K-Homology and [Q,R]=0

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Introduction

My goal is to explain the *quantization commutes with reduction problem* from the perspective of K-theory and K-homology.

The problem was formulated by Guillemin and Sternberg in the early 1980's, and solved by Meinrenken, then others, about a dozen years later.

There are basically two approaches to the solution, one geometric (Meinrenken's) and one analytic (due to Tian and Zhang). They mirror two approaches to defining K-homology theory and I'll touch on both, although I'll concentrate mostly on the geometry.

This is joint work with my PhD student Yanli Song.

Many thanks to Eckhard Meinrenken.

What is [Q,R]=0 About?

Reduction is a sort of quotient operation defined for group actions on symplectic manifolds.

Quantization is a concept borrowed from physics, of course. Only very simple examples are relevant here, and index theory techniques can be used to define quantization.

The problem is to show that reduction and quantization are compatible (at least in the simple examples under consideration) in the simplest possible way.

This issue arises in representation theory (via coadjoint orbits or the Borel-Weil theorem). Going in the opposite direction, I'll aim to apply representation theory (the Weyl character formula) to the quantization commutes with reduction problem. The new(?) ingredient is *Bott periodicity.*

Symplectic Manifolds and Moment Maps

$$\begin{array}{lcl} M & = & \text{Symplectic manifold} \\ \{\ ,\ \} & = & \text{Poisson bracket on } M \end{array}$$

Thanks to the Poisson bracket, every function f on M determines a *Hamiltonian vector field* X_f on M via

$$X_f g = \{f, g\}.$$

This process can be reversed, more or less, for a vector field (infinitesimal motion) that preserves the symplectic structure.

The $[Q, R] = 0$ problem concerns *Hamiltonian actions* of a connected compact Lie group G. These are actions that are determined by a *moment map*

$$\mu \colon M \longrightarrow \mathfrak{g}^* ; \qquad \{\mu_X, \mu_Y\} = \mu_{[X,Y]} \quad \forall X, Y \in \mathfrak{g}.$$

Reduction

M = Symplectic manifold with symplectic form ω.
μ = Moment map generating a Hamiltonian action of G.

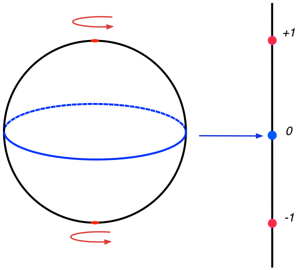
Assume that $0 \in \mathfrak{g}^*$ is a regular value of the moment map.

Then G acts (locally) freely on the submanifold $\mu^{-1}[0] \subseteq M$.

Definition

The *reduction* of M is the manifold (or orbifold)

$$M_{\text{red}} = M /\!\!/ G = \mu^{-1}[0]/G.$$



Lemma

The reduction M_{red} is symplectic: it carries a unique ω_{red} whose pullback to $\mu^{-1}[0]$ is the restriction of ω.

Quantization of Symplectic Manifolds

Following Dirac, the general aim is to construct a Hilbert space $Q(M)$, together with a correspondence

Functions on M $\rightsquigarrow$ Self-adjoint operators on $Q(M)$.

The key requirements are

$$Op(\{f_1, f_2\}) = \frac{i}{2\pi}[Op(f_1), Op(f_2)]$$
$$Op(1) = \textit{Identity operator.}$$

Experience shows that the symplectic manifold must satisfy an integrality requirement (and maybe more). Moreover neither existence nor uniqueness of quantizations is easily resolved.

In fact in the $[Q, R] = 0$ problem we shall settle for the construction of a "virtual" Hilbert space.

Quantization Commutes with Reduction Problem

$$\begin{aligned} M &= \text{Closed, symplectic manifold} \\ \mu &= \text{Moment map generating a Hamiltonian action of } G \end{aligned}$$

Assume this data is quantizable and consider applying first quantization, then reduction (on the quantized side, the reduction is the space of G-coinvariants):

$$(M, \mu) \rightsquigarrow Q(M) \rightsquigarrow Q(M)^G.$$

Alternatively, consider first reduction, then quantization:

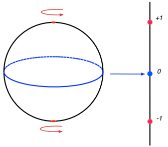
$$(M, \mu) \rightsquigarrow M /\!\!/ G \rightsquigarrow Q(M /\!\!/ G).$$

Problem (Guillemin and Sternberg)

Show that the two processes give same result.

