U}_{n+1}$, and the constants R_n tend to infinity. It follows that each member of the cover $\mathcal{U}_n$ is contained within a member of $\mathcal{U}_{n+1}$, and from now on we shall include a choice of such as part of the structure of an anti-Čech system. Passing to the nerves of the covers $\mathcal{U}_n$, this extra data determines 'coarsening' maps

$$|\mathcal{U}_1| \to |\mathcal{U}_2| \to |\mathcal{U}_3| \to \dots$$

by associating to each member of the cover $\mathcal{U}_n$ the member of the cover $\mathcal{U}_{n+1}$ chosen to contain it.[2]

(2.1) Definition: *We define the coarse M-homology of X to be the direct limit*

$$MX_*(X) = \lim_{\to} M_*(|\mathcal{U}_n|).$$

If $f\colon X \to Y$ is a coarse map, and if $\mathcal{U}_n$ and $\mathcal{V}_m$ are anti-Čech systems for X and Y respectively, then for each n there is an m_n such that if $U \in \mathcal{U}_n$ then $f[U]$ is contained in a member of $\mathcal{U}_{m_n}$. After selecting one such member of $\mathcal{U}_{m_n}$ for each member of $\mathcal{U}_n$ we get a proper map

$$f_n\colon |\mathcal{U}_n| \to |\mathcal{V}_{m_n}|.$$

Defining the maps f_n inductively, we can arrange that the diagrams

$$\begin{array}{ccc} |\mathcal{U}_n| & \to & |\mathcal{U}_{n+1}| \\ f_n \downarrow & & \downarrow f_{n+1} \\ |\mathcal{V}_{m_n}| & \to & |\mathcal{U}_{m_{n+1}}| \end{array}$$

[1]**Added in proof:** Since writing this paper we have learned of unpublished calculations of Ferry and Pedersen which make use of similar ideas.

[2]Recall that the nerve of a cover $\mathcal{U} = \{U_\alpha\}$ is the simplicial complex with vertices $[U_\alpha]$ labelled by members of $\mathcal{U}$, and a p-simplex $[U_{\alpha_0} \dots U_{\alpha_p}]$ for each $(p+1)$-tuple in $\mathcal{U}$ with $U_{\alpha_0} \cap \dots \cap U_{\alpha_p} \neq \emptyset$.

commute. Passing to the direct limit we obtain an induced map

$$f_*\colon MX_*(X) \to MX_*(Y).$$

It is independent of the choices involved in the definition of the maps f_n.

Applying this observation to the identity map on X we note that two different choices of anti-Čech system on X will give rise to canonically isomorphic direct limits $MX_*(X)$, which justifies our omission of the anti-Čech system $\mathcal{U}$ in the notation $MX_*(X)$.

We have made MX_* into a functor on the category UBB.

(2.2) PROPOSITION: *Two bornotopic maps $f, g\colon X \to Y$ induce the same transformation on MX_*.*

PROOF: For each n one can choose an m_n such that the maps f_n and g_n in the above construction both map to the same $|\mathcal{V}_{m_n}|$ and are linearly homotopic. The induced maps on homology are therefore the same. □

We shall occasionally find it useful to confine our attention to coarse maps which are continuous. Imposing this requirement on morphisms we obtain the *continuous coarse category* UBC, a subcategory of UBB. The following definition is best viewed within this context.

(2.3) DEFINITION: *Let X and Y be proper metric spaces. A* coarse homotopy *from X to Y is a continuous and proper map*

$$h\colon X \times [0,1] \to Y$$

such that for every $R > 0$ there exists $S > 0$ with

$$d(x,x') \le R \quad \Rightarrow \quad d(h(x,t), h(x',t)) \le S, \quad \text{for all } t \in [0,1].$$

Two coarse maps $f_0, f_1\colon X \to Y$ are coarsely homotopic if there is a coarse homotopy $h\colon X \times [0,1] \to Y$ such that

$$f_0(x) = h(x,0) \quad \text{and} \quad f_1(x) = h(x,1), \quad \text{for all } x \in X.$$

A coarse map $f\colon X \to Y$ is a coarse homotopy equivalence if there is a coarse map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are coarsely homotopic to the identity maps on Y and X, respectively.

REMARK: It is possible to relax the continuity requirement in this definition to *pseudocontinuity*, that is continuity 'on a certain scale'; see the remarks to Section 1 in [15]. This point will be of some importance when we discuss hyperbolic spaces later in this paper.

(2.4) THEOREM: *Let M be a generalized homology theory. Then coarse M-homology, MX_*, is also a coarse homotopy invariant functor on UBC.*

PROOF: Let $h\colon X \times [0,1] \to Y$ be a coarse homotopy, and let $\mathcal{U}_n$ and $\mathcal{V}_m$ be anti-Čech systems for X and Y respectively. From the definition of coarse homotopy it follows that for each n there is an m_n such that h induces a proper homotopy $h_n\colon |\mathcal{U}_n| \to |\mathcal{V}_{m_n}|$. The result follows. □

3. Comparison of homology and coarse homology

Let X be a proper metric space, let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open cover and let $\{\varphi_\alpha\}$ be a partition of unity subordinate to this cover. Define a map $\kappa\colon X \to |\mathcal{U}|$ by

$$\kappa(x) = \sum_\alpha \varphi_\alpha(x)[U_\alpha].$$

To explain this formula, note that for each x we have $\sum \varphi_\alpha(x) = 1$, and those finitely many vertices $[U_\alpha]$ for which $\varphi(x) \neq 0$ span a simplex in $|\mathcal{U}|$. So $\kappa(x)$ describes, in barycentric coordinates, a point of $|\mathcal{U}|$.

Suppose that we apply this construction to the first cover $\mathcal{U}_1$ in an anti-Čech system. Two different choices of partition of unity will give rise to maps which are properly homotopic, and so induce the same map on homology. Passing to the direct limit we obtain a canonical *coarsening map*

$$c\colon M_*(X) \to MX_*(X).$$

(This is dual to the map c considered in [26] in a cohomology context.) We wish to inquire when c is an isomorphism.

Since the passage from X to $|\mathcal{U}_1|$, and then $|\mathcal{U}_2|$, and so on, obliterates the 'small scale' topology of X, it is natural to confine our attention to *uniformly contractible* spaces. These are defined by the requirement that for each $R > 0$ there be some $S > R$ such that $B(x;R)$ is contractible within $B(x;S)$, for every $x \in X$. One might conjecture that if X is uniformly contractible then the coarsening map c is an isomorphism. However, this is not so: Dranishnikov, Ferry and Weinberger [6] have constructed an example of a uniformly contractible space X for which the coarsening map in K-homology is not an isomorphism. But we shall show that if X is a *bounded geometry complex* (defined below) then uniform contractibility does imply that the coarsening map is an isomorphism.

Recall that a *path metric* on a space X is a metric such that the distance between any two points of X is the infimum of the lengths of the continuous paths connecting them. A *path metric space* is a metric space whose metric has this property.

(3.1) Definition: *A path metric space X is called a* metric simplicial complex *if it is a simplicial complex and its metric coincides on each simplex with the usual spherical metric.*

The spherical metric on the standard n-simplex Δ^n is obtained by regarding it as the set of points of $S^n \subseteq \mathbf{R}^{n+1}$ with nonnegative coordinates. Any locally finite simplicial complex can be given a complete metric that makes it into a metric simplicial complex.

The following result is proved in [25, Section 3].

