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Amenable group actions and the Novikov conjecture

By *Nigel Higson* and *John Roe* at University Park

Abstract. Guoliang Yu has introduced a property of discrete metric spaces which guarantees the existence of a uniform embedding into Hilbert space. We show that the metric space underlying a finitely generated discrete group has this property if and only if the action of the group on its Stone-Čech compactification is topologically amenable. It follows from Yu's work that if BG is a finite complex, and if G acts amenably on some compact Hausdorff space, then the Novikov higher signature conjecture is true for G.

1. Introduction

A function $f : X \to Y$ between metric spaces is a *uniform embedding* if for every $R > 0$ there is a constant $S > 0$ such that

$$d(x_1, x_2) \leqq R \quad \Rightarrow \quad d\big(f(x_1), f(x_2)\big) \leqq S$$

and

$$d\big(f(x_1), f(x_2)\big) \leqq R \quad \Rightarrow \quad d(x_1, x_2) \leqq S,$$

for every $x_1, x_2 \in X$. Thus in the language of coarse geometry [10], f is a uniform embedding if it is a coarse equivalence from X to $f[X]$. Gromov has drawn attention to the problem of embedding separable metric spaces into Hilbert space [6]. In response to a question of Gromov [6], p. 213, Bekka, Cherix and Valette proved in [4] that every countable amenable group admits an affine, isometric action on a Hilbert space H with the property that for every $v \in H$, $\|g \cdot v\| \to \infty$ as $g \to \infty$ (let us call such actions *metrically proper*). It follows that the metric space underlying a finitely generated amenable group admits a uniform embedding into Hilbert space. As it happens, many other groups admit metrically proper actions on Hilbert space, for instance free groups and Coxeter groups [9], while still more groups admit uniform embeddings into Hilbert space.

The first author proved with Kasparov [7] that if a group G admits a metrically proper action on Hilbert space then the Baum-Connes conjecture [3], and hence in particular the Novikov conjecture, holds for G. In a recent paper [15], Guoliang Yu has combined the main idea of [7] with techniques of coarse geometry to prove that the *coarse* Baum-Connes conjecture [8], [13] is true for any bounded geometry metric space which admits a uniform embedding into Hilbert space. By a well-known 'descent' argument he deduces from this that if BG is a finite complex, and if G, with its word length metric, admits a uniform embedding into Hilbert space, then the Novikov conjecture is true for G.

Until recently there was no known example of a separable metric space which does not admit a uniform embedding into Hilbert space (compare [6], p. 218). There is still noknown example of a finitely generated group which does not embed. In view of this, Yu's result on the Novikov conjecture is very striking. The purpose of the present note is to emphasize even further the importance of Yu's theorem by analyzing one class of groups to which it applies. These are groups whose underlying metric spaces satisfy a Følner-type condition that Yu calls property A in [15]. Just as amenable groups may be shown to admit a metrically proper action on Hilbert space, so also may property A groups be shown to admit a uniform embedding into Hilbert space (see [15], Theorem 2.2). We shall prove the following result:

1.1. Theorem. *Let G be a finitely generated group. The metric space underlying G has Yu's property* A *if and only if the left translation action of G on its Stone-Čech compactification is topologically amenable.*

The notion of topological amenability [2] is reviewed in Section 2. In view of the universal property of the Stone-Čech compactification, the following is an immediate consequence of Yu's main result:

1.2. Theorem. *Let G be a discrete group for which BG is a finite complex. If G admits an amenable action on some compact Hausdorff space then the Novikov higher signature conjecture holds for G.*

Various examples of amenable actions are presented in Section 4 of this note. Because of the breadth of these examples, Theorem 1.2 covers a good deal of what is known concerning the Novikov conjecture. In fact there is a certain amount of evidence for, and no known counterexample to, the (probably rash) conjecture that *every* countable group admits a topologically amenable action on some compact Hausdorff space.

2. Amenable group actions

A recent monograph of Anantharaman-Delaroche and Renault [2] explores in detail a variety of different notions of amenability for topological groupoids. The definitions below specialize their basic Definition 2.2.7 of [2] to the context of a countable group acting on a compact Hausdorff space.

