Can. J. Math.Vol. 43 (2), 1991 pp. 322–330

THE WEYL-VON NEUMANN THEOREM FOR MULTIPLIERS OF SOME *AF*-ALGEBRAS

NIGEL HIGSON AND MIKAEL RØRDAM

Introduction. A well known theorem of Weyl-von Neumann asserts that if X is a self-adjoint operator acting on a separable Hilbert space, then there is a decomposition $1 = \sum e_n$ of the identity into finite rank projections so that we may write

$$X = \sum \lambda_n e_n + y,$$

where the λ_n are scalars and y is a *compact* operator with small norm. In other words, X can be *approximately* diagonalized. In this paper we consider the following question: given an AF-algebra I and a self-adjoint element X of $\mathcal{M}(I)$, the multiplier algebra of I, can we express X in the above form, where now the e_n are projections in I (and $\sum e_n = 1$ in the sense of strict convergence) and $y \in I$? This reduces to the Weyl-von Neumann Theorem in the case $I = \mathcal{K}$

We shall answer this question affirmatively in the case that I is simple and has a unique trace (up to scaling). Our approach is based upon the observation, which seems to have been made by a number of people (see especially [9]), that the problem is equivalent to showing that $\mathcal{M}(I)$ has one of a number of basic structural properties. See Section 1. These properties can then be analyzed in terms of the ideal structure of $\mathcal{M}(I)$, which in the case at hand is very straightforward.

Our techniques would carry over to a somewhat larger class of AF- and other algebras I (for example, simple AF-algebras with only finitely many extremal traces), and indeed we have no doubt that the answer to the question is affirmative for general AF-algebras. However, in order to make the exposition as clear as possible we shall consider only simple I with unique trace.

This research was initiated whilst the first author was visiting Odense Universitet. He would like to thank the second author for arranging the visit, as well as the entire mathematics department, in particular U. Haagerup, for their hospitality. Both authors would like to thank I. Putnam for several helpful conversations.

After this paper was first typed, we discovered that there is a considerable overlap between this article and work of others in this area (we are very grateful to S. Zhang for sending us preprints of his articles [9], [10] and to G. Pedersen for discussions and a draft of [3]). In fact, our Theorem 4.4 is contained within [9]. However, our arguments are,

---

The work of the first author is partially supported by NSF (USA).
Received by the editors September 21, 1989.

for the most part, rather different, and since they are perhaps simpler and more direct, we hope that our article is still worthy of the reader's attention.

1. **Equivalent formulations.** We shall make extensive use of the following result. For a proof see [2,3,8].

THEOREM 1.1. *The following three conditions on a C^*-algebra A are equivalent.*

(i) Each hereditary subalgebra of A has an approximate unit consisting of projections.

(ii) Every self-adjoint element in A is a norm limit of invertible self-adjoint elements.

(iii) Every self-adjoint element in A is a norm limit of self-adjoint elements in A with finite spectrum. ∎

(In (ii), if A does not have a unit then replace A by the C^*-algebra obtained by adjoining a unit.)

We shall say that A has *property FS* (= *finite spectrum*, a reference to (iii)) if it satisfies one of the above conditions. We shall move from one condition to another without comment.

Lemma 1.3 below shows that the Weyl-von Neumann Theorem for $\mathcal{M}(I)$ reduces to showing that $\mathcal{M}(I)$ has property FS.

LEMMA 1.2. *Let I be a separable C^*-algebra with property FS, and let P be a projection in $\mathcal{M}(I)$. There is a sequence $\{p_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in I such that*

$$P = \sum_{n=1}^{\infty} p_n,$$

where the sum converges in the strict topology.

PROOF. The C^*-algebra PIP is a hereditary subalgebra of I, and so there is a sequence $\{p_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in PIP such that $P = \sum_{n=1}^{\infty} p_n$, the convergence being in the strict topology of $\mathcal{M}(PIP)$. But this implies strict convergence $\mathcal{M}(I)$, for given $x \in I$ we have that

$$\begin{aligned} \|\sum_{n=M}^{N} p_n x\|^2 &= \|\sum_{m=M}^{N}\sum_{n=M}^{N} p_m x x^* p_n\| \\ &= \|\sum_{m=M}^{N}\sum_{n=M}^{N} p_m P x x^* P p_n\| \\ &= \|\sum_{n=M}^{N} p_n (P x x^* P)^{\frac{1}{2}}\|^2 \end{aligned}$$

and $(Pxx^*P)^{\frac{1}{2}} \in PIP$. ∎

LEMMA 1.3 (SEE [9]). *Let I be a separable C^*-algebra with property FS. The following are equivalent:*

