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Mapping Surgery to Analysis I: Analytic Signatures★

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Abstract. We develop the theory of analytically controlled Poincaré complexes over C^*-algebras. We associate a *signature* in C^*-algebra K-theory to such a complex, and we show that it is invariant under bordism and homotopy.

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1. Introduction

Let M be a smooth, closed, oriented Riemannian manifold of dimension $4k$. The *signature operator* on M is a certain linear, elliptic, partial differential operator whose Fredholm index is the cohomological signature of M. It can be defined for M of any dimension n, but unless $n=4k$ its Fredholm index is zero. However for any n the signature operator has a 'higher index,' lying in the C^*-algebra K-theory group $K_n(C_r^*(\pi_1 M))$, and this higher index is decidedly non-trivial. It is closely related to the 'higher signatures' of M studied in manifold theory, and a basic result, which is fundamental for the application of C^*-algebras to the Novikov higher signature conjecture [2,9] is that this higher index, like the ordinary signature of M, is an *oriented homotopy invariant* [7,8].

This is the first of three articles which will explore in some detail the relationship between the K-theoretic higher index of the signature operator and the L-theoretic signature of M, around which surgery theory [21] is organized and which is the usual context for discussions of manifolds and their signatures. Kasparov and others [1,9,10] have studied a C^*-algebraic *assembly map*

$$\mu\colon K_*(M)\to K_*(C_r^*\pi_1 M)$$

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which maps K-homology to C^*-algebra K-theory and takes the class of the signature operator in $K_n(M)$ to the higher index of the signature operator in $K_n(C^*_r\pi_1 M)$. The assembly map fits into a long exact sequence

$$K_{n+1}(C^*_r\pi_1 M) \longrightarrow K_{n+1}(D^*_\pi M) \longrightarrow K_n(M) \longrightarrow K_n(C^*_r\pi_1 M),$$

and we asserted in [5], albeit in a slightly different context, that this can be regarded as an analytic counterpart to the surgery exact sequence [21]. In particular we asserted that the term $K_*(D^*_\pi M)$ can be regarded as an analytic counterpart to the structure set $\mathcal{S}(M)$ in surgery theory. Our aim is to provide justification for this by constructing explicitly a commutative[1] diagram from the usual surgery exact sequence to our analytic surgery exact sequence:

$$\begin{array}{ccccccc}
L_{n+1}(\pi_1 M) & \longrightarrow & \mathcal{S}(M) & \longrightarrow & \mathcal{N}(M) & \longrightarrow & L_n(\pi_1 M) \\
\downarrow \gamma & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
K_{n+1}(C^*_r\pi_1 M) & \longrightarrow & K_{n+1}(D^*_\pi M) & \longrightarrow & K_n(M) & \longrightarrow & K_n(C^*_r\pi_1 M)
\end{array}$$

The diagram (all of whose parts will be explained later on) provides the fullest possible account of homotopy invariance for the higher index of the signature operator. For instance an element of $\mathcal{S}(M)$ is a homotopy equivalence of oriented manifolds $M' \sim M$; we shall give an explicit geometric construction of a corresponding element in $K_{n+1}(D^*_\pi M)$ and show that its image in $K_n(M)$ is the difference of the classes of the signature operators of M and M'. Homotopy invariance of the index of the signature operator is thus a consequence of exactness in the bottom row of the diagram.

In order to carry out our constructions we need to associate 'signature invariants' in C^*-algebra K-theory groups to various kinds of 'Poincaré duality complexes', and we need to check that when the complex in question arises from a smooth manifold, the signature invariant that we have constructed agrees with the higher index of the (de Rham) signature operator. The construction is somewhat elaborate. In this paper we shall show how to associate a C^*-algebraic signature to what we call an *analytically controlled Hilbert–Poincaré complex* – a complex of Hilbert spaces and linear maps which admits a 'Poincaré duality' operator. The construction is an abstract one at the level of operator K-theory and analysis, and should be thought of as an analytic version of the construction of the signature of an algebraic Poincaré complex by Mischenko and Ranicki.

In the second paper of this series we shall show how such analytically controlled Hilbert–Poincaré complexes can be obtained from natural *geometric* constructions, the simplest example being the de Rham complex of a

[1]Strictly speaking, the diagram commutes only modulo certain powers of 2. See paper III in this series for details.

complete, bounded geometry Riemannian manifold.[2] In the third and final paper we shall use these geometric constructions to define our map from surgery theory (specifically, from the structure set $\mathcal{S}(M)$ associated to a compact smooth manifold M) to K-theoretic invariants of C^*-algebras, and we shall complete the construction of the commutative diagram alluded to above. A preliminary version of our main results appears in [20].

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2. Complexes of Hilbert Modules

Let us fix, for the first several sections, a C^*-algebra C. We shall be considering complexes of Hilbert C-modules[3]

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n. \tag{1}$$

The C^*-algebra C will be unital, and the differentials b_j will be bounded, adjointable operators. Later on in the paper we shall relax these hypotheses and develop a somewhat different set of hypotheses.

The *homology* of the complex is, of course, the sequence of quotient spaces obtained upon dividing the kernel of b_j by the image of b_{j+1}. Note that, since the differentials need not have closed range, our homology spaces are *not* necessarily Hilbert modules themselves.

Let E denote the direct sum of all the modules E_j, and denote by b the corresponding direct sum of the operators b_j; note that $b^2 = 0$.

PROPOSITION 2.1. *The complex* (2.1) *is acyclic (that is, its homology groups are all zero) if and only if the self-adjoint operator $B = b^* + b$ is invertible*

Note that according to a theorem of Mischenko [11, Chapter 3], the terminology 'invertible' is unambiguous: a self-adjoint operator on a Hilbert module E gives a bijective map from E to itself if and only if it is invertible in the C^*-algebra $\mathfrak{B}(E)$.

Proof. Suppose that the homology of the complex (E, b) is zero. To prove that B is invertible it suffices to check that B is surjective (the kernel of a self-adjoint operator is orthogonal to the image, so surjectivity implies injectivity). The range of b, being equal to its kernel, is closed. Furthermore,

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[2] The idea of considering C^*-algebraic signatures in this context was introduced by Novikov [17, 18] and was further developed by Kasparov and Mischenko.

[3] We refer to the book [11] for the basic theory of Hilbert modules and their operators.

by another theorem of Mischenko, if an operator has closed range then so does its adjoint, and so

$$E = \mathrm{Im}(b^*) \oplus \mathrm{Ker}(b) = \mathrm{Im}(b^*) \oplus \mathrm{Im}(b).$$

Surjectivity follows easily from this. Conversely, if B is invertible, and if $bv = 0$, then from $v = Bw$, for some w, we get

$$\|b^*w\|^2 = \|\langle bb^*w, w\rangle\| = \|\langle bBw, w\rangle\| = 0,$$

and hence $v = Bw = bw$. Thus every cycle v is a boundary bw, and hence the complex (E, b) is acyclic. □

DEFINITION 2.2. *A* chain map *from one complex of Hilbert modules to another, denoted*

$$A\colon (E, b) \to (E', b'),$$

is a family of adjointable operators $A_j\colon E_j \to E'_j$ such that $b'_j A_j = A_{j-1} b_j$ for all j. The mapping cone complex *associated to A is the complex*

$$E''_0 \xleftarrow{b_1} E''_1 \xleftarrow{b_2} \cdots \xleftarrow{b_{n+1}} E''_{n+1}.$$

for which

$$E''_j = E_{j-1} \oplus E'_j \quad \textit{and} \quad b''_j = \begin{pmatrix} b_{j-1} & 0 \\ A_{j-1} & -b'_j \end{pmatrix} \qquad (j = 1, \ldots, n+1).$$

Here we set $E_{-1} = 0$ and $E'_{n+1} = 0$.