Example: Branching Problems

H	Compact connected Lie group
$M \subseteq \mathfrak{h}^*$	Integral coadjoint orbit
G	Subgroup of H
μ	Projection from $\mathfrak{h}^*$ to $\mathfrak{g}^*$



According to the Kirillov philosophy (and the Borel-Weil theorem), integral coadjoint orbits in $\mathfrak{h}^*$ correspond, under quantization to irreducible representations of H.

How does the representation decompose into irreducible representations of H?

The $[Q, R] = 0$ problem gives formulas ...

For example, if 0 is not in the image of $\mu \colon M \to \mathfrak{g}^*$, then the trivial representation of G has zero multiplicity in $Q(M)$.

Kähler Quantization

The simplest type of quantization, mathematically speaking, is *Kähler quantization* ...

M = Complex manifold, closed for simplicity.
L = Holomorphic line bundle on M.

Definition

$$Q(M, L) := \{\text{Holomorphic sections of } L\}.$$

Of course L may not have *any* holomorphic sections, but assume it has many. Define

$$\mathbb{P} = \textit{Projective space of lines in } Q(M, L)^*.$$

There is a *coherent states map*, or *Kodaira embedding*

$$M \longrightarrow \mathbb{P}.$$

Observables in Kähler Quantization

$M \subseteq \mathbb{P}$	Complex submanifold.
L	Dual of the tautological bundle.

There is a canonical connection ∇ on L, and its curvature defines a Kähler structure on M. In particular the curvature defines a symplectic form through the *prequantization condition*

$$\omega = \frac{i}{2\pi}\nabla^2.$$

Given $f\colon M \to \mathbb{R}$ define

$$Op(f) = \frac{i}{2\pi}\nabla_{X_f} + f,$$

on smooth sections of L. Here X_f is the *Hamiltonian vector field* determined by f. Then

$$Op(\{f_1, f_2\}) = \frac{i}{2\pi}[Op(f_1), Op(f_2)]$$

and $Op(1) = I$ as required.

Prequantization

M = Symplectic manifold with symplectic form ω.
L = Complex hermitian line bundle on M.

Assume given a connection ∇ on L with curvature

$$\omega = \frac{i}{2\pi}\nabla^2$$

(so in particular ω is integral). Then we can define prequantum operators $Op(f)$ using the formula from the complex case.

From the point of view of geometric quantization theory the problem is now to reduce the domain of the $Op(f)$ to an appropriate size by equipping (M, L) with a *polarization* (e.g. a complex structure).

Quantization and Index Theory

Bott pointed out that quantization (in the simplified context we are studying) can be studied from the perspective of K-theory and index theory.

A bonus: *any closed symplectic manifold M with a prequantum line bundle L can be quantized this way.*

Every symplectic manifold M may be equipped with an *almost complex structure* and every almost complex, or even *weakly complex* manifold, admits a $\overline{\partial}$-operator.

Definition

$$Q(M, L) = Index(\overline{\partial}_L) \in R(G).$$

Theorem (Meinrenken)

If M is any closed quantizable symplectic manifold with a Hamiltonian G-action, then $[Q, R] = 0$.

Topological K-Theory

Of course, there are close connections between index theory and K-theory ...

If $f\colon M \to N$ is any continuous map between (weakly) complex manifolds, then there is an associated *wrong-way map* in Atiyah-Hirzebruch K-theory

$$f_*\colon K^0(M) \longrightarrow K^0(N).$$

The construction is *functorial* and *homotopy invariant*.

Wrong-way maps are defined using Bott periodicity. *They also encode Bott periodicity.* For if V is a complex vector space, then consider

$$V \longrightarrow pt \longrightarrow V.$$

Functoriality and homotopy invariance imply the associated wrong-way maps are isomorphisms.