(i) for every self-adjoint $X \in \mathcal{M}(I)$, every projection $p \in I$, and every $\varepsilon > 0$ there is a projection $q \in I$ such that $q \geq p$ and $\|[q, X]\| < \varepsilon$;

(ii) for every self-adjoint $X \in \mathcal{M}(I)$ and every $\varepsilon > 0$ there is a family $\{e_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in I such that $\sum e_n = 1$ (convergence in the strict topology), such that we may write $X = \sum \lambda_n e_n + y$, where $\lambda_n \in \mathbf{R}, y \in I$ and $\|y\| < \varepsilon$; and

(iii) $\mathcal{M}(I)$ has property FS.

PROOF. (i)$\Rightarrow$(ii) A simple induction argument shows that we can write $X = \sum f_k X f_k + y_1$ for some sequence $\{f_k\}$ of projections in I with $\sum f_k = 1$, and where $y_1 \in I, \|y_1\| < \varepsilon/2$. Using the fact that I, and hence $f_k I f_k$, has property FS, we can perturb each $f_k X f_k$ by an element of norm less that $\varepsilon 2^{-(k+1)}$ to a self-adjoint element $x_k \in f_k I f_k$ with finite spectrum. The spectral projections of all the x_k together then form a suitable family $\{e_n\}$.

(ii)$\Rightarrow$(iii) This is clear.

(iii)$\Rightarrow$(i) For fixed $p \in I$, the set of self-adjoint $X \in \mathcal{M}(I)$ satisfying (i) for all $\varepsilon > 0$ is norm closed, and so it suffices to prove (i) for X with finite spectrum. Write $X = \sum_{i=1}^{n} \lambda_i P_i$, where $P_i \in \mathcal{M}(I)$ are projections, $P_1 + \cdots + P_n = 1$, and $\lambda_i \in \mathbf{R}$. From Lemma 1.2, we get $X = \sum_{j=1}^{\infty} \mu_j e_j$ where the $e_i \in I$ are projections, $\sum_{j=1}^{\infty} e_j = 1$ and $\mu_j \in \mathbf{R}$ (in fact, $\mu_j = \lambda_i$ for some i). Put $f_n = \sum_{j=1}^{n} e_j$. Then $\lim_{n\to\infty} \|(1 - f_n)p\| = 0$. For n large enough, f_n is equivalent to a projection $q \in I$ with $q \geq p$ and $\|f_n - q\|$ small. Since $[f_n, X] = 0$, $\|[q, X]\|$ is small. ∎

We note that the Weyl-von Neumann Theorem for *normal* elements is much more complicated, if it is true at all, in the general situations we are considering.

2. **Comparison theory in $\mathcal{M}(I)$.** From here on, I will denote a non-unital, simple AF-algebra which has a unique semi-finite trace τ, up to scaling. We may extend τ to a trace function on $\mathcal{M}(I)^+$ by the formula

$$\tau(X) = \sup \tau(e_n X e_n),$$

where $\{e_n\}_{n=1}^{\infty}$ is any approximate unit for I.

PROPOSITION 2.1. *Let P and Q be projections in $\mathcal{M}(I)$.*

(i) If $\tau(P) < \tau(Q)$, then $P \precsim Q$; and

(ii) if neither of P and Q is in I, and if $\tau(P) = \tau(Q)$, then $P \sim Q$.

PROOF. Write $P = \sum p_n$ and $Q = \sum q_n$, as in Lemma 1.2. In case (i), by regrouping the q_n (that is, replacing the q_n with sums of the form $\sum_M^N q_n$) we may assume that $\tau(p_n) < \tau(q_n)$. In case (ii), by regrouping both the p_n and the q_n, we may assume that $\sum_{n=1}^{N} \tau(p_n) < \sum_{n=1}^{N} \tau(q_n)$, for every N, and also $\sum_{n=1}^{N} \tau(q_n) < \sum_{n=1}^{N+1} \tau(p_n)$ (this construction requires that both $\sum \tau(p_n)$ and $\sum \tau(q_n)$ have infinitely many non-zero terms, which is where we use the fact that $P, Q \notin I$, as well as the fact that I is simple, so that τ is faithful). Now, it is well known (see [4]) that for projections p and q in I, if $\tau(p) < \tau(q)$