It is a simple and well-known matter of algebra to check that:

LEMMA 2.3. *A map of complexes A is an isomorphism on homology if and only if its mapping cone complex is acyclic.* □

We shall use this basic observation several times in the paper. A simple application is as follows. From the complex (E, b) in (2.1) we obtain a *dual complex*

$$E_n \xleftarrow{b_n^*} E_{n-1} \xleftarrow{b_{n-1}^*} \cdots \xleftarrow{b_1^*} E_0 \tag{2}$$

which we shall denote (E, b^*).

LEMMA 2.4. *A chain map $A\colon (E, b) \to (E', b')$ is an isomorphism on homology if and only if the same is true of the adjoint map $A^*\colon (E', b'^*) \to (E, b^*)$.*

Proof. The chain map A induces an isomorphism on homology if and only if its mapping cone complex is acyclic. Since the mapping cone complex of the adjoint map identifies with the adjoint of the mapping cone complex of A, the lemma reduces to showing that a complex is acyclic if and only if its adjoint complex is acyclic. But this last assertion is an immediate consequence of Proposition 2.1. □

3. Hilbert–Poincaré Complexes

The following definition, which is adapted from the algebraic theory of surgery, particularly Mischenko's notion of an algebraic Poincaré complex [12, 13], explains what we mean by a 'complex of Hilbert modules which obeys Poincaré duality.' Because the spaces with which we are working are vector spaces over $\mathbb{C}$, many of the complexities of the algebraic theory of surgery disappear (in surgery one works with modules over various rings – usually integral group rings). So our definition will look simpler than Mischenko's. Note also that in the context of vector spaces over $\mathbb{C}$ the various algebraic notions of Poincaré complex – in particular the 'symmetric complexes' of Mischenko [12] and the 'quadratic complexes' of Ranicki [19] – agree with one another.

DEFINITION 3.1. An n-dimensional *Hilbert–Poincaré complex* (over the C^*-algebra C) is a complex of finitely generated (and therefore projective) Hilbert C-modules

$$E_a \xleftarrow{b} E_{a+1} \xleftarrow{b} \cdots \xleftarrow{b} E_{n-a} \tag{3}$$

together with adjointable operators $T\colon E_p \to E_{n-p}$ such that

(i) if $v \in E_p$ then $T^*v = (-1)^{(n-p)p}Tv$;
(ii) if $v \in E_p$ then $Tb^*v + (-1)^p bTv = 0$; and
(iii) T induces an isomorphism from the homology of the dual complex

$$E_{n-a} \xleftarrow{b^*} E_{n-a-1} \xleftarrow{b^*} \cdots \xleftarrow{b^*} E_a \tag{4}$$

to the homology of the complex (E, b).

Notice the indexing of the above complexes. In geometric examples one has $a = 0$, so that the complexes are indexed in the same way as the complexes (2.1) and (2.2). But it is occasionally useful to consider other (integral) values of a. A systematic way of doing this is to consider two-way infinite complexes with the property that $E_j = 0$, for all but finitely many j. Our arguments will be consistent with this approach.

We shall refer to the operator T as a *duality* operator on the complex (E, b) since it induces 'Poincaré duality' on the homology of (E, b). Geometric examples related to the Poincaré duality of manifolds will be given in the second paper of this series.

Remark 3.2. One might think that in the definition of Hilbert–Poincaré complex it would be more appropriate to require that T be a chain homotopy equivalence from (E, b^*) to (E, b) (rather than just a homology isomorphism). But it is not hard to see that this apparently stronger definition is equivalent to the one we have given.

Let (E, b, T) be an n-dimensional Hilbert–Poincaré complex over C. We are going to associate to it a *signature* in the K-theory group $K_n(C)$. Because of Bott periodicity there are only two cases to consider: n even and n odd.

DEFINITION 3.3. Let (E, b, T) be an n-dimensional Hilbert–Poincaré complex and let

$$n = \begin{cases} 2l & \text{if } n \text{ is even,} \\ 2l+1 & \text{if } n \text{ is odd.} \end{cases}$$

Denote by $S\colon E \to E$ the bounded operator such that

$$Sv = i^{p(p-1)+l}Tv \quad (v \in E_p)$$

where $i = \sqrt{-1}$.

By introducing the operator S we eliminate most of the ± 1 signs from the definition of a Hilbert–Poincaré complex:

LEMMA 3.4. *If (E, b, T) is a Hilbert–Poincaré complex and S is defined from T as above then $S = S^*$ and $bS + Sb^* = 0$.*

Proof. This is a straightforward calculation. □

Our construction of the signature is based on the following observation.

LEMMA 3.5. *Let (E, b, T) be a Hilbert–Poincaré complex and let $B = b^* + b$. The self-adjoint operators $B \pm S\colon E \to E$ are invertible.*

Proof. Consider the mapping cone complex of the chain map $S\colon (E, -b^*) \to (E, b)$. Its differential is the operator

$$b_S = \begin{pmatrix} b & 0 \\ S & b^* \end{pmatrix}.$$

Since S is an isomorphism on homology the mapping cone complex is acyclic, and therefore the operator $B_S = b_S{}^* + b_S$ on $E \oplus E$ is invertible. Now bearing in mind that S is self-adjoint we see that

$$B_S = \begin{pmatrix} b^* + b & S \\ S & b + b^* \end{pmatrix}.$$

The symmetry which exchanges the two copies of E in the direct sum $E \oplus E$ commutes with B_S, and since

$$\begin{pmatrix} b^* + b & S \\ S & b + b^* \end{pmatrix} \begin{pmatrix} v \\ v \end{pmatrix} = \begin{pmatrix} (b^* + b + S)v \\ (b^* + b + S)v \end{pmatrix},$$

we see that B_S identifies with $B + S$ on the $+1$ eigenspace of the exchange symmetry. Similarly it identifies with $B - S$ on the -1 eigenspace. □

Using this lemma we can define our notion of signature. As we observed above, there are two cases:

DEFINITION 3.6. Let (E, b, T) be an *odd*-dimensional Hilbert–Poincaré complex. Its *signature* is the class in $K_1(C)$ of the invertible operator

$$(B + S)(B - S)^{-1} \colon E_{ev} \to E_{ev},$$

where $E_{ev} = \oplus_p E_{2p}$.

