The Atiyah-Singer Index Theorem

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1 Elliptic equations

The Atiyah-Singer index theorem is concerned with the existence and uniqueness of solutions to linear partial differential equations of *elliptic type*. To understand this concept, consider the two equations

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0 \quad \text{and} \qquad \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

They differ only by the factor $i = \sqrt{-1}$, but nevertheless their solutions have very different properties. Any function of the form $f(x, y) = g(x-y)$ is a solution to the first equation, but in the analogous general solution $g(x+iy)$ of the second equation, g must be an *analytic* function of the *complex* variable $z = x + iy$, and it was already known in the nineteenth century that such functions are very special. For example, the first equation has an infinite-dimensional set of bounded solutions, but Liouville's theorem in complex analysis asserts that the only bounded solutions of the second equation are the constant functions.

The differences between the solutions of the two equations can be traced to the differences between the *symbols* of the equations, which are the polynomials in real variables ξ, η that are obtained by substituting $i\xi$ for $\partial/\partial x$ and $i\eta$ for $\partial/\partial y$. Thus the symbols of the two equations above are

$$i\xi + i\eta \quad \text{and} \qquad i\xi - \eta,$$

respectively. An equation is said to be *elliptic* if its symbol is zero only when $\xi = \eta = 0$; thus the second equation is elliptic but the first is not. The fundamental *regularity theorem*, which is proved using Fourier analysis, states that an elliptic partial differential equation (subject to suitable boundary conditions, if needed) has a finite-dimensional solution space.

2 Topology of elliptic equations and the Fredholm index

Consider now the general first-order linear partial differential equation

$$a_1 \frac{\partial f}{\partial x_1} + \cdots + a_n \frac{\partial f}{\partial x_n} + bf = 0,$$

in which f is a vector-valued function and the coefficients a_j and b are complex matrix-valued functions. It is *elliptic* if its *symbol*

$$i\xi_1 a_1(x) + \cdots + i\xi_n a_n(x)$$

is an invertible matrix for every $\xi \neq 0$ and every x. The regularity theorem applies in this generality, and it allows us to form the *Fredholm index* of an elliptic equation (with suitable boundary conditions), which is the number of linearly independent solutions of the equation minus the number of linearly independent solutions of the *adjoint equation*

$$-\frac{\partial}{\partial x_1}(a_1^* f) - \cdots - \frac{\partial}{\partial x_n}(a_n^* f) + b^* f = 0.$$

The reason for introducing the Fredholm index is that it is a *topological invariant* of elliptic equations. This means that continuous variations in the coefficients of an elliptic equation leave the Fredholm index unchanged (in contrast the number of linearly independent solutions of an equation can vary as the coefficients of the equation vary). The Fredholm index is therefore constant on each connected component of the set of all elliptic equations, and this raises the prospect of using topology to determine the structure of the set of all elliptic equations as an aid to computing the Fredholm index. This observation was made by Gelfand in the 1950s. It lies at the root of the Atiyah-Singer index theorem.

3 An example

To see in more detail how topology can be used to determine the Fredholm index of an elliptic equation let us consider a specific example. Consider elliptic equations for which the coefficients $a_j(x)$ and $b(x)$ are *polynomial* functions of x, with a_j of degree $m-1$ or less and b of degree m or less. The expression

$$i\xi_1 a_1(x) + \cdots + i\xi_n a_n(x) + b(x)$$

is then a polynomial in both x and ξ of degree m or less. Let us strengthen the hypothesis of ellipticity by assuming that the terms in this expression that have degree exactly m (jointly in x and ξ) define an invertible matrix whenever *either* x *or* ξ is nonzero. Let us also agree to consider only solutions f of the equation or its adjoint which are *square-integrable*, which means that

$$\int |f(x)|^2 \, dx < \infty.$$

All these extra hypotheses are types of boundary conditions (the behaviors of the equation and its solutions at infinity are controlled), and collectively they imply that the Fredholm index is well-defined.

A simple example is the equation

$$\frac{df}{dx} + xf = 0. \tag{1}$$

The general solution to this ordinary differential equation is the 1-dimensional space of multiples of the function $e^{-x^2/2}$, which is square-integrable. In contrast, the solutions of the adjoint equation

$$-\frac{df}{dx} + xf = 0$$

are multiples of the function $e^{+x^2/2}$, which is not square-integrable. Thus the index of this differential equation is equal to 1.

Returning to the general equation, the degree m terms in

$$i\xi_1 a_1(x) + \cdots + i\xi_n a_n(x) + b(x)$$

determine a map from the unit sphere in (x, ξ)-space to the set $GL(k, \mathbb{C})$ of invertible $k \times k$ complex matrices. Moreover, every such map comes from an elliptic equation (possibly of a more general type than we have discussed up to now, but an equation to which the basic regularity theorem guaranteeing the existence of the Fredholm index applies). It therefore becomes important to determine the topological structure of the space of all maps from the sphere S^{2n-1} into $GL(k, \mathbb{C})$.