then $p \precsim q$. Thus in case (i) there exist partial isometries $v_n \in I$ such that $v_n^* v_n = p_n$ and $v_n v_n^* \leq q_n$, whilst in case (ii) there exist v_n such that

$$\begin{array}{ll} v_1^* v_1 = p_1, & v_1 v_1^* \leq q_1 \\ v_2^* v_2 \leq p_2, & v_2 v_2^* = q_1 - v_1 v_1^* \\ v_3^* v_3 = p_2 - v_2^* v_2, & v_3 v_3^* \leq q_2 \\ v_4^* v_4 \leq p_3, & v_4 v_4^* = q_2 - v_3 v_3^* \end{array}$$

and so on. In either case, the series $V = \sum_{n=1}^{\infty} v_n$ is strictly convergent. In case (i) we have $V^* V = P$, and $VV^* \leq Q$, and in case (ii), $V^* V = P$ and $VV^* = Q$. ∎

3. **Ideals in $\mathcal{M}(I)$ and quotients.** For the rest of the paper we shall denote by J the norm-closure of the set of elements $X \in \mathcal{M}(I)$ with $\tau(X^* X) < \infty$. G. Elliot [5] and H. Lin [6] prove that J is an ideal in $\mathcal{M}(I)$, that $0 \subseteq I \subseteq J \subseteq \mathcal{M}(I)$, and that $\mathcal{M}(I)$ has no other ideals than these. Moreover, $I \neq J$ if and only if I is non elementary (i.e., $I \not\cong \mathcal{K}$); and $J \neq \mathcal{M}(I)$ if and only if I is not finite (in the sense that τ is unbounded on the positive unit-ball of I), and this again is equivalent to I being stable. These results hold because I is assumed to be simple (and AF) with a *unique* trace.

LEMMA 3.1. *Let $X \in \mathcal{M}(I)$. There is an approximate unit $\{u_n\}_{n=1}^{\infty}$ for I such that $\lim_{n\to\infty} \|u_n X - X u_n\| = 0$ and $u_n u_{n-1} = u_{n-1}$ for all n. In fact, we may choose $\{u_n\}_{n=1}^{\infty}$ so that there is an approximate unit $\{e_n\}_{n=1}^{\infty}$ of projections in I such that for each n, $u_n e_n = e_n$ and $u_n e_{n+1} = u_n$.*

PROOF. Let $\{f_n\}_{n=1}^{\infty}$ be any approximate unit of projections for I. The argument of [1] (see also [7, Theorem 3.12.14]) shows that there is an approximate unit $\{w_n\}_{n=1}^{\infty}$ contained in conv$\{f_n\}$ such that $\|w_n X - X w_n\| \longrightarrow 0$. Thus we can choose some $u_1 \in$ conv$\{f_n\}$ with $\|u_1 X - X u_1\| < 2^{-1}$. Setting $e_1 = f_1$ we have $u_1 e_1 = e_1$. For sufficiently large N, any element u of conv$\{f_N, f_{N+1}, \ldots\}$ satisfies $uu_1 = u_1$, and so by the argument of [1] again, we can choose some u_2 in conv$\{f_N, f_{N+1}, \ldots\}$ such that $\|u_2 X - X u_2\| < 2^{-2}$. For $e_2 = f_N$ we have $e_2 u_1 = u_1$ and $u_2 e_2 = u_2$. Iterating this procedure, we obtain the desired approximate unit. ∎

PROPOSITION 3.2. *If $X \geq 0$ is an element of $\mathcal{M}(I)$, but not of J, then the hereditary subalgebra of $\mathcal{M}(I)$ generated by X contains an infinite trace projection.*

PROOF. Choose $\{u_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ as in Lemma 3.1, for which $\|u_n X - X u_n\|$ is so small that $\|d_n X - X d_n\| < 2^{-n}$, where $d_n = (u_n - u_{n-1})^{\frac{1}{2}}$ (and $u_0 = 0$). Then $\sum d_n X d_n = X + y_1$, where $y_1 = \sum_{n=1}^{\infty} d_n [X, d_n] \in I$. Let $p_n = e_{n+1} - e_{n-1}$ (where $e_0 = 0$). Note that $p_n d_n = d_n$ and that the projections $p_{2n-1} (n = 1, 2, \ldots)$ are pairwise disjoint, as are the projections $p_{2n} (n = 1, 2, \ldots)$. By perturbing each $d_n X d_n$, within $p_n I p_n$, by a suitable operator z_n, with say $\|z_n\| < 2^{-n}$, we may write