If a is an invertible self-adjoint element in a C^*-algebra A, we will use the terminology *positive projection* of a for the projection $\varphi(a)$, where φ is a continuous function on the spectrum of a equal to 1 on the positive part of the spectrum and 0 elsewhere.

DEFINITION 3.7. Let (E, b, T) be an *even*-dimensional Hilbert–Poincaré complex. Its *signature* is the class in $K_0(C)$ determined by the formal difference $[P_+] - [P_-]$ of the positive projections of $B + S$ and $B - S$.

PROPOSITION 3.8. *If a Hilbert–Poincaré complex over C is acyclic, its signature is zero.*

Proof. Since neither of the complexes (E, b) or (E, b^*) has any homology, the operator tT is a duality operator for all real t. So Lemma 3.5 shows that the operators $B \pm tS$ are invertible, for all t. In particular the operators $B + S$ and $B - S$ are path connected through self-adjoint, invertible operators and the operator $(B + S)(B - S)^{-1} \colon E_{ev} \to E_{ev}$ is path-connected to the identity through invertible operators. This proves the lemma. □

The connection with the usual notion of signature is brought out clearly by the next result:

PROPOSITION 3.9. *For a Hilbert–Poincaré complex over C, of dimension $n=2l$, whose homology vanishes except in dimension l, the signature that we have defined is equal to the formal difference of the positive and negative projections of the invertible self-adjoint operator S acting on the middle-dimensional homology.*

It is implicit in this statement that (in this case) the middle-dimensional homology can be identified as a (finitely generated) Hilbert C-module.

Proof. The result is obvious for a Hilbert–Poincaré complex that is concentrated in dimension l – that is, for which all the Hilbert spaces E_j, $j \neq l$, vanish. Thus it will suffice to show that a Hilbert–Poincaré complex whose homology vanishes except in dimension l can be written as the direct sum of two such complexes, one of which is concentrated in dimension l and the other of which is acyclic.

Let (E, b, T) be a Hilbert–Poincaré complex whose homology is concentrated in dimension l. Let E' be the subcomplex defined by setting $E'_j = 0$ for $j \neq l$ and $E'_l = \mathrm{Ker}(b_l) \cap \mathrm{Ker}(b^*_{l+1})$. Since b_l and b^*_{l+1} have closed range, Mischenko's Theorem shows that E'_l is an orthogonally complemented submodule. Both E' and its orthogonal complement are Hilbert–Poincaré subcomplexes.

The inclusion of E' into E is a homology isomorphism. To see this, we need only check dimension l. If $x \in E'_l$ and x vanishes in the homology of E, then $x = b_{l+1}y$ for some y. Then $b^*_{l+1}b_{l+1}y = 0$ and, taking the inner product with y, we find that $x = 0$. So the inclusion $E' \to E$ is injective on homology. On the other hand, suppose that $z \in E_l$ with $b_l z = 0$. Write $z = x + y$ with $x \in E'_l = \mathrm{Ker}(b_l) \cap \mathrm{Ker}(b^*_{l+1})$ and $y \in \mathrm{Ker}(b_l) \cap \mathrm{Ker}(b^*_{l+1})^\perp$. Since b^*_{l+1} has closed range, it follows that b_{l+1} has closed range also and $\mathrm{Ker}(b^*_{l+1})^\perp = \mathrm{Im}(b_{l+1})$. We conclude that z is homologous to $x \in E'_l$ and so the inclusion $E' \to E$ is surjective on homology.

Now $E = E' \oplus (E')^\perp$ as Hilbert–Poincaré complexes; since the inclusion $E' \to E$ is a homology isomorphism, the complex $(E')^\perp$ is acyclic. This completes the proof. □

4. Homotopy Invariance of the Signature

We introduce an equivalence relation of *homotopy* on Hilbert–Poincaré complexes as follows:

DEFINITION 4.1. Let (E, b, T) and (E', b', T') be a pair of n-dimensional Hilbert–Poincaré complexes. A *homotopy equivalence* between them is a chain map $A\colon (E, b) \to (E', b')$, comprised of adjointable operators, which induces an isomorphism on homology, and for which the two chain maps

$$ATA^*, T'\colon (E'_{n-*}, b'^*) \to (E'_*, b')$$

induce the same map on homology.

Remark 4.2. We would obtain the same notion of equivalence if we required A to be a chain equivalence, and required ATA^* and T' to be chain homotopic. Compare Remark 3.2.

Our objective in this section is to prove the following homotopy invariance principle:

THEOREM 4.3. *If (E, b, T) and (E', b', T') are homotopy equivalent, n-dimensional Hilbert–Poincaré complexes then their signatures in $K_n(C)$ are equal.*

We begin by analyzing a simpler type of homotopy equivalence.

DEFINITION 4.4. Let (E, b) be a complex of Hilbert modules. An *operator homotopy* of Hilbert–Poincaré structures on (E, b) is a norm continuous family of adjointable operators T_s $(s \in [0, 1])$ such that each (E, b, T_s) is a Hilbert–Poincaré complex.

LEMMA 4.5. *Operator homotopic Hilbert–Poincaré complexes have the same signature.*

Proof. Consider first the odd-dimensional case. Form the self-adjoint operators S_s from T_s, as in Definition 3.3. From the *resolvent identity*

$$(B \pm S_{s_1} \pm i)^{-1} - (B \pm S_{s_2} \pm i)^{-1} = (B \pm S_{s_1} \pm i)^{-1}(S_{s_2} - S_{s_1})(B \pm S_{s_2} \pm i)^{-1},$$

we see that the resolvent operators of $B \pm S_s$ vary norm-continuously with s. It follows that $(B + S_s)(B - S_s)^{-1}$ varies norm-continuously with s, so the lemma follows from the homotopy invariance property of K-theory.

The proof in the even-dimensional case is similar. □

LEMMA 4.6. *If a duality operator T on a Hilbert–Poincaré complex (E, b) is operator homotopic to $-T$ then the signature of (E, b, T) is zero.*

Proof. We shall consider the odd-dimensional case; as with the previous lemma the even-dimensional case is similar. Let T_s be an operator homotopy connecting T to $-T$. Repeating the argument used to prove the previous lemma, we see that $(B+S)(B-S_s)^{-1}$ is a norm-continuous path connecting the invertible operator representing the signature of (E, b, T) to the identity. So the class in K-theory of that invertible operator is zero. □

Proof of Theorem 4.3. We are going to argue that the signature of the direct sum complex $(E\oplus E', b\oplus b', T\oplus -T')$ is zero. This will be sufficient since it is easy to check that

$$\begin{aligned}\operatorname{Sign}(E\oplus E', b\oplus b', T\oplus -T') &= \operatorname{Sign}(E, b, T) + \operatorname{Sign}(E', b', -T')\\ &= \operatorname{Sign}(E, b, T) - \operatorname{Sign}(E', b', T').\end{aligned}$$