(3.2) Proposition: *Let X be a complete path metric space, $\mathcal{U}$ an open cover of X that has positive Lebesgue number and such that the sets of $\mathcal{U}$ have bounded diameter. Then the map $\kappa\colon X \to |\mathcal{U}|$, defined above, is a bornotopy-equivalence.*

(3.3) Lemma: *let $f\colon X \to Y$ be a coarse map. Suppose that X is a finite-dimensional metric simplicial complex and that Y is uniformly contractible. Then there exists a continuous map $g\colon X \to Y$ that is bornotopic to f. Moreover, if f is already continuous on a subcomplex X', then we may take $g = f$ on X'.*

Proof: We construct g by induction over the skeleta X_k of (X, X'). The base step is provided by setting $g = f$ on $X' \cup X_0$. Assume then that g has been defined on $X' \cup X_k$. Then g is defined on the boundary of each $k+1$-simplex Δ of (X, X'), and as Y is uniformly contractible, $g|\partial\Delta$ can be extended to a map $\Delta \to Y$ whose image lies within a bounded distance of the image of the vertex set of Δ. Proceeding thus inductively, after finitely many stages we obtain a continuous map $g\colon X \to Y$ which coincides with f on $X' \cup X_0$, and which has the property that there is a constant $C > 0$ such that $d(g(x), g(x')) < C$ whenever $x \in X_0$ is a vertex of a simplex containing x'. Since X_0 is coarsely dense, g is bornotopic to f. □

(3.4) LEMMA: *Let X be a finite-dimensional metric simplicial complex and Y a uniformly contractible space; then any two bornotopic continuous coarse maps from X to Y are properly homotopic.*

PROOF: Let $h\colon X \times [0,1] \to Y$ be a bornotopy, and apply Lemma 3.3 to h which is continuous on the subcomplex $X \times \{0,1\}$ of $X \times [0,1]$. □

(3.5) COROLLARY: *If two uniformly contractible, finite dimensional metric simplicial complexes are bornotopy equivalent then they are proper homotopy equivalent.*

The following notion is due to Fan [7].

(3.6) DEFINITION: *A proper metric space X has* bounded coarse geometry *if there is some $\varepsilon > 0$ such that for each $R > 0$ there is $C > 0$ such that the ε-capacity of any ball of radius R is at most C.*

Recall [21] that the ε-capacity of a set Y is the maximum number of elements in an ε-separated subset of Y.

One can show that bounded coarse geometry implies that for all sufficiently large ρ there is a universal bound on the ρ-entropy and the ρ-capacity of any subset of X in terms of its diameter.

Bounded coarse geometry has the following consequence which will be important for us.

(3.7) LEMMA: *If X is a space of bounded coarse geometry, then for any $R > 0$ there is $S > 0$ such that X has an open cover $\mathcal{U}$ with:*

- *The Lebesgue number of $\mathcal{U}$ is at least R;*
- *The cover $\mathcal{U}$ is of finite order (that is, its nerve is finite-dimensional);*
- *The sets of $\mathcal{U}$ have diameter less than S.*

The proof is straightforward.

For brevity, we will abbreviate the phrase 'metric simplicial complex with bounded coarse geometry' to 'bounded geometry complex.' It is easy to check that *every bounded geometry complex is finite dimensional.*

(3.8) PROPOSITION: *Let X be a uniformly contractible, bounded geometry complex, and let MX_* be the coarse homology theory associated to a generalized homology theory M_* as above. Then the natural map $c\colon M_*(X) \to MX_*(X)$ is an isomorphism.*

PROOF: We will construct an anti-Čech system by induction as follows. Let $\mathcal{U}_1$ be any cover of X of the kind described in Lemma 3.7, and let $f_1\colon X \to |\mathcal{U}_1|$ be the map $\kappa_{\mathcal{U}_1}$. By 3.2, f_1 is a bornotopy equivalence; so it admits a bornotopy inverse $g_1\colon |\mathcal{U}_1| \to X$. Since X is flabby and $|\mathcal{U}_1|$ is finite-dimensional, Lemmas 3.3 and 3.4 show that g_1 may be assumed to be continuous and to be a left proper homotopy inverse of f_1.

The map $f_1 \circ g_1$ is bornotopic to the identity map on $|\mathcal{U}_1|$. It is therefore possible to find a second cover $\mathcal{U}_2$ of the kind described in Lemma 3.7, which coarsens $\mathcal{U}_1$ with coarsening map $f_2 \colon |\mathcal{U}_1| \to |\mathcal{U}_2|$ and which has $f_2 \circ f_1 \circ g_1$ properly homotopic to f_2 by a linear homotopy. Proceeding inductively we may obtain an anti-Čech system

$$X \xrightarrow{f_1} |\mathcal{U}_1| \xrightarrow{f_2} |\mathcal{U}_2| \xrightarrow{f_3} \cdots$$

which has the following properties:

- The maps f_i are continuous;
- The maps $h_i = f_i \circ \cdots \circ f_1$ admit left proper homotopy inverses g_i;
- The maps f_{i+1} and $h_{i+1} \circ g_i$ are properly homotopic.

It follows (using the proper homotopy invariance of M-homology) that $(h_i)_* : M_*(X) \to M_*(|\mathcal{U}_i|)$ is an isomorphism onto the image of $(f_i)_*$. Thus the induced map to the direct limit is an isomorphism. □

Remark: Let X be a proper metric space. We define a *coarsening* of X to be a uniformly contractible, bounded geometry complex Y equipped with a bornotopy-equivalence $X \to Y$. A space X that has a coarsening might be called *coarsenable*. It follows from Corollary 3.5 that a coarsening of X, if it exists, is unique up to a proper homotopy equivalence (which is at the same time a bornotopy equivalence).

Moreover, by Lemmas 3.3 and 3.4, coarsening is functorial: a coarse map between two spaces induces a unique proper homotopy class of continuous coarse maps between their coarsenings. Thus one may define the coarse M-homology of a space X simply to be the ordinary M-homology of a coarsening of X, and indeed one may make the analogous definition for cohomology also. This definition has the disadvantage of applying only to the category of coarsenable spaces, which seems to be rather hard to characterize by an internal description.

One situation in which the notion of coarsening can be made concrete, however, is that in which X is (the underlying metric space of) a finitely generated discrete group Γ. The usual Baum-Connes conjecture for Γ relates to the equivariant K-homology of a certain space $\underline{E}\Gamma$, the universal space for proper Γ-actions. A model for $\underline{E}\Gamma$ as a Γ-finite simplicial complex, if one exists, will automatically be a bounded geometry complex in our sense, and will in fact be a coarsening of Γ. Thus the coarse K-homology of $|\Gamma|$ is a 'nonequivariant' version of the left-hand side of the ordinary Baum-Connes conjecture for Γ.

4. Cones

In this section we shall analyze the coarse homology of an *open cone*. Recall that if Y is a compact subset of the unit sphere in a normed space then the open cone on Y, denoted $\mathcal{O}Y$, is the set of all non-negative multiples of points in Y.

(4.1) Definition: *Let $r \colon \mathbf{R}^+ \to \mathbf{R}^+$ be a contractive map*[3] *such that $r(0) = 0$ and $r(\infty) = \infty$, and let $\mathcal{O}Y$ be an open cone. The* radial contraction *associated to r is the map $\rho \colon \mathcal{O}Y \to \mathcal{O}Y$ defined by*

$$\rho(ty) = r(t)y, \quad y \in Y.$$

Any radial contraction is coarsely homotopic to the identity map, and therefore induces the identity on coarse homology. On the other hand, radial contractions can be used to force more or less arbitrary maps from open cones to obey a growth

[3]That is, a Lipschitz map with Lipschitz constant less than or equal to 1.

condition; this is the content of the next lemma. These two properties taken together make radial contractions extremely useful in computations involving coarse homology.