$$X + y = \sum x_n,$$

where $y = y_1 + \sum z_n \in I$, and $x_n = d_n X d_n + z_n$ is a positive element in $p_n I p_n$ with finite spectrum. Since $X \notin J$ it follows that $X + y \notin J$, and so (at least) one of $X_e = \sum x_{2n}$ or $X_o = \sum x_{2n+1}$ is not an element of J. Let us say $X_e \notin J$, and show first that the hereditary subalgebra generated by X_e contains an infinite trace projection. From $X_e \notin J$ it follows easily that X_e is not a norm limit of elements of $\mathcal{M}(I)^+$ of finite trace. From this it follows that for small enough $\varepsilon > 0$, the spectral projection P_ε of X_e corresponding to $[\varepsilon, \infty)$ (defined in $\mathcal{M}(I)$ since X_e is an orthogonal strict sum of elements of finite spectrum) has infinite trace. But all the P_ε are in the hereditary subalgebra generated by X_e. Now, the hereditary subalgebra generated by X_e is contained in the hereditary subalgebra A' generated by $X_e + X_0 = X + y$, so A' contains an infinite trace projection P'. The images in $\mathcal{M}(I)/I$ of the hereditary subalgebra A generated by X, and of A' are equal; therefore $A/A \cap I$ contains the image of P'. Since $A \cap I$ is an AF-algebra, this image lifts to a projection P in A (see [4]). It is easily verified that $\tau(P) = \infty$.

PROPOSITION 3.3. *If $X \geq 0$ is an element of J, but not of I, then the hereditary subalgebra of J generated by X contains a projection in $J \setminus I$.*

PROOF. Repeat the above decomposition of X into the sum $X = X_e + X_0 + y$, with say $X_e \notin I$. For suitable $\varepsilon > 0$ we have $\mathrm{dist}(X_e, I) > \varepsilon$, and so since $\|X_e - X_e P_\varepsilon\| \leq \varepsilon$ it follows that $P_\varepsilon \notin I$ for such ε. The rest of the above argument now produces a projection P in the hereditary subalgebra generated by X such that $P - P_\varepsilon \in I$. Since $P_\varepsilon \notin I$ it follows that $P \notin I$. ∎

These two propositions give more information than we actually need, which is the following corollary.

COROLLARY 3.4. *The C^*-algebra $\mathcal{M}(I)/J$ is purely infinite, as is PJP/PIP for every finite trace projection $P \in \mathcal{M}(I) \setminus I$.*

PROOF. Recall that a unital C^*-algebra different from $\mathbb{C}$ is said to be *purely infinite* if every non-zero hereditary subalgebra contains a projection equivalent to 1. For $\mathcal{M}(I)/J$ this follows immediately from Propositions 3.2 and 2.1. As for PJP/PIP, by Proposition 3.3 every hereditary subalgebra contains a non-zero projection, the image of a projection $Q \in PJP \setminus PIP$. Now, it follows from Lemma 1.2 that there is a projection $p \in PIP$ such that $\tau(P - p) < \tau(Q)$, and so by Proposition 2.1 there is a partial isometry W such that $WW^* \leq Q$ and $W^*W = P - p$. If V denotes the image of W in PJP/PIP then V is an isometry and so VV^* is a suitable projection in the hereditary subalgebra. ∎

The remaining two propositions in this section generalize to J two basic properties of AF-algebras. We need the following lemma.

LEMMA 3.5. *Let $p \in I$ be a projection and let $x \in (pIp)^+$. For any $\varepsilon > 0$ there is a projection $q \leq p$ with $\|(1 - q)x\| < \varepsilon$ and $\tau(q) < \frac{3}{\varepsilon}\tau(x)$.*

PROOF. Using the fact that τ is norm continuous on pIp and the fact that pIp is AF, we can reduce to the case where x lies in some finite dimensional C^*-subalgebra. Take q to be the spectral projection for x corresponding to $[\varepsilon/2, \infty)$. ∎

PROPOSITION 3.6. *Suppose that $X \in \mathcal{M}(I)^+$ and $\tau(X) < \infty$. For any $\varepsilon > 0$ there exists a projection $Q \in \mathcal{M}(I)$ with $\|(1-Q)X\| < \varepsilon$ and $\tau(Q) < \infty$.*