Throughout this note, Z will denote a countable set.

2.1. Definition. Denote by $\mathrm{prob}(Z)$ the set of Borel probability measures on Z, or in other words, the set of functions $b : Z \to [0,1]$ such that $\sum_{z \in Z} b(z) = 1$. We shall view prob(Z) as a subset of $\ell^1(Z)$ and equip it with the weak*-topology (recall that $\ell^1(Z)$ is the Banach space dual of $c_0(Z)$). We shall denote by $\|\cdot\|_1$ the usual norm on $\ell^1(Z)$.

If Z is equipped with an action of a group G (in our main example Z will be equal to G and the action will be left-translation) then there is an induced action of G on prob(Z) defined by $gb(z) = b(g^{-1}z)$.

2.2. Definition. Let G be a countable discrete group and let X be a compact Hausdorff space on which G acts by homeomorphisms. The action is said to be (*topologically*) *amenable* if there exists a sequence of weak*-continuous maps $b^n : X \to \mathrm{prob}(G)$ such that for every $g \in G$,

$$\lim_{n\to\infty} \sup_{x\in X} \|gb^n_x - b^n_{gx}\|_1 = 0.$$

Anantharaman-Delaroche and Renault consider in [2] only spaces X which are second countable, whereas we shall consider larger spaces. A primary purpose of their monograph is to show that various definitions of amenability are equivalent to one another. For instance, if G is a countable group acting on a compact and second countable space then the results 2.2.11, 3.3.8 and 6.2.15 of [2] show that our Definition 2.2 is equivalent to a broad variety of other definitions, some measure theoretic and some functional analytic. One should be cautious about carrying these results over to the non-second countable case. But in this note, when dealing with non-second countable spaces, we shall only use Definition 2.2 and no other notion of amenability.

Recall that the *Stone-Čech compactification* of a set Z is a compact Hausdorff space βZ, equipped with an inclusion of the discrete space Z as an open dense subset, which has the universal property that any function from Z into a compact Hausdorff space X extends to a continuous map from βZ into X. We are going to study the translation action of a countable group G on its Stone-Čech compactification. To construct this, fix $g \in G$ and consider the map $L_g : G \to G$ which is left translation by g. By the universal property of the Stone-Čech compactification there is a unique continuous map $\tilde{L}_g : \beta G \to \beta G$ making the diagram

$$\begin{array}{ccc} G & \xrightarrow{L_g} & G \\ \big\downarrow & & \big\downarrow \\ \beta G & \xrightarrow[\tilde{L}_g]{} & \beta G \end{array}$$

commute. The map $g \mapsto \tilde{L}_g$ is an action of G on βG by homeomorphisms.

Observe that if X is any compact Hausdorff G-space then there is a G-map from βG to X, obtained by first mapping G into X along an orbit of the G-action and then extending this map to βG using the universal property of the Stone-Čech compactification. It is clear from Definition 2.2 that if G acts amenably on a compact space X, and if $Y \to X$ is any map of G-spaces, then the action of G on Y is amenable. Hence:

2.3. Proposition. *If G admits an amenable action on any compact Hausdorff space then its action on the Stone-Čech compactification βG is amenable.* $\square$

3. Property A

In this section we shall suppose that the (countable) set Z has been equipped with a metric whose underlying topology is the discrete topology of Z.

3.1. Definition. Denote by $\mathrm{fin}(Z)$ the set of all finite, non-empty subsets of Z.

3.2. Definition (See [15], Definition 2.1)**.** The discrete metric space Z has *property* A if there are maps $A_n : Z \to \mathrm{fin}(Z \times \mathbb{N})$, where $n = 1, 2, \ldots$, such that

(1) for each n there is some $R > 0$ for which

$$A_n(z) \subset \{(z', j) \in Z \times \mathbb{N} \mid d(z, z') < R\},$$

for every $z \in Z$; and

(2) for every $K > 0$,

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z) \triangle A_n(w)|}{|A_n(z) \cap A_n(w)|} = 0.$$

Whether or not a discrete metric space has property A depends only on the coarse equivalence class of the metric on Z (see [10] for a discussion of coarse geometry). So if for example G is a finitely generated group, and if we equip it with the word-length metric associated to some finite generating set, then whether or not G, as a metric space, has property A is independent of the choice of generating set.