Since T' and ATA^* induce the same map on homology the path

$$\begin{pmatrix} T & 0 \\ 0 & (s-1)T' - sATA^* \end{pmatrix} \quad (s\in[0,1]).$$

is an operator homotopy connecting the duality operator $T\oplus -T'$ on the direct sum complex $(E\oplus E', b\oplus b')$ to $T\oplus -ATA^*$. Following this, the path

$$\begin{pmatrix} \cos(s)T & \sin(s)TA^* \\ \sin(s)AT & -\cos(s)ATA^* \end{pmatrix} \quad (s\in[0,\pi/2])$$

is an operator homotopy connecting $T\oplus -ATA^*$ to $\left(\begin{smallmatrix} 0 & TA^* \\ AT & 0 \end{smallmatrix}\right)$ (to see that the operators in the path really are duality operators, note that if α, $\alpha^\dagger$ and τ are inverse to A, A^* and T at the level of homology then

$$\begin{pmatrix} \cos(s)\tau & \sin(s)\tau\alpha \\ \sin(s)\alpha^\dagger\tau & -\cos(s)\alpha^\dagger\tau\alpha \end{pmatrix}$$

is inverse to the displayed operator at the level of homology). The last duality operator in this path is homotopic to its additive inverse along the path

$$\begin{pmatrix} 0 & \exp(is)AT \\ \exp(-is)TA^* & 0 \end{pmatrix} \quad (s\in[0,\pi]).$$

The theorem now follows from Lemmas 4.5 and 4.6. □

EXAMPLE 4.7. We shall develop geometric applications at length in the second paper of this series. But to provide some context, the following concrete application may be useful. Suppose that M is a closed, oriented manifold equipped with a triangulation (M might be smooth or just piecewise linear – it does not matter which). Form the complex

$$C_0 \xleftarrow{b_1} C_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} C_n$$

which computes the simplicial homology of the *universal cover* $\widetilde{M}$ of M, with complex coefficients. The spaces C_j are finitely generated, free $\mathbb{C}[\pi_1 M]$-modules, and the differentials are $\mathbb{C}[\pi_1 M]$-linear. The proof of Poincaré duality furnishes the complex with a duality operator T as in Definition 3.1 (the adjoint T^* is computed with respect to the natural bases of the spaces C_j; the self-adjointness condition (i) is ensured by averaging T and T^*, which is one reason why we work over $\mathbb{C}$, not $\mathbb{Z}$). If we now induce up to $C = C^*_r(\pi_1 M)$ we obtain a Hilbert–Poincaré structure $T \otimes I$ on the complex

$$C_0 \otimes C^*_r(\pi_1 M) \xleftarrow{b_1 \otimes I} C_1 \otimes C^*_r(\pi_1 M) \xleftarrow{b_2 \otimes I} \cdots \xleftarrow{b_n \otimes I} C_n \otimes C^*_r(\pi_1 M)$$

(the tensor products are over $\mathbb{C}[\pi_1 M]$). Theorem 4.3 shows that the signature is an oriented homotopy invariant of M. Further details will be given in the second paper of this series.

5. Analytically Controlled Complexes

It is well known that there exists a second way of associating a 'higher signature' in $K_n(C^*_r(\pi_1 M))$ to a smooth, closed, oriented manifold: one can form the signature operator on M and associate to it its K-theoretic index, as for example in [14]. It is natural to ask whether or not this is the same invariant. The answer is that it is indeed the same, and in this section we shall develop some tools to prove this. The complete argument will be given in the second paper of this series, where other important applications of the same tools will also be given. We shall however sketch parts of the argument here as we go along.

One way of obtaining the K-theoretic higher index of the signature operator is to begin with the de Rham complex of square-integrable, differential forms on the universal cover of M, equip it with its natural duality operator (determined by the Hodge star operation), and repeat the steps taken in Sections 3 and 4 of this paper to obtain a homotopy invariant signature in K-theory. With this in mind, we shall now alter the basic setup with which we began the paper, so as to incorporate complexes of Hilbert spaces in which the differentials are (possibly) unbounded operators. To see the relationship between this set-up and our earlier one involving Hilbert modules over C, suppose that $\rho\colon C \to \mathfrak{B}(H)$ is a faithful representation of the C^*-algebra C. For each Hilbert *module* E over C we can form the Hilbert *space* $E \otimes_\rho H$, and this process gives a functor which transforms complexes of Hilbert modules into complexes of Hilbert spaces. Moreover, an operator T on a Hilbert module E is invertible if and only if the operator $T \otimes 1$, on the Hilbert space $E \otimes_\rho H$, is invertible; thus a complex of Hilbert modules is acyclic, or a chain map of Hilbert modules is a homology isomorphism, if and only if the induced complex of Hilbert spaces is acyclic, or the induced chain map of Hilbert spaces is a homology isomorphism[4]. It follows that a Poincaré complex of Hilbert modules induces a Poincaré complex of Hilbert spaces.

We shall therefore begin anew with a complex

$$H_0 \xleftarrow{b_1} H_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} H_n \tag{5}$$

in which the spaces H_j are Hilbert spaces, and the differentials b_j are closed, possibly unbounded operators. We shall assume that successive operators in the complex are composable (that is, the image of one is contained within the domain of the next). As before, we shall denote by b the direct sum of the b_j acting on the direct sum H of all the H_j. It is a closed operator (its domain is the direct sum of the domains of the b_j), it is composable with itself, and $b^2 = 0$.

LEMMA 5.1. *Let* (H, b) *be a complex of Hilbert spaces, as defined above. The operator* $B = b^* + b$ (*with domain the intersection of the domains of* b^* *and* b) *is densely defined and self-adjoint.*

Proof. This is obvious if the differential b is a bounded operator. In the unbounded case, write H as $H' \oplus H''$, where H' is the kernel of b. Then b assumes the form

$$\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

where β is a closed operator from H'' to H'. One checks that B is the operator associated to the matrix

$$\begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix}$$

which is certainly densely defined and self-adjoint. □

PROPOSITION 5.2. *The complex* (5) *is acyclic if and only if the self-adjoint operator* $B = b^* + b$ *is invertible.*

Proof. The proof is the same as that of Proposition 2.1. (In the unbounded case 'invertible' means that B maps its domain one-to-one onto H. By the closed graph theorem the inverse is then a bounded linear operator.) □

In the definition of chain map between complexes (compare Definition 2.2) we now require that the operators $A\colon H_j \to H_j'$ be bounded and map the domain of b into the domain of b'. With this, the basic facts about mapping cone complexes stated in Section 2 carry over, with the same proofs.

[4]This observation was of crucial importance in [6].

We define the concept 'Hilbert–Poincaré complex' in the new context exactly as we did in Definition 3.1. We insist that the duality operators $T\colon H_p \to H_{n-p}$ be bounded and everywhere defined, that they map the domain of b^* into the domain of b, and that the equation

$$Tb^*v + (-1)^p bTv = 0,$$

which appears as item (ii) of Definition 3.1, holds for all vectors v in the domain of b^*. Lemma 3.5 remains true in this context, interpreting the word 'invertible' as in the proof of proposition 5.2 above; again, the proof is the same.