PROOF. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of mutually orthogonal projections in I such that $\sum_{n=1}^{\infty} p_n = 1$. By regrouping the p_n (as in Proposition 2.1) we may assume that for all n, $\|p_n X \sum_{|m-n|>2} p_m\| < \varepsilon 2^{-n}$ (compare [5]). For $Y = \sum_{|m-n|\le 2} p_n X p_m$ we have $\|X - Y\| < \varepsilon$. By Lemma 3.5, for each n there is a projection $q_n \le p_n$ such that $\|(1 - q_n)p_n X p_n\| < \varepsilon$ and $\tau(q_n) < \frac{3}{\varepsilon}\tau(p_n X p_n)$. The sum $Q = \sum q_n$ converges in the strict topology and

$$\tau(Q) < \frac{3}{\varepsilon}\sum_{n=1}^{\infty}\tau(p_n X p_n) = \frac{3}{\varepsilon}\tau(X) < \infty.$$

Furthermore,

$$\begin{aligned}
&\|(1-Q)Y\| \\
&\quad \le \|(1-Q)\textstyle\sum p_n X p_{n-1}\| + \|(1-Q)\sum p_n X p_n\| + \|(1-Q)\sum p_n X p_{n+1}\| \\
&\quad = \sup_n \|(1-q_n)p_n X p_{n-1}\| + \sup_n \|(1-q_n)p_n X p_n\| + \sup_n \|(1-q_n)p_n X p_{n+1}\|.
\end{aligned}$$

The middle term is no more than ε, by construction of the q_n. As for the other two terms, we have that

$$\begin{aligned}
\|(1-q_n)p_n X p_{n\pm 1}\|^2 &= \|(1-q_n)p_n X p_{n\pm 1} X p_n (1-q_n)\| \\
&\le \|(1-q_n)p_n X^2 p_n (1-q_n)\| \\
&\le \|X\| \cdot \|(1-q_n)p_n X p_n (1-q_n)\| \\
&\le \|X\|\varepsilon.
\end{aligned}$$

Thus $\|(1-Q)Y\| \le (1 + 2\|X\|)\varepsilon$, and so $\|(1-Q)X\| \le (2 + 2\|X\|)\varepsilon$. ∎

We remark that the classification of the ideals of $\mathcal{M}(I)$ follows easily from this and Corollary 3.4.

PROPOSITION 3.7. *Every projection in $\mathcal{M}(I)/J$ lifts to a projection in $\mathcal{M}(I)$.*

PROOF. Let $\bar{P}$ be a non-trivial projection in $\mathcal{M}(I)/J$. Applying Proposition 3.2 to any positive lifting of $\bar{P}^{\perp}$, we see that there is an infinite trace projection $Q \in \mathcal{M}(I)$ whose image in $\mathcal{M}(I)/J$ is orthogonal to $\bar{P}$. We can write Q as an orthogonal sum $Q = Q_2 + Q_3 + \cdots$ of infinite trace projections, the sum converging in the strict topology. Setting $Q_1 = Q^{\perp}$, which is also of infinite trace, and fixing a system of partial isometries between Q_1 and Q_n, we shall represent elements of $\mathcal{M}(I)$ as infinite matrices, with repsect to the decomposition $1 = \sum Q_i$, with entries in $Q_1\mathcal{M}(I)Q_1$. Let X be any lifting of $\bar{P}$ with $1 \ge X \ge 0$ (not necessarily a projection). Since $Q_1 X Q_1$ is also such a lifting, we may assume $X \in Q_1\mathcal{M}(I)Q_1$. Let $g_1, g_2, \ldots$ be a sequence of continuous, non-negative functions on $[0, 1]$ such that (i) $\operatorname{supp}(g_n) \subset [x_{n+2}, x_n]$, where $1 = x_1, x_2, \ldots$ is a sequence of points in $(\frac{1}{2}, 1]$ decreasing to $\frac{1}{2}$; and (ii) $\sum g_n = 1$ on $(\frac{1}{2}, 1]$ (note that for any x, at most two of the $g_n(x)$ are non-zero). Define functions f_n in terms of the g_n by

$$f_n(x) = \begin{cases} \left(g_n(x) - g_n(x)^2\right)^{\frac{1}{2}}, & x_{n+2} \le x \le x_{n+1}, \\ 0, & \text{otherwise}. \end{cases}$$