Index Theorem, Quantization and Topology

Consider $p\colon M \to pt$. Atiyah and Hirzebruch invented K-theory to prove that if $[E] \in K^0(M)$, then

$$p_*([E]) = \int_M Todd(TM) \wedge ch(E),$$

which implies the integrality of the right-hand side.

Atiyah and Singer proved that

$$Index(\overline{\partial}_E) = \int_M Todd(TM) \wedge ch(E) = p_*([E]).$$

More generally,

$$Index(\overline{\partial}_E) = p_*([E]) \in K^0_G(pt) = R(G).$$

So quantization, as we are studying it, has a purely topological aspect, deriving from Bott periodicity.

K-Homology

Various topics in index theory become more conceptual when viewed from the perspective of *K-homology.* This is the dual theory to K-theory, characterized by functorial pairings

$$K_0(X) \otimes K^0(X \times Y) \stackrel{\textit{slant product}}{\longrightarrow} K^0(Y)$$

(compare integration over the fiber in homology/cohomology).

Roughly speaking, K-homology conveniently encodes the theory of wrong-way maps (which encodes Bott periodicity).

The K-homology groups (and their equivariant counterparts) can be defined using homotopy theory, geometry or analysis. I'll mostly focus on the geometric approach.

Geometric Cycles for K-Homology (after Baum)

Any class in $K_0(X)$ can be represented by a *geometric cycle* (M, E, f), where

- M is a closed (weakly) complex manifold,
- E is a complex vector bundle on M,
- f is a continuous map from M into X.

The product with K-theory works like this (I've written it when $Y = pt$):

$$K^0(X) \xrightarrow{\text{pullback along } f} K^0(M) \xrightarrow{\text{tensor with } E} K^0(M) \xrightarrow{\text{pushforward to } pt} K^0(pt).$$

In fact there is a *geometric definition* of K-homology, given by geometric cycles modulo a suitable equivalence relation.

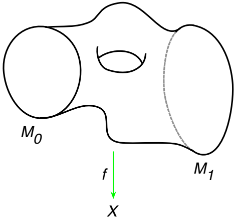
Equivalence Relation on Geometric Cycles

The equivalence relation among cycles that defines K-homology geometrically has two major components:

Bordism. If (M_0, E_0, f_0) and (M_1, E_1, f_1) are bordant over X, then

$$[M_0, E_0, f_0] = [M_1, E_1, f_1]$$

in $K_0(X)$.



Bundle Modification. Here is an example. If $\widehat{M}$ is a bundle of projective spaces over M, then

$$[M, E, f] = [\widehat{M}, \widehat{E}, \widehat{f}]$$

in $K_0(X)$, where $\widehat{E}$ is the pullback of E and $\widehat{f}$ is the composition of f with the projection from $\widehat{M}$ to M.

Analytic Cycles for K-Homology

Following a suggestion of Atiyah, Kasparov gave a functional-analytic definition of the K-homology group $K_0(X)$, based on analytic cycles (H, F), where:

- *H* is a Hilbert space with an action of $C(X)$, and
- *F* is a pseudolocal Fredholm operator on H.

Kasparov's equivalence relation is homotopy.

There is a *quantization isomorphism*

$$Q\colon K_{0,geometric}(X) \longrightarrow K_{0,analytic}(X).$$

It associates to a geometric cycle (M, E, f) an analytic cycle constructed from the $\overline{\partial}$-operator on M.

Reduction Map in K-Homology

The geometric and analytic constructions both generalize to the equivariant case (when G is compact), giving groups $K_0^G(X)$.

The equivariant K-homology groups are modules over the representation ring $R(G)$.

Lemma

If the action of G on a space Z is trivial, then the natural map

$$R(G) \otimes K_0(Z) \longrightarrow K_0^G(Z)$$

is an isomorphism.