(4.2) Lemma: *Let $\mathcal{OY}$ be an open cone, as above, and let Z be any metric space. Let $f\colon \mathcal{OY} \to \mathcal{Z}$ be a continuous (or pseudocontinuous) and proper map. Then there exists a radial contraction $\rho\colon \mathcal{OY} \to \mathcal{OY}$ such that $f \circ \rho$ is a coarse map.*

Proof: Fix $\varepsilon > 0$ such that, for any $x \in \mathcal{OY}$, the inverse image $f^{-1}(B(f(x);\varepsilon))$ is a neighbourhood of x. (Any ε will do if f is continuous; pseudocontinuity means, by definition, that there exists an ε with this property.) For each $s \geq 0$ let $K_s = \{x \in \mathcal{OY} : \|\S\| \leq \int\}$ and let $\delta(s)$ be a Lebesgue number for the covering

$$\{f^{-1}(B(f(x);\varepsilon)) : x \in K_s\}$$

of K_s. Then $\delta(s)$ is a monotone decreasing function of s. Define a function $r(t)$ by the equation

$$t = \int_0^{r(t)} \frac{(s+1)}{\min\{\delta(s), 1\}}\, ds.$$

By elementary calculus, r is a contractive map. Consider the radial contraction ρ defined by r. By construction, for any $R > 0$ there exists $r_0 > 0$ such that if $x, x' \in \mathcal{OY}$ with $d(x,x') < R$ and $\|x\| > r_0$, then $d(\rho(x),\rho(x')) < 1/\delta(\max\{\|x\|,\|x'\|\})$, and hence $d(f\circ\rho(x), f\circ\rho(x')) < \varepsilon$. Thus, whenever $d(x,x') < R$, one has $d(f \circ \rho(x), f\circ\rho(x')) < S$, where

$$S = \varepsilon + \operatorname{diam} f(K_{r_0}).$$

Hence $f \circ \rho$ is a coarse map. □

For example, suppose that $h\colon Y \to Y'$ is a homeomorphism. The induced map $\mathcal{O}h\colon \mathcal{O}Y \to \mathcal{O}Y'$ need not be a coarse map; in fact, it will be so if and only if the original h is Lipschitz. Nevertheless, by the above result there exists a radial contraction ρ such that both $\rho \circ h$ and $h^{-1} \circ \rho$ are coarse maps. Since ρ is coarsely homotopic to the identity, these maps are coarse homotopy equivalences[4] between $\mathcal{O}Y$ and $\mathcal{O}Y'$. This allows one to extend the scope of the definition of 'open cone'. Let Y be any finite-dimensional compact metrizable space; then it is a classical theorem that Y is homeomorphic to a subset of a sphere in a Euclidean space (see Theorem V.2 in [18]). The open cone $\mathcal{O}Y$ can be defined as the open cone on any such homeomorphic image, and will be well defined up to coarse homotopy equivalence.

We should now like to calculate the coarse M-homology of such an open cone. If Y is a finite complex, then $\mathcal{O}Y$ can be triangulated as a uniformly contractible bounded geometry complex, and so the calculation will follow from 3.8. However, we will need to consider open cones on more unpleasant metrizable spaces also; our proof of the Baum-Connes conjecture for a hyperbolic space will proceed *via* the open cone on its Gromov boundary.

It will be necessary to assume that the homology theory M satisfies the *strong excision axiom* [29, 22, 19]. We recall that this axiom states that for any pair (X, A) of compact metric spaces, the natural map $M_*(X, A) \to M_*(X \setminus A)$ is an isomorphism. For example, Steenrod homology satisfies strong excision, but

[4] In one or two places in the literature one can find statements which might mislead the unwary into believing that $\mathcal{O}Y$ and $\mathcal{O}Y'$ are *bornotopy* equivalent. This is not, in general, the case.

singular homology does not. Of more direct relevance to this paper is the fact that Kasparov's K-homology [20] satisfies the strong excision axiom.

(4.3) PROPOSITION: *If M_* is a generalized homology theory satisfying the strong excision axiom, then the coarsening map*

$$c\colon M_*(\mathcal{OY}) \to \mathcal{MX}_*(\mathcal{OY})$$

is an isomorphism for any finite-dimensional compact metric space Y.

PROOF: We consider $\mathcal{OY}$ to be embedded in $\mathbf{R}^n$. Form an anti-Čech system as follows: the cover $\mathcal{U}_i$ is made up of all the nonempty intersections $B \cap \mathcal{OY}$, where B runs over the set of open balls in $\mathbf{R}^n$ with centres at the integer lattice points and radius 3^i. Let X_i be the geometric realization of the nerve of $\mathcal{U}_i$. We also let $X_0 = \mathcal{OY}$ itself. There are obvious coarsening maps

$$X_0 \to X_1 \to X_2 \to X_3 \to \dots$$

and $MX_*(\mathcal{OY})$ is by definition the direct limit of the M-homology of this sequence. We make the following claims about this construction.

- *Claim 1:* Each X_i can be compactified to a space $\overline{X_i}$ obtained by adding a copy of Y as a set of points at infinity.[5]
- *Claim 2:* The coarsening maps can be extended by the identity on Y to give continuous maps $\overline{X_i} \to \overline{X_{i+1}}$.
- *Claim 3:* The extended map $\overline{X_i} \to \overline{X_{i+1}}$ is nullhomotopic.

Granted these three claims, the result follows. For consider the commutative diagram

$$\begin{array}{ccccccc}
\tilde{M}_*(Y) & \longrightarrow & \tilde{M}_*(\overline{X_i}) & \longrightarrow & M_*(X_i) & \xrightarrow{\partial} & \tilde{M}_{*-1}(Y) \\
\downarrow{=} & & \downarrow & & \downarrow & & \downarrow{=} \\
\tilde{M}_*(Y) & \longrightarrow & \tilde{M}_*(\overline{X_{i+1}}) & \longrightarrow & M_*(X_{i+1}) & \xrightarrow{\partial} & \tilde{M}_{*-1}(Y)
\end{array}$$

in which the rows are the long exact sequences in reduced M-homology arising from the pairs $(\overline{X_i}, Y)$ and the columns are the coarsening maps. Because we are using reduced homology, Claim 3 implies that the second vertical map is in fact zero. A diagram chase then shows that ∂ gives an isomorphism between $\mathrm{Im}(M_*(X_i) \to M_*(X_{i+1}))$ and $\tilde{M}_{*-1}(Y)$. Passing to the direct limit we obtain another commutative diagram

$$\begin{array}{ccc}
M_*(\mathcal{OY}) & \xrightarrow{\partial} & \tilde{M}_{*-1}(Y) \\
c\downarrow & & =\downarrow \\
MX_*(\mathcal{OY}) & \xrightarrow{\partial} & \tilde{M}_{*-1}(Y)
\end{array}$$

where both ∂'s are isomorphisms, and therefore c is an isomorphism too.

It remains to prove the claims. Claims 1 and 2 are straightforward; to check Claim 3, let us notice that there are continuous maps $\alpha_i\colon X_i \to \mathrm{Pen}(\mathcal{OY}; \in \cdot \ni^{\rangle})$ and $\beta_i\colon \mathrm{Pen}(\mathcal{OY}; \in \cdot \ni^{\rangle}) \to \mathcal{X}_{\rangle+\infty}$ defined as follows: α_i sends each vertex of X_i to

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[5] In the case of X_0, the compactification is simply the usual closed cone on Y, and is therefore contractible. The point of the construction is to obtain a 'pro'-version of this fact for the sequence of coarsenings X_i.

the centre of the corresponding ball in $\mathbf{R}^n$ and extends by linearity, and $\beta_i(x)$ is defined to be

$$\frac{\sum_p \varphi(|x-p|)\cdot p}{\sum_p \varphi(|x-p|)}$$

where the sum ranges over all integer lattice points p and $\varphi(r)$ is a positive continuous function equal to 1 for r small and equal to 0 for $r > 3^i$. (Notice that, because $x \in \mathrm{Pen}(\mathcal{OY}; \in \cdot \ni^{\rangle})$, the points p for which $\varphi(x-p) > 0$ do indeed define the vertices of a simplex in X_{i+1}.) Clearly $\mathrm{Pen}(\mathcal{OY}; \in \cdot \ni^{\rangle})$ can be compactified by adding Y at infinity, and both α_i and β_i extend continuously by the identity. Moreover, $\beta_i\alpha_i$ is homotopic to the coarsening map (by a linear homotopy). It is therefore enough to prove that the continuous extension of α_i is nullhomotopic, and this is so because $\overline{\mathrm{Pen}(\mathcal{OY}; \in \cdot \ni^{\rangle})}$ is star-shaped about 0. $\square$

5. K-Homology and Paschke duality

In this section we will recall the basic definitions of Kasparov's K-homology theory, together with the duality theory of Paschke that relates K-homology and K-theory.