Since at each x_k every g_n is either 0 or 1, the f_n are continuous on [0, 1], The following relations are easily verified:

$$f_n f_m = 0 \qquad \text{if } n \neq m$$
$$g_{n+1} f_n + g_n f_n = f_n,$$
$$f_{n+1}^2 + g_n^2 + f_n^2 = g_n.$$

From these it follows that the element

$$p = \begin{pmatrix} g_1(X) & f_1(X) & & \\ f_1(X) & g_2(X) & f_2(X) & \\ & f_2(X) & g_3(X) & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

is a projection (it is easily seen that the matrix does indeed define an element of $\mathcal{M}(I)$). Since $g_1(1) = 1$ and $g_1(0) = 0$, the element $g_1(X)$ is a lifting of $\bar{P}$, so it suffices to show that the element of $\mathcal{M}(I)$, obtained by removing the $g_1(X)$ term from P, is an element of J. In fact, since $f_1(1) = f_1(0) = 0$, we have that $f_1(X) \in J$, and so it suffices to show that the positive element R obtained from P by deleting the terms $g_1(X)$ and $f_1(X)$ is in J. We will show that R is a norm limit of positive elements with finite trace. All the functions $f_n, g_n (n \geq 2)$ are supported within $[\frac{1}{2}, x_2]$, so there is a continuous function h on [0, 1] with $h \geq 0$, and $hf_n = f_n$, $hg_n = g_n$ for all $n \geq 2$. We have that $h(X) \in J$, and so there is, for every $\varepsilon > 0$, an $X_\varepsilon \in Q_1 \mathcal{M}(I) Q_1^+$ with $\tau(X_\varepsilon^2) < \infty$ and $\|h(X) - X_\varepsilon\| < \varepsilon$. The element R_ε obtained from R by multiplying each entry on the left and right by X_ε satisifies $\|R_\varepsilon - R\| < 2\varepsilon$ and $\tau(R_\varepsilon) = \sum_{n=2}^{\infty} \tau\left(X_\varepsilon g_n(X) X_\varepsilon\right)$. Since

$$\sum_{n=2}^{N} X_\varepsilon g_n(X) X_\varepsilon = X_\varepsilon \left(\sum_{n=2}^{N} g_n(X) \right) X_\varepsilon \leq X_\varepsilon^2$$

we see that $\tau(R_\varepsilon) < \infty$. ∎

4. **Property** *FS* **for** $\mathcal{M}(I)$. The following two quite general lemmas reduce the main theorem to the properties of $\mathcal{M}(I)$ and J that we have already established.

LEMMA 4.1. ([9], Part III, Proposition 2.33). *Let D be a unital C^*-algebra and let L be an ideal in D such that every projection in D/L lifts to a projection in D. If L and D/L have property FS then so does D.*

PROOF. The fact that projections lift from D/L to D implies that every self-adjoint, invertible $\bar{s} \in D/I$ lifts to a self-adjoint invertible in D. Indeed, by polar decomposition we can write $\bar{s} = \bar{t}(\bar{p} - \bar{p}^\perp)\bar{t}$, ($\bar{t} = |\bar{s}|^{\frac{1}{2}}$), and since $\bar{t}$ certainly lifts to some positive invertible t, $\bar{s}$ lifts to $t(p - p^\perp)t$, where p lifts $\bar{p}$. Given that D/L has property FS, we see that any self-adjoint element $x \in D$ may be approximated by elements of the form $s + y$, with s invertible and $y \in L$. Thus it suffices to approximate every $s + y$ by self-adjoint invertibles. Writing $s = (p - p^\perp)|s|$, we have that

$$s + y = |s|^{\frac{1}{2}}(p - p^\perp + |s|^{-\frac{1}{2}} y |s|^{-\frac{1}{2}})|s|^{\frac{1}{2}}$$

and so putting $z = |s|^{-\frac{1}{2}} y |s|^{-\frac{1}{2}}$ we see that it suffices to approximate each element of the form $p - p^{\perp} + z$, $(z \in L)$, by self-adjoint invertibles. Both pLp and $p^{\perp}Lp^{\perp}$ are hereditary subalgebras of L, and so for any $\varepsilon > 0$ there exist projections $q_1 \in pLp$ and $q_2 \in p^{\perp}Lp^{\perp}$ such that $\|(1 - q_1)pz^2p\| < \varepsilon^2$ and $\|(1 - q_2)p^{\perp}z^2p^{\perp}\| < \varepsilon^2$. Let $q = q_1 + q_2$. This projection commutes with p, and almost supports z:

$$\begin{aligned}\|(1 - q)z\| &\leq \|(1 - q)pz\| + \|(1 - q)p^{\perp}z\| \\ &= \|(1 - q_1)pz\| + \|(1 - q_2)p^{\perp}z\| < 2\varepsilon.\end{aligned}$$