Apply this to $Z = X/G$ to *define a reduction map in K-homology*

$$K_0^G(X) \longrightarrow K_0^G(X/G) \cong R(G) \otimes K_0(X/G) \longrightarrow K_0(X/G).$$

The last map is projection onto multiples of $[1_G] \in R(G)$.

Reduction in Analytic K-Homology

The reduction map

$$R\colon K_0^G(X) \longrightarrow K_0(X/G)$$

just defined is easy to calculate in analytic K-homology (because the K-homology isomorphism in the lemma is easy to invert).

Indeed, if (H, F) is an equivariant analytic cycle, then

$$R\colon [H, F] \mapsto [H^G, F|_{H^G}].$$

If $(H, F) = Q(M, E)$ is the quantization of a geometric cycle over $X = pt$, then this gives the multiplicity of the trivial representation in Bott's quantization of (M, E).

In other words, quantization in K-homology, followed by reduction in K-homology, is the RQ half of the $[Q, R] = 0$ story.

[Q,R]=0 Problem in K-Homology

The diagram

$$\begin{array}{ccc} K^G_{0,geometric}(X) & \xrightarrow{Q} & K^G_{0,analytic}(X) \\ \downarrow R & & \downarrow R \\ K_{0,geometric}(X/G) & \xrightarrow[Q]{} & K_{0,analytic}(X/G) \end{array}$$

commutes.

This is a tautology!

The problem is to *give a concrete geometric description of the reduction map on the left-hand side* (at the level of cycles, at least for certain cycles).

The problem may not have a good solution in general, but it does when G is a torus, and when (M, E) is prequantum data (this is the $[Q, R] = 0$ theorem).

Localization of the Representation Ring

Let T be a torus (eventually it will be a maximal torus in a compact connected G). Then

$$R(T) \cong \textit{A ring of Laurent polynomials}.$$

Notation: I'll write characters of T (i.e. basis elements of $R(T)$) additively:

$$T \ni z \mapsto z^\alpha \in U(1).$$

Now define

$$\begin{aligned} R(T)_{\text{loc}} &= R(T) \text{ with all } (1 - z^\alpha)^{-1} \text{ adjoined } (\alpha \neq 0) \\ &\cong \textit{A ring of rational functions.} \end{aligned}$$

Localization in K-Homology

Localized equivariant K-homology is defined algebraically by extension of scalars:

$$K_0^T(X)_{\mathrm{loc}} = R(T)_{\mathrm{loc}} \otimes_{R(T)} K_0^T(X).$$

But *localized equivariant K-homology can be defined geometrically too.* We use the same (M, E, f) cycles and equivalence relation, but *we require only M^T compact rather than M compact.* The correspondence

$$\text{Geometry} \leftrightarrow \text{Algebra}$$

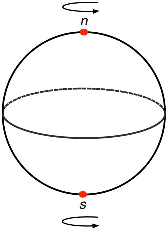
is

$$\mathbb{C}_\alpha \leftrightarrow (1 - z^{-\alpha})^{-1}.$$

This determines a map from the algrebraically defined groups to the geometric groups, which is an isomorphism.

Example: The Riemann Sphere

$M = S^2$
$T = U(1)$, acting by rotations
$L =$ Line bundle with weights z^n and z^s at the poles.



The $K_0^t(pt)_{loc}$-cycle (M, L) is bordant to $(T_s M, \mathbb{C}_s) \sqcup (T_n M, \mathbb{C}_n)$.

Therefore

$$[M, L] = \frac{z^s}{1 - z} + \frac{z^n}{1 - z^{-1}} \in K_T(pt)_{\mathrm{loc}}.$$

Thus for instance

$$Index(\overline{\partial}_L) = z^s + \cdots + z^n \in R(T)$$

if $s \leq n$.

Example: Weyl Character Formula

$G =$ Compact, connected group.
$T =$ Maximal torus.
$M = G/T =$ A complex manifold (after choosing positive roots).
$L = G$-equivariant line bundle with weight φ at eT.