Let X be a locally compact metrizable space. By an *X-module*, we will mean a separable Hilbert space equipped with a representation of the C^*-algebra $C_0(X)$ of continuous functions (tending to zero at infinity) on X. We shall say that an X-module is *non-degenerate* if the representation of $C_0(X)$ is non-degenerate, and as in [16] we shall say that an X-module is *standard* if it is non-degenerate and no non-zero function in $C_0(X)$ acts as a compact operator.

Let T be a bounded operator on an X-module H_X. By definition, T is *locally compact* if the operators $T\varphi$ and φT are compact for every $\varphi \in C_0(X)$. We say that T is T *pseudolocal* if $\varphi T \psi$ is a compact operator for all pairs of continuous functions on X with compact and disjoint supports. As Kasparov remarks (see Proposition 3.4 in [20]), T is pseudolocal if and only if the commutator $\varphi T - T\varphi$ is compact, for every $\varphi \in C_0(X)$. This makes it clear that the set of all pseudolocal operators on H_X is a C^*-algebra containing the set of all locally compact operators as a closed two-sided ideal. We shall use the notation

$$\Psi_0(X, H_X) = \text{pseudolocal operators on } H_X$$

and

$$\Psi_{-1}(X, H_X) = \text{locally compact operators on } H_X,$$

which is meant to suggest that pseudolocal operators should be thought of as abstract pseudodifferential operators of order ≤ 0, and locally compact operators as abstract pseudodifferential operators of order ≤ -1. Compare [1, 20].

Kasparov's K-homology groups $K_i(X)$ are generated by certain cycles modulo a certain equivalence relation. A cycle for $K_0(X)$ is given by a pair (H_X, F), comprised of an X-module H_X and a pseudolocal operator F on H_X such that $FF^* - I$ and $F^*F - I$ are locally compact.[6] A cycle for $K_1(X)$ is given by a similar pair (H_X, F), but with the additional requirement that F be self-adjoint. In both cases the equivalence relation on cycles is given by homotopy of the operator F, unitary equivalence, and direct sum of 'degenerate' cycles, these being cycles for

[6] Kasparov's theory allows for more a complicated sort of cycle, comprised of a pseudolocal operator $F\colon H_X \to H'_X$ between two different Hilbert spaces, but there is a simple trick to convert such a cycle to one of the simpler ones we are considering. See Section 1 of [14].

which $F\varphi - \varphi F$, $\varphi(F^*F - 1)$, and so on, are not merely compact but actually zero. See [20] for further details.

The K-homology of X may be related to the K-theory of the algebras of abstract pseudodifferential operators on X. Let H_X be an X-module. Then for $i = 0, 1$ there are maps

$$K_i(\Psi_0(X, H_X)/\Psi_{-1}(X, H_X)) \to K_{1-i}(X) \tag{5.1}$$

defined as follows. In the odd case ($i = 1$), we map a unitary U in $\Psi_0(X)/\Psi_{-1}(X)$, representing an odd K-theory class, to the even K-homology cycle (H_X, F), where F is any lifting of U to $\Psi_0(X)$. Similarly in the even case ($i = 0$), we map a projection P, representing an even K-theory class, to the odd K-homology cycle $(H_X, 2Q - 1)$, where Q is any self-adjoint lifting of P. It is readily checked that these operations respect the various equivalence relations and give a well-defined homomorphism of abelian groups. Now Paschke [23] (see also [14]) proved the following result:

(5.2) PROPOSITION: *If H_X is a standard X-module, then the map 5.1 is an isomorphism.*

Suppose now that a standard X-module H_X has been fixed. One can in fact identify not only the K-theory of the quotient $\Psi_0(X)/\Psi_{-1}(X)$ but also the K-theory of the algebras $\Psi_0(X)$ and $\Psi_{-1}(X)$ individually. It turns out that $K_i(\Psi_{-1}(X))$ is always zero, except when $i = 0$ and X is compact in which case it is $\mathbf{Z}$, and from this and Paschke's duality theorem it follows that $K_i(\Psi_0(X))$ is isomorphic to $\tilde{K}_{1-i}(X)$, the *reduced* K-homology of X.

It is also possible to interpret the boundary map in K-homology in terms of Paschke duality. We will summarize the results of [14] in a special case which suffices for our purposes. Suppose that X is locally compact but not compact, and that $\overline{X} = X \cup Y$ is a compactification of X, that is a compact space containing X as a dense open subspace. Then H_X can be thought of as an $\overline{X}$-module, because any representation of $C_0(X)$ on a Hilbert space has a unique extension to a representation of $C(\overline{X})$. It therefore makes sense to consider the algebra $\Psi_0(\overline{X}) \cap \Psi_{-1}(X)$, and one easily sees that this is an ideal in $\Psi_0(\overline{X})$. The relative form of Paschke's duality is then the following

(5.3) PROPOSITION: *The K-theory groups of the algebras $\Psi_0(\overline{X}) \cap \Psi_{-1}(X)$, $\Psi_0(\overline{X})$, and $\Psi_0(\overline{X})/(\Psi_0(\overline{X}) \cap \Psi_{-1}(X))$ are isomorphic to the reduced K-homology groups of Y, $\overline{X}$, and X, with a dimension shift in each case. Moreover, the isomorphisms transform the six-term exact sequence in K-theory arising from the C^*-algebra extension*

$$0 \to \Psi_0(\overline{X}) \cap \Psi_{-1}(X) \to \Psi_0(\overline{X}) \to \Psi_0(\overline{X})/(\Psi_0(\overline{X}) \cap \Psi_{-1}(X)) \to 0$$

into the six-term exact sequence in reduced K-homology arising from the pair $(\overline{X}, Y)$.

6. FORMULATION OF THE BAUM-CONNES CONJECTURE

In previous papers we have associated to each proper metric space X a C^*-algebra $C^*(X)$ [16, 15] whose K-theory is functorial for maps in UBB and is a bornotopy invariant. Our objective in this section is to construct an 'analytic index' map

$$\mu\colon K_*(X) \to K_*(C^*(X)). \tag{6.1}$$

This map is analogous to the assembly map in bounded L-theory.

We begin by briefly recalling some relevant definitions. Let X and Y be proper metric spaces. We refer the reader to Section 4 of [16] for the definition of the *support* of a bounded linear operator from an X-module to a Y-module. It is a closed subset of $X \times Y$, and generalizes the support of the distributional kernel in the C^∞ context. We recall that a bounded linear operator T on an X-module has *finite propagation* if there exists some $R > 0$ such that

$$(x, x') \in \operatorname{Supp}(T) \quad \Rightarrow \quad d(x, x') \leq R.$$

The least such R is called the *propagation* of T. The set of all locally compact, finite propagation operators on a non-degenerate X-module H_X is a $*$-algebra, and we denote by $C^*(X, H_X)$ the C^*-algebra obtained by closing it in the operator norm.

It is shown in Section 4 of [16] that if H_X is any non-degenerate X-module and H'_X is a standard X-module then there is a canonical homomorphism of K-theory groups

$$K_*(C^*(X, H_X)) \to K_*(C^*(X, H'_X)).$$

Moreover, if H_X is also standard, then this map is an isomorphism. So, at the level of K-theory at least, the C^*-algebra $C^*(X, H_X)$ does not depend on the choice of standard X-module H_X. For this reason we shall often suppose that that a particular standard X-module has been chosen, and write $C^*(X)$ in place of $C^*(X, H_X)$.

To define our index map (6.1) it will be convenient to introduce one more C^*-algebra. The set of all pseudolocal, finite propagation operators on a non-degenerate X-module H_X is a $*$-algebra, and we define

$$D^*(X, H_X) = \text{norm closure of the pseudolocal, finite propagation operators.}$$

As is the case for $C^*(X, H_X)$, the K-theory of $D^*(X, H_X)$ depends only slightly on the choice of H_X. We shall discuss this point in the next section. Here we make the following important observation.