Before we can form the signature invariant of one of our 'unbounded' Hilbert–Poincaré complexes we need to determine in which K-theory group the invariant ought to lie. For this purpose let us fix a C^*-subcategory[5] $\mathfrak{A}$ of the category of all Hilbert spaces and bounded linear maps (this is an additive subcategory whose morphism sets $\mathrm{Hom}_{\mathfrak{A}}(H_1, H_2)$ are Banach subspaces of the bounded linear operators from H_1 to H_2, and are closed under the adjoint operation). Let us also fix an ideal $\mathfrak{J}$ in $\mathfrak{A}$ (apart from the fact that we no longer require identity morphisms, this is a C^*-subcategory with the additional property that any composition of a morphism in $\mathfrak{J}$ with a morphism in $\mathfrak{A}$ is a morphism in $\mathfrak{J}$).

DEFINITION 5.3. An unbounded, self-adjoint, Hilbert space operator D is *analytically controlled* over $(\mathfrak{A}, \mathfrak{J})$ if

(i) the Hilbert space on which it is defined is an object of $\mathfrak{J}$;
(ii) the resolvent operators $(D \pm i)^{-1}$ are morphisms of $\mathfrak{J}$; and
(iii) the operator $D(1+D^2)^{-\frac{1}{2}}$ is a morphism of $\mathfrak{A}$.

Remark 5.4. For a *bounded* operator, the definition simply means that D is a morphism in $\mathfrak{J}$. Requirements (ii) and (iii) together are equivalent to saying that $f(D)$ belongs to $\mathfrak{J}$ for every $f \in C_0(\mathbb{R})$, and that $f(D)$ belongs to $\mathfrak{A}$ for every function f continuous on $[-\infty, \infty]$.

DEFINITION 5.5. A complex of Hilbert spaces (H, b) is analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ if the self-adjoint operator $B = b + b^*$ is analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ in the sense of Definition 5.3. A Hilbert–Poincaré complex is analytically controlled over $(\mathfrak{A}, \mathfrak{J})$ if

(i) the complex (H, b) is analytically controlled in the sense just defined, and
(ii) the duality operator T is a morphism in the category $\mathfrak{A}$.

[5]See [3,15,16] for C^*-categories and their K-theory.

EXAMPLE 5.6. In our geometric example, the objects of $\mathfrak{A}$ and $\mathfrak{J}$ are Hilbert spaces equipped with compatible representations of both $\pi_1 M$ and $C_0(\widetilde{M})$. The morphisms of $\mathfrak{A}$ are norm limits of $\pi_1 M$-equivariant, bounded, *finite-propagation* operators (see [5]), while the morphisms of $\mathfrak{J}$ are in addition *locally compact* (see [5] again). The L^2-de Rham homology complex for the universal cover $\widetilde{M}$ is then analytically controlled over $(\mathfrak{A}, \mathfrak{J})$, and the Hodge star operator makes it into a controlled Hilbert–Poincaré complex. All this will be developed in greater detail in the second paper of this series.

We are almost ready to define the signature invariant of an analytically controlled Hilbert–Poincaré complex. But first we need to carry out one or two perturbation computations.

LEMMA 5.7. *If a self-adjoint operator D is analytically controlled and if S is a bounded self-adjoint operator in $\mathfrak{A}$ then the resolvents of the self-adjoint operator $D+S$ (that is, the operators $(D+S\pm i)^{-1}$) belong to $\mathfrak{J}$.*

Proof. The formula

$$(D+S\pm i)=(I+S(D\pm i)^{-1})(D\pm i)$$

shows that $(I+S(D\pm i)^{-1}$ is invertible. As a result, we can write

$$(D+S\pm i)^{-1}=(D\pm i)^{-1}(I+S(D\pm i)^{-1})^{-1}$$

which expresses $(D+S\pm i)^{-1}$ as the product of an operator in $\mathfrak{J}$ and a multiplier of $\mathfrak{J}$. □

LEMMA 5.8. *Let D be an analytically controlled self-adjoint operator and let $g\colon [-\infty, \infty] \to \mathbb{R}$ be a continuous function. If S is a bounded self-adjoint operator in $\mathfrak{A}$ then the difference $g(D)-g(D+S)$ lies in $\mathfrak{J}$.*

Proof. It suffices to prove the lemma for $g(x)=x(1+x^2)^{-1/2}$. From the integral representation

$$g(x)=\frac{2}{\pi}\int_1^\infty \frac{t}{\sqrt{t^2-1}} x(x^2+t^2)^{-1}\,\mathrm{d}t,$$

it is easy to calculate that

$$\begin{aligned} g(D)-g(D+S) &= \frac{1}{\pi}\int_1^\infty \frac{t}{\sqrt{t^2-1}} R(t) S R_S(t)\,\mathrm{d}t \\ &\quad + \frac{1}{\pi}\int_1^\infty \frac{t}{\sqrt{t^2-1}} R(-t) S R_S(-t)\,\mathrm{d}t, \end{aligned}$$

where $R(t)=(D+it)^{-1}$ and $R_S(t)=(D+S+it)^{-1}$. These are norm convergent, improper Riemann integrals and the integrands lie within $\mathfrak{J}$. So the integrals lie there too. □

Combining these results we obtain in particular

PROPOSITION 5.9. *If D is an analytically controlled self-adjoint operator and S is a bounded self-adjoint operator in $\mathfrak{A}$, then $D+S$ is analytically controlled also.* □

Suppose now that (H, b, T) is an n-dimensional Hilbert–Poincarè complex analytically controlled over $(\mathfrak{A}, \mathfrak{J})$. Its signature invariant will lie in the K-theory group $K_n(\mathfrak{J})$. To avoid getting too involved with the K-theory of C^*-categories let A be the C^*-algebra of $\mathfrak{A}$-endomorphisms of H, and let J be the ideal of $\mathfrak{J}$-endomorphisms. We will define a signature invariant in $K_n(J)$; the C^*-categorical signature invariant is the image of this one under the natural map $K_n(J) \to K_n(\mathfrak{J})$. As before there are two cases to consider, and we will begin with the case where n is odd.

Because Lemma 3.5 remains true for our unbounded complexes (as we noted above), we may form the invertible operator $F = (B+S)(B-S)^{-1} \in A$. Since $B-S$ is analytically controlled and invertible, $(B-S)^{-1} \in J$ and therefore $I - F = -2S(B-S)^{-1} \in J$ also. So we can make the following definition:

DEFINITION 5.10. Let (H, b, T) be an odd-dimensional analytically controlled Hilbert–Poincaré complex. Its *signature* is the class in $K_1(J)$ of the invertible operator

$$(B+S)(B-S)^{-1}\colon H_{ev} \to H_{ev}.$$

which belongs to the unitalization J^+ of J.

In the even-dimensional case, let $P_+, P_- \in A$ be the positive projections of the invertible self-adjoint operators $B+S$, $B-S$. It follows from Lemma 5.8 that their difference $P_+ - P_-$ is an element of J. But a pair of projections in A whose difference belongs to J gives rise to a 'formal difference' element $[P_+] - [P_-] \in K_0(J)$. Hence we can make the following definition:

DEFINITION 5.11. Let (H, b, T) be an even-dimensional analytically controlled Hilbert–Poincaré complex. Its *signature* is the class in $K_0(\mathfrak{J})$ determined by the formal difference $[P_+] - [P_-]$ of the positive projections of $B+S$ and $B-S$.