Once again, the fixed set M^T is finite and M is bordant to a collection of tangent spaces. So

$$[M, L] = \sum_{w \in W} [T_w M, L_w] = \sum_{w \in W} \frac{z^{w(\phi)}}{\prod_{\alpha > 0}(1 - z^{-w(\alpha)})} \in K_0^T(pt)_{\mathrm{loc}}.$$

As a result

$$Index(\partial_L) = \frac{\sum_{w \in W}(-1)^w z^{w(\phi+\rho)-\rho}}{\prod_{\alpha>0}(1 - z^{-\alpha})} \in R(T).$$

A Rigidity Theorem

$(M, E, f) =$ Geometric cycle for $K_0^T(X)$.

The restriction of E to the fixed set $M^T \subseteq M$ decomposes as

$$E|_{M^T} = \sum E|_{M^T}^{\varphi} \otimes \mathbb{C}_{\varphi}.$$

The φ for which $E|_{M^T}^{\varphi} \neq 0$ are the *weights* of $E|_{M^T}$.

Theorem

If $\varphi = 0$ is the only weight of $E|_{M^T}$, then the class

$$[M, E, f] \in K_0^T(X/T) = R(T) \otimes K_0(X/T)$$

is concentrated at $1_T \in R(T)$.

Translation: *The reduction of $[M, E, f] \in K_0^T(X)$ is $[M, E, f] \in K_0(X/T)$.*

A Vanishing Theorem

Assume for simplicity T is a circle group.

Theorem

If all the weights φ of $E|_{M^T}$ are positive, then the reduction of $[M, E, f] \in K_0^T(X)$ is $0 \in K_0(X/T)$.

Proof.

Decompose the normal bundle N of M^T into weight bundles:

$$N = \sum N^\alpha \otimes \mathbb{C}_\alpha.$$

Assume for simplicity that each component N^α is a trivial bundle. Then, according to the correspondence between the geometric and algebraic versions of localized K-homology,

$$[M, E] = z^\varphi \prod (1 - z^{-\alpha})^{-1} \otimes [M^T, E|^\varphi_{M^T}]$$

in $R(T)_{\text{loc}} \otimes_{R(T)} K_0(X/T)$. This "rational function" is regular, and indeed vanishes, at $z = 0$. □

[Q,R]=0 for the Circle Group

$(M, E, f) =$ Geometric cycle for $K_0^T(X)$.

I'll assume that E is a line bundle (c.f. splitting principle). I want to compute the image under the reduction map

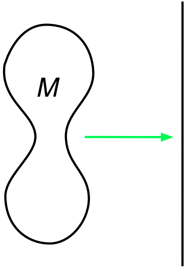
$$K_0^T(X) \longrightarrow K_0(X/T).$$

Define a *(generalized) moment map*

$$\mu \colon M \longrightarrow \mathbb{R}$$

using the generator D_V of the T-action on sections of E and a connection ∇ on E:

$$D_V = \nabla_V + \frac{i}{2\pi}\mu.$$



Lemma

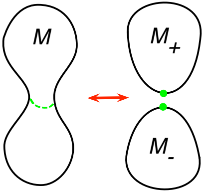
(a) The values of μ on M^T are the (integral) weights of $E|_{M^T}$.
(b) If $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, the T-action on $\mu^{-1}[\alpha]$ is (locally) free.

Cutting

$\alpha \in \mathbb{R}$ nonintegral regular value of μ_X.

Cut M into two smooth manifolds with common boundary $\mu_X^{-1}[\alpha]$.

Collapse T-orbits in boundaries to obtain closed manifolds (or orbifolds) $M_\pm$.



More precisely,

$$M_\pm = \{(m, z) \in M \times \mathbb{C} : \mu_X(m) \pm |z|^2 = \alpha\}/T.$$

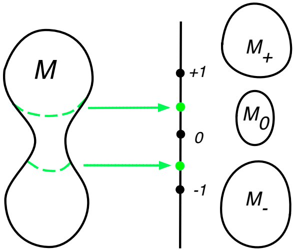
This is a bordism of weakly complex manifolds:

$$M \sim M_+ \sqcup M_-,$$

and if E is descended to $E_\pm$ on $M_\pm$, then

$$[M, E, f] = [M_+, E_+, f_+] + [M_-, E_-, f_-] \in K_0^T(X/T).$$

Step 1 of [Q,R]=0



Cut at regular values.