(6.2) LEMMA: *Let X be any proper metric space. The inclusion of $D^*(X, H_X)$ into the C^*-algebra of pseudolocal operators $\Psi_0(X, H_X)$ induces an isomorphism of quotient C^*-algebras*

$$D^*(X, H_X)/C^*(X, H_X) \cong \Psi_0(X, H_X)/\Psi_{-1}(X, H_X).$$

This should be compared with the simple fact that every properly supported pseudodifferential operator can be perturbed by a properly supported smoothing operator so as to have support confined to a strip near the diagonal in $X \times X$.

PROOF: It suffices to show that every pseudolocal operator T can be written as a sum of a finite propagation operator and a locally compact operator.

Choose a partition of unity ψ_j^2 subordinate to a locally finite open cover of X by sets of uniformly bounded diameter. The series

$$T' = \sum_j \psi_j T \psi_j$$

converges in the strong topology. Indeed the partial sums are uniformly bounded in norm and $\sum_j \psi_j T \psi_j v$ converges (in fact it is a finite sum) for any vector v in the dense subset $C_c(X)H \subset H$. Clearly T' is an operator of finite propagation. On the other hand, if φ is a function of compact support, then

$$(T' - T)\varphi = \sum_j [\psi_j, T]\psi_j\varphi$$

is a finite sum of compact operators, hence is compact. Similarly, $\varphi(T - T')$ is compact, and thus $T - T'$ is locally compact. □

Now fix a standard X-module H_X, and consider the long exact K-theory sequence associated to the extension

$$0 \to C^*(X, H_X) \to D^*(X, H_X) \to D^*(X, H_X)/C^*(X, H_X) \to 0. \qquad (6.3)$$

The boundary map in this sequence is a map

$$K_{i-1}(D^*(X, H_X)/C^*(X, H_X)) \to K_i(C^*(X, H_X)).$$

But by the lemma above together with Paschke duality (5.2), the first group that appears here is simply $K_i(X)$. Thus we have obtained a homomorphism

$$\mu\colon K_i(X) \to K_i(C^*(X))$$

which is our analytic index map.

If X is compact, then $C^*(X)$ is just the algebra of compact operators and so $K_0(C^*(X)) \cong \mathbf{Z}$. In this case the map μ associates to each operator in $K_0(X)$ its usual Fredholm index in $K_0(C^*(X)) \cong \mathbf{Z}$. On the other hand, if X is a complete Riemannian manifold, D is a Dirac operator on a Clifford bundle S over X, and $[D]$ denotes its K-homology class, then $\mu[D]$ is the index of D in $K_*(C^*(X))$ as defined in [26]. To prove this one needs to verify that if Ψ is a 'chopping function' (as defined in [26]), then $\Psi(D)$ belongs to the algebra $D^*(X, L^2(S))$. This can be accomplished by a straightforward finite propagation speed argument.

We can now formulate a first version of the coarse Baum-Connes conjecture.

(6.4) Conjecture: *If X is a uniformly contractible bounded geometry complex then the map $\mu\colon K_*(X) \to K_*(C^*(X))$ is an isomorphism.*

In [26] this was conjectured for all uniformly contractible spaces, but the example of Dranishnikov, Ferry and Weinberger cited earlier shows that this more wide-ranging conjecture is false.

A more general version of the conjecture removes the hypothesis of uniform contractibility. To formulate it, recall that in the paper [16], the group $K^*(C^*(X))$ is made into a bornotopy invariant functor on UBB. Now let X be a complete path metric space. By 3.2, X admits an anti-Čech system $\mathcal{U}_i$ consisting of covers whose geometric realizations $|\mathcal{U}_i|$ are all metric simplicial complexes bornotopy-equivalent to X. The maps

$$\mu_i\colon K_*(|\mathcal{U}_i|) \to K_*(C^*(|\mathcal{U}_i|)) \cong K_*(C^*(X))$$

therefore give in the direct limit a map

$$\mu_\infty\colon KX_*(X) \to K_*(C^*(X)). \qquad (6.5)$$

(6.6) Conjecture: *For any complete path metric space of bounded coarse geometry, the map* (6.5) *is an isomorphism.*

This is essentially Conjecture 6.30 of [26]. Using Proposition 3.8, it implies Conjecture 6.4.

It is clear from the definitions that

$$\mu_\infty \circ c = \mu$$

where c is the coarsening map. In the Dranishnikov-Ferry-Weinberger example alluded to above, the map c fails to be injective, and therefore the map μ cannot be injective either. This example therefore leaves intact the possibility that Conjecture 6.6 is true for all spaces whether or not they are of bounded geometry, and indeed we will prove the conjecture for certain spaces of non-bounded geometry in this paper. Nevertheless, it seems safer to restrict the general statement to spaces of bounded geometry for the time being.

Remark: Note that conjecture 6.6 is coarse homotopy invariant, in the following sense. It is easy to see that the maps μ and μ_∞ are natural under coarse maps of X. Now the functors $X \mapsto KX_*(X)$ and $X \mapsto K_*(C^*(X))$ are both coarse homotopy invariant. It follows (by considering the obvious commutative diagram) that if X is coarse homotopy equivalent to X', and the conjecture holds for X, then it holds for X' also. We will make use of a version of this principle in our discussion of hyperbolic metric spaces.

Remark: Suppose that Γ is a finitely generated discrete group; how is the coarse Baum-Connes conjecture for the underlying metric space $|\Gamma|$ of Γ related to the usual Baum-Connes conjecture for Γ? One answer is as follows: we have seen that $KX_*(|\Gamma|) = K_*(\underline{E}\Gamma)$, the K-homology of the universal space for proper actions of Γ. On the analytic side there is a natural action of Γ on $C^*(|\Gamma|)$, and it is not hard to show that the fixed subalgebra $C^*(|\Gamma|)^\Gamma$ is Morita equivalent to the reduced group C^*-algebra $C^*_r(\Gamma)$. In fact there is a commutative diagram

$$\begin{array}{ccc} K_*^\Gamma(\underline{E}\Gamma) & \longrightarrow & K_*(C_r^*(\Gamma)) \\ \downarrow & & \downarrow \\ K_*(\underline{E}\Gamma) & \longrightarrow & K_*(C^*(|\Gamma|)) \end{array}$$

where the top line is the conjectured isomorphism of the usual Baum-Connes conjecture, the bottom line is the conjectured isomorphism of our coarse version, and the vertical arrows represent a process of forgetting the Γ-equivariance of the situation.

Remark: We have identified the boundary map in the K-theory exact sequence corresponding to the extension 6.3 with an assembly map. This suggests very strongly that the whole of this K-theory exact sequence should be thought of as an analytic analogue of the (simple, bounded) surgery exact sequence. In particular, $K_*(D^*(X))$ should be thought of as an analytic analogue of the simple structure set for X bounded over itself, and this can be made precise by relating it to the boundedly controlled model for the structure set discussed in [4]. It is an intriguing problem to determine what kind of 'structures' this 'structure set' is classifying.

7. Proof of the conjecture for open cones, nonpositively curved manifolds, and affine buildings

In this section we shall work in the category UBC.

The following is a slight weakening of the notion of 'rescaleable space' in [26].

(7.1) DEFINITION: *A proper metric space X is* scaleable[7] *if there is a continuous and proper map $f\colon X \to X$, coarsely homotopic to the identity map, such that*

$$d(f(x), f(x')) \leq \frac{1}{2} d(x, x')$$

for all $x, x' \in X$.

Every cone is scaleable, as is every complete, simply connected, non-positively curved Riemannian manifold, every tree and affine building (so long as the trees and buildings are locally finite).

Here is the main result of this section.

(7.2) THEOREM: *If X is a scaleable space then the the index map*

$$\mu\colon K_*(X) \to K_*(C^*(X))$$

is an isomorphism.