Our objective in this section is to prove the following result:

3.3. Theorem. *The action of a finitely generated group G on its Stone-Čech compactification is amenable if and only if G, considered as a metric space with the word-length metric, has property* A.

Most of the proof will be divided between Lemmas 3.5 through 3.8 below, but we begin with an easy calculation:

3.4. Lemma. *Let Z be a discrete metric space and suppose that for $n = 1, 2, \ldots$ there are given maps $A_n : Z \to \mathrm{fin}(Z \times \mathbb{N})$. Then for every $K > 0$,*

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z) \triangle A_n(w)|}{|A_n(z) \cap A_n(w)|} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z) \triangle A_n(w)|}{|A_n(z)|} = 0.$$

In addition, the vanishing of either limit implies that

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(w)|}{|A_n(z)|} = 1. \quad \square$$

A metric space Z has *bounded geometry* if for every $C > 0$ there is an absolute bound on the number of elements in any ball within Z of radius C. The following lemma gives an equivalent form of property A which involves *functions* on Z rather than *subsets* of $Z \times \mathbb{N}$.

3.5. Lemma. *If Z is a discrete metric space of bounded geometry then Z has property* A *if and only if there is a sequence of maps $a^n : Z \to \mathrm{prob}(Z)$ such that*

(1) *for every n there is some $R > 0$ with the property that for every $z \in Z$,*

$$\mathrm{Supp}(a_z^n) \subset \{z' \in Z \mid d(z, z') < R\};$$

and

(2) *for every $K > 0$, $\lim_{n\to\infty} \sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 = 0$.*

Proof. Suppose maps a^n as above exist. Since Z has bounded geometry, the hypothesis (1) implies that for each fixed n, the number of elements of Z within the support of a_z^n is uniformly bounded. By an approximation argument, we may therefore assume that for each n there is a natural number M such that if $z \in Z$, then the function $a_z^n \in \ell^1(Z)$ assumes only values in the range

$$\frac{0}{M}, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M}{M}.$$

Define $A_n(z) \subset Z \times \mathbb{N}$ by the relation

$$(z', j) \in A_n(z) \quad \Leftrightarrow \quad \frac{j}{M} \leqq a_z^n(z').$$

Observe that the $A_n(z)$ satisfy part (1) of Definition 3.2 and that $|A_n(z)| = M$, for every z. In addition,

$$|A_n(z) \triangle A_n(w)| = M \cdot \|a_z^n - a_w^n\|_1 = |A_n(z)| \cdot \|a_z^n - a_w^n\|_1.$$

It follows from Lemma 3.4 and hypothesis (2) that

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z) \triangle A_n(w)|}{|A_n(z) \cap A_n(w)|} = \lim_{n\to\infty} \sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 = 0,$$

and so Z has property A.

Suppose now that Z has property A. Define $a^n : Z \to \mathrm{prob}(Z)$ as follows:

$$a_z^n(z') \cdot |A_n(z)| = |\{j : (z', j) \in A_n(z)\}|.$$

Then

$$\|a_z^n \cdot |A_n(z)| - a_w^n \cdot |A_n(w)|\,\|_1 \leqq |A_n(z) \triangle A_n(w)|$$

and so

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \left\| a_z^n - a_w^n \cdot \frac{|A_n(w)|}{|A_n(z)|} \right\|_1 \leqq \lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z) \triangle A_n(w)|}{|A_n(z)|} = 0.$$

But according second part of Lemma 3.4,

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(w)|}{|A_n(z)|} = 1,$$

and so $\lim\limits_{n\to\infty} \sup\limits_{d(z,w)<K} \|a_z^n - a_w^n\|_1 = 0$, as required. □

We are now going to specialize Z to be a finitely generated group G, equipped with its word-length metric. Note that such a space has bounded geometry.