Let us turn now to the formulation and proof of the homotopy invariance property of the signature. Homotopy equivalence of analytically controlled Hilbert–Poincaré complexes is defined exactly as in Definition 4.1, except that we require in addition that the chain map A should be a morphism of $\mathfrak{A}$.

THEOREM 5.12. *If (H, b, T) and (H', b', T') are homotopy equivalent, n-dimensional, analytically controlled Hilbert–Poincaré complexes then their signatures in $K_n(\mathfrak{J})$ are equal.*

The proof is the same as the proof given in Section 4, using the notion of operator homotopy. The calculation below replaces the proof of Lemma 4.5 in the even-dimensional case (the proof in the odd-dimensional case needs no additional argument).

PROPOSITION 5.13. *Let B be an analytically controlled, unbounded self-adjoint operator and let S_t $(t \in [0,1])$ be a norm-continuous family of analytically controlled, bounded self-adjoint operators. If the operators B and $B+S_t$ are all invertible then the formal difference $[P]-[P_t]$ of the positive spectral projections for B and $B+S_t$ defines a class in $K_0(\mathfrak{J})$ which is independent of t.*

Proof. One can choose a single continuous function $g\colon [-\infty,\infty] \to \mathbb{R}$ such that the operators $g(B)$ and $g(B+S_t)$ are the positive spectral projections of B and $B+S_t$, for all t. Let P be the positive spectral projection of B, considered as a constant function on $[0,1]$ and let Q be the function mapping t to the positive spectral projection of $B+S_t$. By Lemma 5.8 (applied to the pair of categories $(C[0,1]\otimes\mathfrak{A}, C[0,1]\otimes\mathfrak{J})$) the formal difference $[P]-[Q]$ defines a class in $K_0(C[0,1]\otimes\mathfrak{J})$. Evaluation at $s \in [0,1]$ maps this element to the formal difference of the spectral projections of B and $B+S_t$ in $K_0(\mathfrak{J})$. But by the homotopy invariance of K-theory all these evaluation homomorphisms are the same, so all the formal differences, for different t, are the same. □

EXAMPLE 5.14. In our running example, the K-theory of J identifies with the K-theory of the C^*-algebra $C^*_r(\pi_1 M)$. The signature invariant in $K_n(C^*_r\pi_1 M)$ in Example 4.7 from the complex

$$C_0 \otimes C^*_r(\pi_1 M) \xleftarrow{b_1\otimes I} C_1 \otimes C^*_r(\pi_1 M) \xleftarrow{b_2\otimes I} \cdots \xleftarrow{b_n\otimes I} C_n \otimes C^*_r(\pi_1 M)$$

corresponds to the signature invariant constructed from the analytically controlled Hilbert–Poincar'e complex of Hilbert spaces

$$C_0 \otimes \ell^2(\pi_1 M) \xleftarrow{b_1\otimes I} C_1 \otimes \ell^2(\pi_1 M) \xleftarrow{b_2\otimes I} \cdots \xleftarrow{b_n\otimes I} C_n \otimes \ell^2(\pi_1 M)$$

(once again, all tensor products are over the complex group algebra). By well-known arguments (basically integration of differential forms over simplices), this complex is homotopy equivalent to the (homology) L^2-de Rham complex

$$\Omega^0_{L^2}(\widetilde{M}) \xleftarrow{b_1\otimes I} \Omega^1_{L^2}(\widetilde{M}) \xleftarrow{b_2\otimes I} \cdots \xleftarrow{b_n\otimes I} \Omega^n_{L^2}(\widetilde{M})$$

for the universal cover (even as analytically controlled Hilbert–Poincaré complexes). By the homotopy invariance theorem, it follows that the 'simplicial' signature of M is equal to its 'de Rham' signature, and from here it is easy to check that the simplicial signature coincides with the index of the signature operator. For more details see the second paper in this series [4].

6. Euler Characteristics

An analytically controlled complex of Hilbert spaces (H, b) has an *Euler characteristic* in the K-theory group $K_0(J)$ which generalizes the usual Euler characteristic of a complex of finite dimensional vector spaces. This is a simpler invariant than the signatures we have been studying up to now, but for completeness we give a brief account of it.

Take the operator $B = b^* + b$ on H and form its Cayley transform

$$U = (B + i)(B - i)^{-1}.$$

Consider also the symmetry (i.e. the self-adjoint unitary) S on H which is $+1$ on the spaces H_{2j} and -1 on the spaces H_{2j+1}. A straightforward calculation shows that

(i) $SU = U^*S$, so that the operator SU is also a symmetry; and
(ii) the difference $S - SU$ lies in J.

Now to every symmetry S there is a corresponding projection $P = \frac{1}{2}(S + 1)$, and to every pair of symmetries whose difference lies in J there corresponds a pair of projections whose difference lies in J. The formal difference of such a pair of projections defines an element of the group $K_0(J)$. Thus the complex (H, b) determines a pair of symmetries, which determines a pair of projections, whose formal difference determines a class in $K_0(J)$. This class is the Euler characteristic.

PROPOSITION 6.1. *If a complex is acyclic, its Euler characteristic is zero.*

Proof. By Proposition 2.1, if a complex (H, b) is acyclic then $B = b + b^*$ is an invertible operator. Consider now the one-parameter family of complexes formed by replacing b by tb, $t \in [1, \infty)$, and thus B by tB. The corresponding operators $U_t = (tB + i)(tB - i)^{-1}$ vary continuously in norm, so the K-theory class defined by the formal difference

$$[S] - [SU_t]$$

is independent of t and equal to the Euler characteristic. But since B is invertible, $U_t \to 1$ in norm as $t \to \infty$, by the spectral theorem, and thus the Euler characteristic vanishes. □

PROPOSITION 6.2. *In the case that b is bounded, the Euler characteristic is equal to the alternating sum*

$$\sum_i (-1)^i [H_i] \in K_0(J).$$

Proof. Notice that in the bounded case $J = A$ is unital and thus the objects H_i define classes in $K_0(J)$ (for instance, via the corresponding orthogonal projections).

Follow the argument of the preceding proposition where now $t \in (0, 1]$ and tends to 0. Then the Euler characteristic is represented by the formal difference $[S] - [-S]$ of symmetries, which belong to J (by unitality). In terms of projections, S corresponds to the orthogonal projection onto the even H_j, and $-S$ corresponds to the orthogonal projection onto the odd H_j; the result follows. □

7. Bordism Invariance of the Signature

In this section we shall prove the bordism invariance of the signature. our main result implies, for example, that if a smooth, closed, oriented manifold M is the boundary of a smooth, compact manifold W with the same fundamental group, then the signature of M in $K_n(C_r^*(\pi_1 M))$ is zero.[6]

For simplicity, we shall work with complexes of finitely generated Hilbert modules over a unital C^*-algebra, as we did in the first sections of the paper. This is sufficient for the applications we have in mind. The results of the section extend easily to analytically controlled complexes in which the differentials are *bounded* operators (we shall state the result in this generality at the end of the section), but the story for unbounded differentials is a bit more complicated and will not be discussed.