Thus $\|z - qzq\| < 4\varepsilon$. Since qLq has property FS, there is a self-adjoint invertible $qwq \in qLq$ such that $\|qwq - q(p - p^{\perp} + z)q\| < \varepsilon$. The element $r = q^{\perp}(p - p^{\perp})q^{\perp} + qwq$ is a self-adjoint invertible with $\|r - (p - p^{\perp} + z)\| < 5\varepsilon$. ∎

LEMMA 4.2. (cf. [10]). *If E is a purely infinite C^*-algebra then E has property FS.*

PROOF. Let $x \in E$ be self-adjoint and let $\varepsilon > 0$. Let $g: \mathbf{R} \longrightarrow \mathbf{R}^+$ be a continuous function, supported within $(-\varepsilon/3, \varepsilon/3)$, and equal to 1 near 0. If $g(x) = 0$ then x is invertible (and so is certainly approximable by invertibles); if $g(x) \neq 0$ then there is a projection $p \in \overline{g(x)Eg(x)}$ equivalent to 1. By definition of p, $\|px\| < \varepsilon/3$, and so $\|x - p^{\perp}xp^{\perp}\| < 2\varepsilon/3$. There is some $v \in E$ with $v^*v = 1$ and $vv^* = p$; let $s = vp^{\perp} + p^{\perp}v^* + p - vp^{\perp}v^*$. This is a symmetry ($s = s^* = s^{-1}$) and furthermore $p^{\perp}sp^{\perp} = 0$. The self-adjoint element

$$y = p^{\perp}xp^{\perp} + \frac{\varepsilon}{3}s = \frac{\varepsilon}{3}s(\frac{3}{\varepsilon}sp^{\perp}xp^{\perp} + 1)$$

is invertible, since $(sp^{\perp}xp^{\perp})^2 = 0$, and $\|y - x\| \leq \|x - p^{\perp}xp^{\perp}\| + \frac{\varepsilon}{3}\|s\| < \varepsilon$. ∎

PROPOSITION 4.3. *The ideal J has prpoperty FS.*

PROOF. By Proposition 3.6 it suffices to show that for each finite trace projection P, the C^*-algebra PJP has property FS. But this follows from the fact that PJP/PIP is purely infinite (Corollary 3.4), and the fact that PIP is AF, so that it has property FS and projections lift (see [4]). ∎

THEOREM 4.4. *The C^*-algebra $\mathcal{M}(I)$ has property FS.*

PROOF. This follows immediately from the above results, Corollary 3.4 and Proposition 3.7. ∎

REFERENCES

1. W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. Journal **44**(1977), 329–355.
2. B. Blackadar, *Notes on the structure of projections in simple C^*-algebras*, Semesterbericht Funktionalanalysis, W82, Tübingen, March 1983.
3. L. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, preprint.
4. E. Effros, *Dimensions and C^*-algebras*. CBMS Regional Conf. Ser. in Math., no. 46, A.M.S., Providence, 1981.
5. G. Elliot, *Derivations of matroid C^*-algebras, II*, Ann. Math. **100**(1974), 407–422.

6. H. Lin, *Ideals of multiplier algebras of simple AF C^*-algebras*, Proc. A.M.S. **104**(1988), 239–244.
7. G. K. Pedersen, *C^*-algebras and their auotmorphism groups*. Academic Press, London/New York/San Francisco, 1979.
8. ———, *The linear span of projections in simple C^*-algebras*, J. Operator Theory **4**(1980), 289–296.
9. S. Zhang, *K_1-groups, quasidiagonality and interpolation by multiplier projections*, Trans. A.M.S., to appear.
10. ———, *C^*-algebras with real rank zero and the internal structure of their corona and multiplier algebras, parts I-IV*, preprints.

Department of Mathematics
The Pennsylvania State University
218 McAllister Building
University Park, PA 16802
USA

Department of Mathematics and Computer Science
Odense University
Campusvej 55
DK-5230 Odense M
Denmark