3.6. Lemma. *A finitely generated group G has property* A *if and only if there is a sequence of maps $b^n : G \to \operatorname{prob}(G)$ $(n = 1, 2, \ldots)$ with the following properties*:

(1) *For every n there is a finite subset $F \subset G$ such that* $\operatorname{Supp}(b_g^n) \subset F$, *for every $g \in G$; and*

(2) *for every $g \in G$,*

$$\lim_{n\to\infty} \sup_{h\in G} \|g b_h^n - b_{gh}^n\|_1 = 0.$$

Proof. Assume that G has property A and choose maps $a^n : G \to \operatorname{prob}(G)$ as in Lemma 3.5. It follows from the definition of the word length metric that for every $g \in G$,

$$\lim_{n\to\infty} \sup_{h\in G} \|a_{h^{-1}}^n - a_{h^{-1}g^{-1}}^n\|_1 = 0.$$

Now define $b_h^n = h a_{h^{-1}}^n$. Then

$$\begin{aligned} \|g b_h^n - b_{gh}^n\|_1 &= \|g h a_{h^{-1}}^n - g h a_{h^{-1}g^{-1}}^n\|_1 \\ &= \|a_{h^{-1}}^n - a_{h^{-1}g^{-1}}^n\|_1. \end{aligned}$$

It follows that the maps $b^n : G \to \operatorname{prob}(G)$ have the required properties.

On the other hand, if maps b^n are assumed to exist then the above argument can be reversed to define maps $a^n : G \to \operatorname{prob}(G)$ as in Lemma 3.5. □

Let us now reformulate the definition of amenability in the case of the action of a group G on its Stone-Čech compactification.

3.7. Lemma. *A countable group G acts amenably on its Stone-Čech compactification if and only if there is a sequence of maps $b^n : G \to \operatorname{prob}(G)$ such that*

(1) *for each* n, *the image of the map* b^n *is contained within a weak*-compact subset of* $\mathrm{prob}(G)$; *and*

(2) *for every* $g \in G$,

$$\lim_{n\to\infty} \sup_{h\in G} \|gb_h^n - b_{gh}^n\|_1 = 0.$$

Proof. Suppose that the action is amenable. There are then maps

$$b^n : \beta G \to \mathrm{prob}(G)$$

as in Definition 2.2, and restricting them to $G \subset \beta G$ gives maps as in the statement of the lemma.

If on the other hand maps $b^n : G \to \mathrm{prob}(G)$ are defined, as above, then by item (1) and the universal property of the Stone-Čech compactification the maps b^n extend to maps from βG into $\mathrm{prob}(G)$. We note that

$$\sup_{h\in G} \|gb_h^n - b_{gh}^n\|_1 = \sup_{x\in\beta G} \|gb_x^n - b_{gx}^n\|_1.$$

This is because G is dense in βG, because the map $x \mapsto gb_x^n - b_{gx}^n$, from βG into $\ell^1(G)$, is weak*-continuous, and because the norm function on $\ell^1(G)$ is weak*-semicontinuous (norm-closed balls are weak*-closed). $\square$

Comparing Lemmas 3.6 and 3.7 we see that we have almost proved the main result of this section, Theorem 3.3. Indeed, since the set of probability measures supported on a finite subset $F \subset G$ is weak*-compact, it follows immediately from the two lemmas that if G has property A then G acts amenably on βG. The converse follows from the following simple approximation argument. If $b \in \ell^1(Z)$ and F is a subset of Z then denote by $b|_F$ the function

$$b|_F(z) = \begin{cases} b(z) & \text{if } z \in F, \\ 0 & \text{if } z \notin F. \end{cases}$$

3.8. Lemma. *Let* Z *be a discrete set. For every* w^**-compact subset* B *of* $\mathrm{prob}(Z)$ *and every* $\varepsilon > 0$ *there is a finite set* $F \subset Z$ *such that* $\|b - b|_F\|_1 < \varepsilon$, *for every* $b \in B$.