Let (E, b) be a complex of Hilbert modules (as in Section 2) in which the differentials b are bounded. The following definition is fairly obvious:

DEFINITION 7.1. A (complemented) *subcomplex* of the complex (E, b) is a family of (complemented) Hilbert submodules $E'_p \subset E_p$ such that b maps E'_p into E'_{p-1}, for all p.

We will be considering exclusively complemented subcomplexes, and it will be convenient use the language of orthogonal projections rather that Hilbert submodules. We note that given orthogonal projections $P\colon E_p \to E_p$,

[6]Actually, the 'manifold' W need be nothing more than a Poincaré complex, and by considering mapping cylinders of homotopy equivalences one can see that the bordism invariance result we are about to prove generalizes the homotopy invariance result in Theorem 4.3.

their ranges determine a subcomplex of (E, b) if and only if $PbP = bP$. If a family of orthogonal projections $P\colon E_p \to E_p$ determines a subcomplex in this way then the differential on the subcomplex is of course

$$Pb = b\colon PE_p \to PE_{p-1}.$$

The adjoint of this differential is

$$Pb^*\colon PE_{p-1} \to PE_p$$

(note that $Pb^* \neq b^*$ in general). We shall denote the subcomplex by (PE, Pb) and its dual complex by (PE, Pb^*).

If (PE, Pb) is a subcomplex of (E, b) then there is a corresponding *quotient complex* $(P^\perp E, P^\perp b)$. The inclusion $PE \subset E$ gives a chain map from (PE, Pb) into (E, b), whereas the orthogonal projection $E \to P^\perp E$ gives a chain map from (E, b) onto $(P^\perp E, P^\perp b)$. For computations, it is useful to note that $P^\perp b = P^\perp b P^\perp$.

The following definition, like our definition of algebraic Hilbert–Poincaré complex, is borrowed from the algebraic theory of surgery [12, 19].

DEFINITION 7.2. An $(n+1)$-dimensional *algebraic Hilbert–Poincaré pair* is a complex of finitely generated Hilbert modules

$$E_a \xleftarrow{b} E_{a+1} \xleftarrow{b} \cdots \xleftarrow{b} E_{n-a} \xleftarrow{b} E_{n-a+1} \tag{6}$$

together with a family of bounded adjointable operators $T\colon E_p \to E_{n+1-p}$ and a family of orthogonal projections $P\colon E_p \to E_p$ such that

(i) the orthogonal projections P determine a subcomplex of (E, b); that is, $PbP = Pb$;
(ii) the range of the operator $Tb^* + (-1)^p bT\colon E_p \to E_{n-p}$ is contained within the range of $P\colon E_{n-p} \to E_{n-p}$;
(iii) T induces an isomorphism from the homology of the complex (E, b^*) to the homology of the complex $(P^\perp E, P^\perp b)$ (note that the previous item implies that T is a chain map between these complexes); and
(iv) $T^* = (-1)^{(n+1-p)p} T\colon E_p \to E_{n+1-p}$.

Remark 7.3. As with Hilbert–Poincaré complexes, in the natural geometric examples the index a is zero.

The definition is clearly analogous to Definition 3.1. In fact if $P = 0$ then an $(n+1)$-dimensional algebraic Hilbert–Poincaré pair is just an $(n+1)$-dimensional Hilbert–Poincaré complex. This should of course be compared with the observation that an $(n+1)$-manifold with empty boundary is just a closed $(n+1)$-manifold.

The following computation is carried out in [12].

LEMMA 7.4. *Let (E, b, T, P) be an $(n+1)$-dimensional algebraic Hilbert–Poincaré pair. The operators*

$$T_0 = Tb^* + (-1)^p bT : E_p \to E_{n-p}$$

satisfy the following relations

(i) $T_0^* = (-1)^{(n-p)p} T_0 : E_p \to E_{n-p}$;
(ii) $T_0 = PT_0 = T_0 P$;
(iii) $T_0 b^* v + (-1)^p bT_0 v = 0$ *if* $v \in PE_p$; *and*
(iv) T_0 *induces an isomorphism from the homology of the complex* (PE, Pb^*) *to the homology of the complex* (PE, Pb).

Proof. The first three items are straightforward calculations. To prove the fourth, one considers the following diagram, in which the rows are long-exact sequences of homology and cohomology groups:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^p(PH, Pb^*) & \longrightarrow & H^{p+1}(P^\perp H, P^\perp b^*) & \longrightarrow & H^{p+1}(H, b^*) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} & \\ \cdots \longrightarrow & H_{n-p}(PH, Pb) & \longrightarrow & H_{n-p}(H, b) & \longrightarrow & H_{n-p}(P^\perp H, P^\perp b) & \longrightarrow \cdots \end{array}$$

The middle vertical map is the one given in item (iii) of Definition 7.2. The right-hand vertical map is defined thanks item (iii) and the fact that $T^* = (-1)^{(n+1-p)p} T$. The left-hand vertical map is the one we are to prove is an isomorphism. The diagram commutes, so the result follows from the five-lemma. □

The lemma asserts that T_0 provides the subcomplex (PE, Pb) with the structure of an n-dimensional algebraic Hilbert–Poincaré complex. Accordingly we make the following definition:

DEFINITION 7.5. The algebraic Hilbert–Poincaré complex (PE, Pb, T_0) is the *boundary* of the algebraic Hilbert–Poincaré pair (E, b, T, P).

We are going to prove the following bordism invariance result:

THEOREM 7.6. *If (E, b, T, P) is an $(n+1)$-dimensional algebraic Hilbert–Poincaré pair then the K-theoretic signature of its boundary (PE, Pb, T_0) is zero.*

Proof. Let λ be a real number and define a complex $(\tilde{E}, \tilde{b}_\lambda)$ by

$$\tilde{E}_p = E_p \oplus P^\perp E_{p+1} \quad \text{and} \quad \tilde{b}_\lambda = \begin{pmatrix} b & 0 \\ \lambda P^\perp & -P^\perp b \end{pmatrix}.$$

This is the mapping cone complex for the chain map $\lambda P^{\perp}\colon (E,b)\to(P^{\perp}E, P^{\perp}b)$. If we introduce the operators

$$\tilde{T}=\begin{pmatrix}0 & TP^{\perp}\\ (-1)^{p}P^{\perp}T & 0\end{pmatrix} : \tilde{E}_{p}\to\tilde{E}_{n-p}$$

then for any λ (including $\lambda=0$), the triple $(\tilde{E},\tilde{b}_{\lambda},\tilde{T})$ is an n-dimensional Hilbert–Poincaré complex.