A remarkable theorem of Bott provides the answer. The *Bott periodicity theorem* associates an integer, which we shall call the *Bott invariant*, to each map $S^{2n-1} \to GL(k, \mathbb{C})$. Furthermore, Bott's theorem asserts that provided $k \geq n$, one such map can be continuously deformed into another if and only if the Bott invariants of the two maps agree. In the special case $n = k = 1$, where we are dealing with maps from the one-dimensional circle into the non-zero complex numbers, or in other words closed paths in $\mathbb{C}$ that do not pass through the origin, the Bott invariant is just the classical *winding number*, which measures the number of times such a path winds around the origin. We may therefore regard the Bott invariant as a generalized winding number.

The index theorem for equations of the type that we are considering in this section asserts that the Fredholm index of an elliptic equation is equal to the Bott invariant of its symbol. For instance, in the case of the simple example (1) considered above, the symbol $i\xi + x$ corresponds to the identity map from the unit circle in (x, ξ)-space to the unit circle in $\mathbb{C}$. Its winding number is equal to 1, in agreement with our computation of the index.

The proof of the index theorem depends strongly on Bott periodicity and proceeds as follows. Because elliptic equations are classified topologically by the Bott invariant, and because the Bott invariant and the Fredholm index have analogous algebraic properties, one need only verify the theorem in a single example: that corresponding to a symbol with Bott invariant 1. It turns out that this *Bott generator* can be represented by an n-dimensional generalization of our example (1), and a computation in this case completes the proof.

4 Elliptic equations on manifolds

It is possible to define elliptic equations not just for functions f of n variables, but for functions defined on a MANIFOLD. Particularly accessible to analysis are the elliptic equations on *closed* manifolds, that is, on manifolds that are finite in extent and that have no boundary. For closed manifolds it is not necessary to specify any boundary conditions in order to obtain the basic regularity theorem for elliptic equations (after all, there is no boundary). As a result, every elliptic partial differential equation on a closed manifold has a Fredholm index.

The Atiyah-Singer index theorem concerns elliptic equations on closed manifolds and it has roughly the same form as the index theorem that we studied in the previous section. One builds out of the symbol an invariant called the *topological index* which generalizes the Bott invariant. The Atiyah-Singer index theorem then asserts that the topological index of an elliptic equation is equal to the Fredholm or *analytical* index of the equation. The proof has two stages. In the first, theorems are proved that allow one to transform an elliptic equation on a general manifold into an elliptic equation on a sphere without changing the topological or analytical indices. For example, it may be shown that two elliptic equations on different manifolds that are the common "boundary" of an elliptic equation on a manifold of one higher dimension must have the same topological and analytical indices. In the second stage of the proof the Bott periodicity theorem and an explicit computation are applied to identify the topological and analytical indices of elliptic equations on spheres. Throughout both stages, an important tool is K-THEORY, which is a branch of ALGEBRAIC TOPOLOGY invented by Atiyah and Hirzebruch.

Although the proof of the Atiyah-Singer index theorem makes use of K-theory, the final result can be translated into terms that do not mention K-theory explicitly. In this way one obtains an index formula roughly like this:

$$\text{Index} = \int_M I_M \cdot \text{ch}(\sigma).$$

The term I_M is a DIFFERENTIAL FORM determined by the curvature of the manifold M on which the equation is defined. The term $\text{ch}(\sigma)$ is a differential form obtained from the symbol of the equation.

5 Applications

In order to prove the index theorem, Atiyah and Singer were obliged to study a very broad class of generalized elliptic equations. However the applications they first had in mind were related to the simple equation with which we began this article. Solutions of the equation

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

are precisely the analytic functions of the complex variable $z = x + iy$. There is a counterpart to this equation on any RIEMANN SURFACE and the Atiyah-Singer index formula, applied in this instance, is equivalent to a foundational result about the geometry of surfaces called the *Riemann-Roch theorem*. The Atiyah-Singer index theorem gives a means to generalize the Riemann-Roch theorem to a complex manifold of any dimension.

The Atiyah-Singer index theorem also has important applications outside of complex geometry. The simplest example involves the elliptic equation $d\omega + d^*\omega = 0$ on differential forms on a manifold M. The Fredholm index may be identified with the *Euler characteristic* of M—the alternating sum of the numbers of r-dimensional cells in a cell decomposition of M. For 2-dimensional manifolds the Euler characteristic is the familiar quantity $V - E + F$. In the 2-dimensional case, the index theorem reproduces the GAUSS-BONNET THEOREM, which asserts that the Euler characteristic is a multiple of the total Gaussian curvature.

Even this simple case, the index theorem can be used to produce topological restrictions on the ways a manifold can curve. Many important applications of the index theorem proceed in the same direction. For example, Hitchin used a more refined application of the Atiyah-Singer index theorem to show that there is a 9-dimensional manifold which is homeomorphic to the sphere but which is not positively curved in even the weakest sense (in contrast, the usual sphere is positively curved in the strongest possible sense).

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