Proof. Fix $\varepsilon > 0$, and for each finite set $H \subset Z$ let

$$U_H = \{b \in \mathrm{prob}(Z) \mid \|b|_H\|_1 > 1 - \varepsilon\}.$$

The sets form a weak*-open cover of $\mathrm{prob}(Z)$, and hence the compact set B is covered by finitely many of the U_H. Take F to be the union of the finite sets H associated to the finite cover. $\square$

Proof of Theorem 3.3. Suppose that G acts amenably on βG, and let

$$b^n : G \to \mathrm{prob}(G)$$

be maps as in Lemma 3.7. By Lemma 3.8 we can approximate the maps b^n by maps $\tilde{b}^n : G \to \mathrm{prob}(G)$ satisfying the conditions of Lemma 3.6: for each n we simply define

$$\tilde{b}^n_z = \frac{b^n_z|_F}{\|b^n_z|_F\|_1},$$

for a suitably large finite set $F \subset G$. □

4. Consequences

As we explained in the introduction, Theorem 3.3, combined with the results in Yu's paper [15], has the following consequence:

4.1. Theorem. *If G is a discrete group for which BG is a finite complex, and if G admits a topologically amenable action on a compact Hausdorff space, then the Novikov conjecture is true for G.*

Here are some groups to which the theorem applies.

- If G is amenable then of course the action on a one-point space is amenable.

- If G is a word-hyperbolic group then Adams proves in [1] that the action of G on its Gromov boundary is measure theoretically amenable, for every quasi-invariant measure on the boundary. But it is shown in Chapter 3 of [2] that such 'universal' measurewise amenability is equivalent to topological amenability. See also Appendix B of [2].

- If $\mathscr{G}$ is a connected Lie group then there is a closed, amenable subgroup $\mathscr{H}$ of $\mathscr{G}$ such that the homogeneous space $\mathscr{G}/\mathscr{H}$ is compact [5]. By Theorem 2.2.14 of [2], the action of $\mathscr{G}$ on $\mathscr{G}/\mathscr{H}$ is topologically amenable. By Proposition 5.1.1 of [2], if G is any discrete subgroup of $\mathscr{G}$ then the action of G on $\mathscr{G}/\mathscr{H}$ is also topologically amenable.

- If G acts amenably on a compact space then so does any subgroup (by restriction). Yu shows in [15], Proposition 2.6 that the class of finitely generated groups which admit an amenable action on a compact space is closed under semidirect products.

It would be interesting to prove that Coxeter groups admit amenable actions on compact spaces (although as we noted in the introduction, the Novikov conjecture is already known for them [7]). One possible approach would be to apply the following observation, for which we recall a definition of Gromov [6], Section 1E:

4.2. Definition. A metric space Z has *asymptotic dimension* no more than $d \in \mathbb{N}$, denoted *as* $\dim_+(X) \leqq d$, if for every $D > 0$ there is a cover $\mathscr{U}$ of X for which:

(1) $\sup_{U \in \mathscr{U}} \mathrm{diam}(U) < \infty$; and

(2) $\mathscr{U}$ may be partitioned into $d+1$ families,

$$\mathscr{U} = \mathscr{U}_1 \cup \cdots \cup \mathscr{U}_{d+1},$$

with the property that if U and U' are distinct members of some $\mathcal{U}_j$, and if $x \in U$ and $x' \in U'$, then $d(x, x') > D$.

In an earlier paper [14], Yu proved the Novikov conjecture for groups G with finite classifying space and finite asymptotic dimension. In fact this result is also covered by his more recent work:

4.3. Lemma. *Suppose that Z is a discrete, bounded geometry metric space with finite asymptotic dimension. Then Z has property* A.