This gives an immediate proof of the following cases of Conjecture 6.6.

(7.3) COROLLARY: *If X is an open cone on a finite-dimensional compact metric space Y then the coarse Baum-Connes conjecture 6.6 holds for X.*

PROOF: This follows immediately from the fact that μ is an isomorphism together with the result that coarsening gives an isomorphism on the K-homology of a cone (4.3). □

(7.4) COROLLARY: *If X is a complete, simply connected, non-positively curved Riemannian manifold then the coarse Baum-Connes conjecture (6.6) holds for X.*

PROOF: This argument was already given in [15] and [16]; we use the fact that the exponential map is a coarse homotopy equivalence to reduce to the case of Euclidean space, which (for example) may be thought of as a cone on a sphere. □

(7.5) COROLLARY: *If X is a bounded geometry complex which is either a tree or an affine Bruhat-Tits building then the coarse Baum-Connes conjecture holds for X.*

PROOF: Every tree or affine building is uniformly contractible. So the result follows from 3.8. □

Affine buildings and trees need not be of bounded geometry. However it is not difficult to extend the previous corollary to all buildings and trees by using non-positive curvature to broaden Proposition 3.8 to these cases. Alternatively, one can adapt the argument of the next section so as to apply to buildings. But we shall not pursue this matter here.

We begin by looking the dependence of $K_*(D^*(X, H_X))$ on H_X, and its functoriality in UBC. We shall use the following well-known result of Voiculescu [30].

(7.6) THEOREM: *Let A be a separable C^*-algebra and let H and H' be two separable Hilbert spaces equipped with non-degenerate representations of A. If the representation of A on H' has the property that no non-zero element of A acts as a compact operator on H then there is an isometry $V\colon H \to H'$ such that $Va - aV$ is a compact operator for every $a \in A$.* □

[7] This definition is closely related to 'Lipschitz contractibility' in the sense of Gromov [12, page 25].

In the concrete cases of interest to us it is usually possible to construct such isometries V explicitly, but Voiculescu's theorem provides a convenient general framework.

(7.7) LEMMA: *Let X and Y be proper metric spaces, let H_X be a non-degenerate X-module and let and H_Y be a standard Y-module. Let*

$$H_Y^\infty = H_Y \oplus H_Y \oplus H_Y \oplus \ldots .$$

If $f\colon X \to Y$ is a continuous coarse map then there is an isometry

$$W\colon H_X \to H_Y^\infty$$

such that

$$\mathrm{Supp}(W) \subseteq \{(x,y) : d(f(x),y) < R\},$$

for some $R > 0$, and $\varphi W - W\varphi \circ f$ is compact, for every $\varphi \in C_0(Y)$.

PROOF: Let $\mathcal{U} = \{U_j\}$ and $\mathcal{V} = \{V_k\}$ be locally finite open covers of X and Y by sets of uniformly bounded diameter with the property that f maps each U_j into some V_{k_j}. Let

$$H_X^j = \overline{C_0(U_j)H_X} \quad \text{and} \quad H_Y^k = \overline{C_0(V_k)H_Y}.$$

By Voiculescu's theorem (applied to $A = C(\overline{V_{k_j}})$), for each j there is an isometry $W_j\colon H_X^j \to H_Y^{k_j}$ such that $\varphi W_j - W_j\varphi \circ f$ is compact, for every $\varphi \in C_0(Y)$.

If $\psi_1^2, \psi_2^2, \ldots$ is any partition of unity subordinate to $\mathcal{U}$. then we can define W by

$$Wf = W_1\psi_1 f \oplus W_2\psi_2 f \oplus \ldots \quad ,$$

which has all the required properties. □

Conjugation with W gives a map

$$\mathrm{Ad}(W)\colon D^*(X, H_X) \to D^*(Y, H_Y^\infty).$$

If W and W' are two isometries satisfying the conclusion of the lemma then $W'W^* \in D^*(Y, H_Y^\infty)$. It follows that the induced maps $\mathrm{Ad}(W)_*$ and $\mathrm{Ad}(W')_*$ on K-theory are equal (see Lemma 3.2 of [16], for example).

Let us apply these remarks to the identity map $1\colon X \to X$. If H and H' are two standard X-modules then we obtain canonical maps

$$K_*(D^*(X, H^\infty)) \leftrightarrow K_*(D^*(X, H'^\infty))$$

which are inverse to one another. In view of this, when convenient we shall drop the term H_X^∞ from our notation, writing $D^*(X)$ in place of $D^*(X, H_X^\infty)$. Selecting one standard module for each X we obtain a functor on UBC, and the same construction makes $K_*(D^*(X)/C^*(X))$ a functor on UBC. By Paschke duality (5.2), this latter functor is simply Kasparov's K-homology, $K_{*-1}(X)$.

(7.8) LEMMA: *The functor $K_*(D^*(X))$ is coarse homotopy invariant.*

PROOF: Let $h\colon X \times [0,1] \to Y$ be a coarse homotopy, and denote by $Z = X \times_h [0,1]$ the product space $X \times [0,1]$ endowed with the warped product metric introduced in [15]. As noted in that paper, to prove that h_0 and h_1 induce the same map on K-theory it suffices to show that the projection map $Z \to X$ induces an isomorphism on K-theory. But consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & C^*(Z) & \to & D^*(Z) & \to & D^*(Z)/C^*(Z) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^*(X) & \to & D^*(X) & \to & D^*(X)/C^*(X) & \to & 0 \end{array}$$

where the vertical maps are given by conjugation with an isometry chosen as in Lemma 7.7. By the main theorem in [15], the first vertical map induces an isomorphism on K-theory, and by the identifications

$$K_*(D^*(Z)/C^*(Z)) \cong K_{*-1}(Z) \quad \text{and} \quad K_*(D^*(X)/C^*(X)) \cong K_{*-1}(X),$$

the third vertical map also induces an isomorphism on K-theory. So by the Five Lemma the middle vertical map also induces an isomorphism on K-theory. □

(7.9) Lemma: *The map $\mu\colon K_*(X) \to K_*(C^*(X))$ is an isomorphism if and only if $K_*(D^*(X)) = 0$.*

Proof: The map μ, as defined in the previous section, is precisely the identification $K_{*-1}(X) \cong K_*(D^*(X)/C^*(X))$ followed by the boundary map in K-theory for the short exact sequence

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0.$$

So the lemma follows from the long exact sequence in K-theory. □

Proof of Theorem 7.2: By the previous lemma it suffices to show that if X is a scaleable space then $K_*(D^*(X)) = 0$. Let $f\colon X \to X$ be a contraction of X, as in Definition 7.1. For simplicity, let us suppose that f is one-to-one (as it is for cones, if we make the obvious choice for f). We shall deal with the slight extra complications of the general case at the end of this proof. Let S be a countable dense subset of X which is invariant under f. Form the standard X-module

$$H = \ell^2 S \oplus \ell^2 S \oplus \dots,$$

and let

$$H' = H \oplus H \oplus \dots.$$

There is an obvious inclusion of H as the first summand in H', and a corresponding inclusion $I\colon D^*(X,H) \to D^*(X,H')$. It suffices to show that this inclusion induces the zero map on K-theory, since by the remarks following Lemma 7.7 it also induces an identification $K_*(D^*(X,H)) \cong K_*(D^*(X,H'))$. Define an isometry on $\ell^2 S$ by mapping the basis vector corresponding to $s \in S$ to the basis vector corresponding to $f(s) \in S$ (it is here that we are using our assumption that f be one-to-one). Taking the direct sum of this isometry with itself infinitely many times we obtain an isometry V of H, and an isometry V' of $\mathcal{H}'$. Conjugation with V' induces the map f_* on $K_*(D^*(X,H'))$, and thanks to coarse homotopy invariance this is the identity map.