If $\lambda\neq 0$ then the formula $Av=v\oplus 0\in E_{p}\oplus E_{p+1}$ defines a chain map $A\colon (PE,bP)\to(\tilde{E},\tilde{b}_{\lambda})$. It is in fact an isomorphism on homology (by basic properties of mapping cone complexes). Moreover, if $\lambda=-1$, then A is a homotopy equivalence of Hilbert–Poincaré complexes

$$(PE, Pb, T_{0})\xrightarrow[\sim]{A}(\tilde{E},\tilde{b}_{-1},\tilde{T}).$$

So the signature of (PE, Pb, T_{0}) is equal to the signature of $(\tilde{E},\tilde{b}_{-1},\tilde{T})$. By the homotopy invariance of K-theory, the signature of $(\tilde{E},\tilde{b}_{-1},\tilde{T})$ is equal to the signature of $(\tilde{E},\tilde{b}_{0},\tilde{T})$. But when $\lambda=0$ the duality operator $\tilde{T}$ is operator homotopic to its additive inverse along the path

$$\begin{pmatrix}0 & \exp(is)TP^{\perp}\\ \exp(-is)(-1)^{p}P^{\perp}T & 0\end{pmatrix}\qquad (s\in[0,\pi]).$$

So the signature is zero by Lemma 4.6. □

Remark 7.7. The *algebraic theory of surgery*, as developed by Ranicki and others, defines the L-groups $L_{n}(\mathfrak{A})$ of an additive category with duality[7] as the algebraic bordism groups of algebraic Poincaré complexes over $\mathfrak{A}$. Thus, our bordism invariance result implies that if $\mathfrak{A}$ is the C^{*}-category of finitely generated, projective C-modules, then the analytic signature defines a forgetful map

$$L_{n}(\mathfrak{A})\to K_{n}(C)$$

from L-theory to K-theory. In paper III of this series we will use a geometric implementation of this map in the classical case, based on the geometric definition of the L-groups in Chapter IX of [21].

We conclude by stating the extension of Theorem 7.6 to the analytically controlled context.

[7]The distinction between symmetric and quadratic L-theory is not pertinent to the present discussion.

DEFINITION 7.8. We shall say that a quadruple (H, b, T, P) is a *analytically controlled Hilbert–Poincaré pair* if (H, b) is an analytically controlled complex of Hilbert spaces (in the sense of Definition 5.5) with bounded differentials, and if the operators T and P are morphisms in $\mathfrak{A}$.

Note that if (H, b, T, P) is analytically controlled then so is its boundary (PH, Pb, T_0).

THEOREM 7.9. *If (H, b, T, P) is an $(n+1)$-dimensional analytically controlled Hilbert–Poincaré pair then the K-theoretic signature of its boundary (PH, Pb, T_0) is zero.* □

References

1. Baum, P., Connes, A. and Higson, N.: Classifying space for proper G-actions and K-theory of group C^*-algebras. In *Proceedings of a Special Session on C^*-Algebras*, Vol. 167: *Contemporary Mathematics*, pp. 241–291. American Mathematical Society, Providence, R.I., 1994.
2. Ferry, S. C., Ranicki, A. and Rosenberg, J.: A history and survey of the Novikov conjecture. In: Ferry, S., Ranicki, A. and Rosenberg, J. (eds.), *Proceedings of the 1993 Oberwolfach Conference on the Novikov Conjecture*, volume 226 of *LMS Lecture Notes*, pp. 7–66. Cambridge University Press, Cambridge, 1995.
3. Ghez, P.: A survey of W^*-categories. In: Kadison, R. V. (ed), *Operator Algebras and Applications*, pp. 137–139. American Mathematical Society, Providence, R.I., 1982. Proceedings of Symposia in Pure Mathematics 38.
4. Higson, N. and Roe, J.: Mapping surgery to analysis II: geometric signatures. This issue.
5. Higson, N. and Roe, J.: The Baum–Connes conjecture in coarse geometry. In: Ferry, S., Ranicki, A. and Rosenberg, J. (eds.), *Proceedings of the 1993 Oberwolfach Conference on the Novikov Conjecture*, Vol. 227: *LMS Lecture Notes*, pp. 227–254. Cambridge University Press, Cambridge, 1995.
6. Higson, N., Roe, J. and Schick, T.: Spaces with vanishing ℓ^2-homology and their fundamental groups (after Farber and Weinberger). *Geomet. Dedicata*, **87** (1–3) (2001), 335–343.
7. Hilsum, M. and Skandalis, G.: Invariance par homotopie de la signature à coefficients dans un fibré presque plat. *J. für die reine und angewandte Mathematik*, **423**: (1992), 73–99.
8. Kaminker, J. and Miller, J. G.: Homotopy invariance of the index of signature operators over C^*-algebras. *J. Operator Theory* **14** (1985), 113–127.
9. Kasparov, G. G.: Operator K-theory and its applications: elliptic operators, group representations, higher signatures, C^*-extensions. In: *Proceedings of the International Congress of Mathematicians, Warsaw 1983*, pp. 987–1000. North-Holland, Amsterdam–New York–Oxford, 1984.
10. Kasparov, G. G.: K-theory, group C^*-algebras, and higher signatures (Conspectus), in: Ferry, S., Ranicki, A. and Rosenberg, J. (eds), *Proceedings of the 1993 Oberwolfach Conference on the Novikov Conjecture*, Vol. 226: *LMS Lecture Notes*, pp. 101–146. Cambridge University Press, Cambridge, 1995.

11. Lance, E. C.: *Hilbert C^*-modules: a toolkit for operator algebraists*, Vol. 210 of *London Mathematical Society Lecture Notes*. Cambridge University Press, Cambridge, 1995.
12. Mischenko, A. S.: Homotopy invariants of non-simply connected manifolds I: Rational invariants. *Mathematics of the USSR – Izvestija*, **4** (1970), 506–519.
13. Mischenko, A. S.: Infinite dimensional representations of discrete groups and higher signatures. *Mathematics of the USSR – Izvestija*, **8** (1974), 85–111.
14. Mischenko, A. S. and Fomenko, A. T.: The index of elliptic operators over C^*-algebras. *Mathematics of the USSR – Izvestija*, **15** (1980), 87–112
15. Mitchener, P. D.: C^*-categories. *Proc. London Math. Soc. (3)* **84** (2) (2002), 375–404.
16. Mitchener, P. D.: KK-theory of C^*-categories and the analytic assembly map. *K-Theory*, **26** (4) (2002), 307–344.
17. Novikov, S. P.: Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of hamiltonian formalism. applications to differential topology and the theory of characteristic classes I. *Mathematics of the USSR – Izvestija*, **4** (1970), 257–292.
18. Novikov, S. P.: Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of hamiltonian formalism. applications to differential topology and the theory of characteristic classes II. *Mathematics of the USSR – Izvestija*, **4** (1970), 479–505.
19. Ranicki, A.: The algebraic theory of surgery I: Foundations. *Proc. Lond. Mathe. Soc.*, **40** (1980), 87–192.
20. Roe, J.: *Index Theory, Coarse Geometry, and the Topology of Manifolds*, Vol. 90 *CBMS Conference Proceedings*. American Mathematical Society, Providence, R.I., 1996.
21. Wall, C. T. C.: *Surgery on Compact Manifolds*. Academic Press, Boston, 1970.