Proof. Suppose that Z has finite asymptotic dimension and let $D > 0$. Associated to the cover $\mathcal{U}$ guaranteed by Definition 4.2 there is a partition of unity $\{\varphi\}$ for Z with these properties:

(1) each φ is Lipschitz, with Lipschitz constant no more than $2/D$;

(2) $\sup_{\varphi} \operatorname{diam}\bigl(\operatorname{Supp}(\varphi)\bigr) < \infty$; and

(3) for every $z \in Z$, no more than $d + 1$ of the values $\varphi(z)$, as φ ranges throughout the partition of unity, are non-zero.

Pick a point z_φ in the support of each φ, and then define

$$a_z^D = \sum_{\varphi} \varphi(z) \delta_{z_\varphi} \in \operatorname{prob}(Z),$$

where $\delta_{z_\varphi} \in \ell^1(Z)$ denotes the point mass at z_φ. It follows from (1) and (3) that the maps $a^D : Z \to \operatorname{prob}(Z)$ have the property that

$$\lim_{D \to \infty} \sup_{d(z,w) < K} \|a_z^D - a_w^D\|_1 = 0.$$

It follows from (2) that for every D there is a constant R such that

$$\operatorname{Supp}(a_z^D) \subset \{z' \in Z \mid d(z, z') < R\}.$$

Therefore, by Lemma 3.5, the metric space Z has property A. $\square$

It is clear that there is some room for maneuver in the above argument; we could allow d to be a slowly increasing function of D, rather than a constant. This suggests a whole family of 'secondary' notions of asymptotic dimension (compare Gromov's remarks in [6], page 29). We do not develop these ideas further here.

5. Other issues

As we noted, one might make the following conjecture: *every countable group G acts amenably on some compact space, and hence on its Stone-Čech compactification.* Now one of the equivalent definitions of amenability, taken from [2] is that for every quasi-invariant measure on X there should exist an equivariant, measurable map from X to the state space of the C^*-algebra $\ell^\infty(G)$. But if $X = \beta G$ then X is, by at least one definition of the Stone-Čech compactification, a subset of the state space of $\ell^\infty(G)$ (namely it is the set of pure states). So there is not only a measurable, equivariant map $\beta G \to \ell^\infty(G)^*$, there is a continuous one! Unfortunately, in this context, continuity does not apparently imply measurability (and even if it did, we should still have to take into account the non-second countability of βG before applying any results from [2]).

In order to investigate this further it might be useful to pass to the world of metric spaces (much in the spirit of this note and of Yu's work). If Z is a discrete, bounded geometry metric space then denote by $B(X)$ the set of $Z \times Z$ complex matrices $[b_{zw}]$ such that $\sup_{z,w} |b_{zw}| < \infty$ and such that for each matrix there is an $R > 0$ for which if $d(z,w) > R$ then $b_{zw} = 0$. The set $B(W)$ is a $*$-algebra under the usual matrix operations. It is represented naturally on $\ell^2(Z)$ and we denote by $C_u^*(Z)$ its C^*-algebra completion in this representation. Up to Morita equivalence this is the "rough C^*-algebra" of Z [10].

If Z is the metric space underlying a finitely generated group then it is not hard to see that $C_u^*(Z)$ is isomorphic to the reduced crossed product C^*-algebra $C(\beta G) \rtimes G$. Now if G acts amenably on βG then this C^*-algebra is nuclear. So the above conjecture prompts us to ask the following question: *if Z is a discrete, bounded geometry metric space then is the rough C^*-algebra of Z a nuclear C^*-algebra*? Besides the connection to the Novikov conjecture, a positive answer to this question would imply that the reduced C^*-algebra of a countable discrete group is exact; this is problem 6 of Wassermann's list of open questions about exact C^*-algebras [12], Chapter 10. It seems likely that property A implies nuclearity, but we have not investigated this matter fully.

In conclusion we note that the isomorphism $C(\beta G) \rtimes G \cong C_u^*(Z)$, and also our theorem concerning Yu's property A, links Yu's work with other recent work of J.-L. Tu [11], who proves the analogue of the Baum-Connes conjecture for amenable groupoids, again using the techniques of [7].

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Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
e-mail: higson@math.psu.edu
e-mail: roe@math.psu.edu

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