Consider now the following map from operators on H to operators on H':

$$I \oplus \mathrm{Ad}(V) \oplus \mathrm{Ad}(V^2) \oplus \cdots : T \mapsto T \oplus \mathrm{Ad}(V)(T) \oplus \mathrm{Ad}(V^2)(T) \oplus \cdots .$$

We observe that this maps $D^*(X,H)$ into $D^*(X,H')$. The reason is that if T has propagation R then $\mathrm{Ad}(V)(T)$ has propagation at most $R/2$, thanks to the fact that f contracts distances by at least $1/2$. So if φ and ψ are functions on X with disjoint compact supports then $\varphi \,\mathrm{Ad}(V^n)(T)\psi = 0$ for large enough n. Having made this observation, the rest of the argument is routine. At the level of K-theory we have

$$\begin{aligned}
(I \oplus \mathrm{Ad}(V) \oplus \mathrm{Ad}(V^2) \oplus \cdots)_* &= I_* + (0 \oplus \mathrm{Ad}(V) \oplus \mathrm{Ad}(V^2) \oplus \cdots)_* \\
&= I_* + (\mathrm{Ad}(V) \oplus \mathrm{Ad}(V^2) \oplus \cdots)_* \\
&= I_* + \mathrm{Ad}(V')_*(I \oplus \mathrm{Ad}(V) \oplus \mathrm{Ad}(V^2) \oplus \cdots)_* \\
&= I_* + (I \oplus \mathrm{Ad}(V) \oplus \mathrm{Ad}(V^2) \oplus \cdots)_*
\end{aligned}$$

We subtract $(I \oplus \mathrm{Ad}(V) \oplus \mathrm{Ad}(V^2) \oplus \cdots)_*$ from everything to get $I_* = 0$.

To complete the proof of the theorem we consider the case where f is not necessarily one-to-one. In this case we replace $\ell^2 S$ with $\ell^2 S \times A$, where A is any countably infinite set, and define an isometry on $\ell^2 S \times A$ by mapping the basis vector labelled by (a, s) to a basis vector corresponding to some $(a', f(s))$, where a' is chosen so that this map on basis vectors is one-to-one. Of course $\ell^2 S \times A$ is an X-module in the obvious way, and we proceed as above. □

REMARK: One can give a somewhat different proof of the conjecture for open cones, more in the spirit of certain topological approaches to the Novikov conjecture, as follows. One proves first that the functor $Y \mapsto K_*(C^*(\mathcal{O}\mathcal{Y}))$ defines a generalized homology theory on the category of compact metrizable spaces, and that there is a natural transformation from this functor to the functor $Y \mapsto K_{*-1}(Y)$ which is an isomorphism on spheres. One shows further that this functor has the continuity properties summarized in the Steenrod axioms [22, 19], and one then appeals to a uniqueness theorem for Steenrod homology theories to complete the proof.

8. Proof of the conjecture for hyperbolic metric spaces

Gromov [11] has introduced a notion of *hyperbolicity* for general metric spaces. A metric space is hyperbolic if it is 'negatively curved on the large scale'. In this section we will show that the coarse Baum-Connes conjecture is true for any (geodesic and locally compact) hyperbolic metric space X.

Our proof will proceed by way of the *Gromov boundary* $Y = \partial_g X$ of the hyperbolic space X. This finite-dimensional compact metrizable space is the boundary of a 'radial compactification' of X, and there are natural 'exponential' and 'logarithm' maps between X and the open cone $\mathcal{O}\mathcal{Y}$ on its Gromov boundary. We intend to show that these maps define coarse homotopy equivalences in a certain sense. Since the coarse Baum-Connes conjecture is coarse homotopy invariant, this will prove the conjecture for X by reducing it to the conjecture for $\mathcal{O}\mathcal{Y}$, which was proved in the preceding section.

We recall a definition of the Gromov boundary. Fix a basepoint $*$ in X and consider the set of geodesic rays in X originating from $*$. Two such rays are said to be *equivalent* if they lie within a finite distance of one another; it can be shown that there is an absolute constant 8δ such that if γ_1 and γ_2 are equivalent, then $d(\gamma_1(t), \gamma_2(t)) < 8\delta$ for all t. The Gromov product of two such rays may be defined by

$$(\gamma_1|\gamma_2) = \lim_{s,t\to\infty} (\gamma_1(s)|\gamma_2(t))$$

where we note that the expression $(\gamma_1(s)|\gamma_2(t))$ is an increasing function of s and t, and the metric on Y has the property that there are constants $A > 0$ and $\varepsilon > 0$ for which

$$A^{-1} d([\gamma_1], [\gamma_2]) < e^{-\varepsilon(\gamma_1|\gamma_2)} < A d([\gamma_1], [\gamma_2]).$$

Details of these constructions may be found in [10, 11].

Let $\mathcal{O}\mathcal{Y}$ denote the open cone on the Gromov boundary Y. We define the *exponential map* $\exp\colon \mathcal{O}\mathcal{Y} \to \mathcal{X}$ in the natural way: to a pair $([\gamma], t)$ assign the point $\gamma(t) \in X$. (In order to have a well-defined map we choose one representative from each equivalence class; the choice of representative cannot affect the exponential map by more than 8δ.) The exponential map is highly expansive, but by 4.2 it is always possible to choose a radial shrinking ρ of $\mathcal{O}\mathcal{Y}$ so that $\exp\circ\rho$ is a coarse map. The main result of this section is then

(8.1) PROPOSITION: *The map*

$$\exp\circ\rho\colon \mathcal{O}\mathcal{Y}\to\mathcal{X}$$

induces an isomorphism on coarse K-homology and on the K-theory of the corresponding C^*-algebras.

To avoid wearisome repetition, let us simply say that a coarse map is a *weak equivalence*[8] if it has the property described in proposition 8.1, namely that it induces an isomorphism on coarse K-homology and on the K-theory of the corresponding C^*-algebras.

(8.2) COROLLARY: *The coarse Baum-Connes conjecture 6.6 is true for any hyperbolic metric space X.*

PROOF: Consider the diagram

$$\begin{array}{ccc} KX_*(\mathcal{O}\mathcal{Y}) & \longrightarrow & K_*(C^*(\mathcal{O}\mathcal{Y})) \\ \downarrow & & \downarrow \\ KX_*(X) & \longrightarrow & K_*(C^*(X)) \end{array}$$

in which the vertical maps are induced by $\exp\circ\rho$, the horizontal maps are the assembly maps μ_∞, and the first horizontal map is an isomorphism by the results of the previous section. □

We will prove proposition 8.1 in two stages: first we will prove that the exponential map is a weak equivalence onto its range, and secondly we will prove that the range of the exponential map is weakly equivalent to the whole space X. (Notice that the exponential map need not in general be surjective; consider the example of a tree formed from the real line by attaching a branch of length $|n|$ to each integer point n.)

Let $R\subseteq X$ denote the range of the exponential map. A map $\log\colon R\to\mathcal{O}\mathcal{Y}$ can be defined by sending a point $x\in R$ to the pair $([\gamma],|x|)$, where γ is some choice of geodesic ray that passes within 8δ of x. Let us check that the choices involved here do not affect the map by more than a bounded amount. If γ_1 and γ_2 are two rays both of which pass within 8δ of x, then by definition, $(\gamma_1|\gamma_2)>|x|-8\delta$ and so the distance in the open cone between $([\gamma_1],x)$ and $([\gamma_2],x)$ is bounded above by

$$A|x|e^{-\varepsilon(|x|-8\delta)}$$

which is bounded by some constant independent of $|x|$.

(8.3) LEMMA: *The map* $\log\colon R\to\mathcal{O}\mathcal{Y}$ *defined above is a coarse map.*

PROOF: It is sufficient to prove that if x_1 and x_2 belong to R and have $|x_1|=|x_2|=r$ say, then there is an estimate of the form

$$d(\log x_1,\log x_2)\le\Phi(d(x_1,x_2))$$

[8]This is really the analogue of the notion of *homology* equivalence in ordinary topology, rather than of weak *homotopy* equivalence. But since fundamental group problems are not really relevant here, we ask the reader to permit us this convenient abuse of language.

for some universal function Φ. Let γ_1 and γ_2 be rays passing within 8δ of x_1 and x_2 respectively; then

$$(\gamma_1|\gamma_2) \geq (x_1|x_2) - 16\delta \geq r - 16\delta - d(x_1, x_2)/2.$$

Therefore

$$d(\log x_1, \log x_2) \leq Are^{-\varepsilon r - 16\delta} e^{\varepsilon d(x_1,x_2)/2}$$

and this is bounded by $Ae^{\varepsilon(16\delta + d(x_1,x_2)/2)}$. □

(8.4) Lemma: *The composite map* $\log \circ \exp \circ \rho \colon \mathcal{OY} \to \mathcal{OY}$ *is weakly equivalent to the identity.*

Proof: The map $\log \circ \exp$ is bornotopy equivalent to the identity, and the map ρ is coarsely homotopy equivalent to the identity. □

Now recall from [15] the notion of a *generalized coarse homotopy*, this being a map which satisfies all the requirements of definition 2.3 except that it need only be pseudocontinuous rather than continuous. If the space Y is a path metric space, then any generalized coarse homotopy from X to Y can be factored into a coarse homotopy followed by a bornotopy-equivalence. Thus a generalized coarse homotopy whose range is a path metric space will be a weak equivalence.

(8.5) Lemma: *The composite map* $\exp \circ \rho \circ \log \colon R \to X$ *is weakly equivalent to the inclusion* $R \hookrightarrow X$.

Proof: Let ρ_t, with $\rho_0 = \rho$ and $\rho_1 =$ identity, be a linear coarse homotopy of ρ to the identity map. It is not hard to check that $\exp \circ \rho_t \circ \log$ is then a generalized coarse homotopy of $\exp \circ \rho \circ \log$ to $\exp \circ \log$, which in turn is bornotopic to the inclusion map. □

We must now prove that the range R of the exponential map is weakly equivalent to the whole space X. Let us begin by noticing that R is *coarsely convex*[9], that is, there is a constant 16δ such that given any two points x_0 and x_1 in R, any geodesic segment $[x_0, x_1]$ lies within 16δ of R. Now we have

(8.6) Lemma: *For any coarsely convex subset W of a hyperbolic metric space X, there is a map $\pi \colon X \to W$ such that $d(x, \pi(x)) \leq d(x, W) + \delta$; moreover, this map is unique up to bornotopy.*

Note that such a map π is 'coarsely contractive', that is, $d(\pi(x_0), \pi(x_1)) < d(x_0, x_1) + 2\delta$.

Proof: It suffices to prove that there is a constant $a > 0$ such that if w_1 and w_2 are points of W within distance $d(x, W) + \delta$ of x, then $d(w_1, w_2) < a$. Let p be the point on the geodesic segment $[w_1, w_2]$ that is closest to x. Then, for $i = 1, 2$,

$$d(x, w_i) < d(x, W) + \delta < d(x, p) + \delta + c$$

where c is the constant implied in the statement that W is coarsely convex. However, by Lemma 17 and Proposition 21 of Chapter 2 of [10],

$$d(x, p) < (w_1|w_2)_x + 4\delta < d(x, p) + 5\delta + c - d(w_1, w_2)/2.$$

Therefore

$$d(w_1, w_2) < 10\delta + 2c$$

as required. □

[9] The importance of this point was indicated by M. Gromov in a helpful conversation with one of the authors, for which we are grateful.

Now take $W = R$ in the above lemma. Define a family of maps $\pi_t\colon X \to X$ by interpolating 'linearly' between π_0, the identity map, and $\pi_1 = \pi$; in other words, $\pi_t(x)$ is the point at distance $td(x, \pi(x))$ along a geodesic segment from x to $\pi(x)$. By the approximate convexity of hyperbolic spaces (Proposition 25 in Chapter 2 of [10]), the maps π_t are uniformly coarsely contractive. In order to show that the π_t give a generalized coarse homotopy between π and the identity map (and thereby to show that the inclusion of R into X is a weak equivalence) it will suffice to prove

(8.7) Lemma: *The map $\Pi\colon X \times [0,1] \to X$ representing the family π_t defined above is proper.*

Proof: Suppose not. Then there is a sequence of points (x_j, t_j) in $X \times [0,1]$ such that the points (x_j, t_j) are all at mutual distance at least 2 whereas the $\pi_{t_j}(x_j)$ remain in a compact set. Extracting a subsequence and using the contractiveness of the π's, we can suppose that there is a sequence x_j which tends to a point $x_\infty \in \partial X$ while $\pi(x_j)$ remains in a compact set. This, however, is impossible. For the point x_∞ is represented by a geodesic ray γ which belongs to R. Let $y_j = \gamma((x_j|\gamma)) \in R$. By the construction of y_j and the thinness of geodesic triangles, there is a constant 16δ such that $d(x_j, y_j) < |x_j| - |y_j| + 16\delta$. Since $\pi(x_j)$ remains in a compact set, there is a constant C such that $d(x_j, \pi(x_j)) > |x_j| - C$. Since $d(x_j, \pi(x_j))$ is (up to an error of 2δ) the shortest distance from x_j to any point of R, $d(x_j, \pi(x_j)) < d(x_j, y_j) + 2\delta$, and it follows that

$$|y_j| < 18\delta + C.$$

But $|y_j| = (x_j|\gamma) \to \infty$ as $j \to \infty$, because x_j tends to the point at infinity represented by the geodesic ray γ, and this contradiction completes the proof. $\square$

Combining Lemmas 8.4, 8.5, and 8.7, we complete the proof of Proposition 8.1, and therefore of the coarse Baum-Connes conjecture for hyperbolic metric spaces.

9. Appendix: Remarks on ideal boundaries

In this section we want to show how the machinery developed in this paper may be applied to clarify the relationship between the analytic index map μ and the K-homology theory of suitable 'coronas' of a space X. This idea was of some importance in the paper [26], and we would like to discuss it here both because it is rather simple from the present point of view and because the article [13] by one of us to which reference was made at the relevant point of [26] is no longer available in the form in which it was cited.

The situation of interest is the following. X is a proper metric space as usual, and $\overline{X} = X \cup Y$ is a *corona compactification* of X, which means that the continuous functions on $\overline{X}$ restrict to functions on X whose gradient, measured relative to the metric on X, tends to zero at infinity. In this situation proposition 5.18 of [26] (compare also section 3 of [28]) states in our notation that $C^*(X) \subseteq \Psi_0(\overline{X}) \cap \Psi_{-1}(X)$, and thus by relative Paschke duality (5.3) we get a map

$$b : K_i(C^*(X)) \to \tilde{K}_{i-1}(Y).$$

The result that we would like to prove states that $b \circ \mu$ is just the boundary map $\partial\colon K_i(X) \to \tilde{K}_{i-1}(Y)$ in reduced K-homology; this result was used in the proof of Proposition 5.29 in [26]. To prove it, we need to note first that $D^*(X) \subseteq \Psi_0(\overline{X})$; this can be proved by exactly the same methods as those used for $C^*(X)$. Now consider the diagram of six-term exact sequences in K-theory associated to the diagram of extensions

$$\begin{array}{ccccc} C^*(X) & \longrightarrow & D^*(X) & \longrightarrow & D^*(X)/C^*(X) \\ \downarrow & & \downarrow & & \downarrow \\ \Psi_0(\overline{X}) \cap \Psi_{-1}(X) & \longrightarrow & \Psi_0(\overline{X}) & \longrightarrow & \Psi_0(\overline{X})/(\Psi_0(\overline{X}) \cap \Psi_{-1}(X)) \end{array}$$

The connecting map associated to the lower sequence is ∂ by the relative Paschke duality theorem, and that associated to the upper sequence is μ by definition. The two quotient algebras both have K-theory isomorphic to the K-homology of X, and the right-hand vertical map induces the identity[10] on $K_*(X)$. The desired result follows immediately.

[10] One can show that the map is actually an isomorphism, but we do not need this fact.

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