By the vanishing theorem, neither M_+ nor M_- contributes to the reduction of $[M, E, f]$.

By the rigidity theorem, the reduction of $[M_0, E_0, f_0]$ is itself.

Remark: Making further cuts, we can find all the "Fourier coefficients" $[M_k, E_k, f_k] \in K_0(X/T)$, giving

$$K_0^T(X) \ni [M, E, f] \mapsto \sum_k [\mathbb{C}_k] \otimes [M_k, E_k, f_k] \in R(T) \otimes K_0(X/T).$$

Step 2 of [Q,R]=0

Summary so far: a bordism argument calculates the image of $[M, E, f]$ in the group $K_0(X/T) \otimes R(T)$, and in particular calculates the reduction of $[M, E, f]$.

In the symplectic situation, we can take an extra step.

Theorem

If M is symplectic and L is a prequantum line bundle, and if 0 is a regular value of the moment map, then the manifold M_0 is a bundle modification with fiber $\mathbb{C}P^1$ of the reduced manifold $M /\!\!/ T$.

Hence

$$R[M, L, f] = [M /\!\!/ T, L /\!\!/ T, f /\!\!/ T] \in K_0(X/T)$$

in this case, as per $[Q, R] = 0$.

Nonabelian Groups

Higher-dimensional tori can be treated by iterating the argument just given.

When G is non-abelian, things are more complicated, and there is probably no simple geometric definition of the reduction map in K-homology.

However, a construction involving Bott periodicity and Weyl's character formula gives a partial answer, and it can be pushed much further in the case of a quantizable symplectic manifold (M, E).

The basic idea: a moment map

$$\mu\colon M \longrightarrow \mathfrak{g}^*$$

can be decomposed into

$$\mu^T\colon M \longrightarrow \mathfrak{t}^* \quad \text{and} \quad \mu^{G/T}\colon M \longrightarrow \mathfrak{g}/\mathfrak{t}^*,$$

and μ^T *is the moment map for a T-action,* while $\mathfrak{g}^*/\mathfrak{t}^*$ *has the structure of a complex T-vector space (via positive roots).*

The Weyl Character Formula and the Bott Element

(M, E) = Prequantum data.

Use the "partial moment map"

$$\mu^{G/T} \colon M \longrightarrow \mathfrak{g}^*/\mathfrak{t}^*$$

to pull back the Bott generator $\Lambda \in K^0_T(\mathfrak{g}^*/\mathfrak{t}^*)$ to M.

Form the product $(M, \Lambda \otimes E)$, then quantize. The result is

$$Q(M, E \otimes \Lambda) = \prod_{\alpha > 0} (1 - z^{-\alpha}) \cdot Q(M, E, f) \in R(T).$$

The multiplicative factor is Weyl's denominator, so from Weyl's formula

Multliplicity of 1_G *in* $Q(M, E) \in R(G)$

$= $ *Multiplicity of* 1_T *in* $Q(M, \Lambda \otimes E) \in R(T)$.

[Q,R]=0 for the Group G = SU(2)

So, can we calculate $Q(M, \Lambda \otimes E) \in R(T)$? Without Λ, we already calculated it. For, say, the group $SU(2)$ one has $\Lambda = (1 - z^{-2})$, and there is a "shift" by $\alpha = 2$ to worry about.

The *symplectic structure* gives the formula

$$g(JX, Y) = \omega(X, Y) = \mu_{[X,Y]},$$

which is *a relation between the geometry of the action and the values of the moment map μ^T that is not available for general manifolds.* This rescues the Bott periodicity approach.

From the geometric formula, the normal bundle to M^T contains $\mathbb{C}_2$ wherever $\mu^T > 0$. So, thanks to the correspondence between algebraic and geometric localization, the term $(1 - z^{-2})$ appears in the denominator of the index formula, canceling Λ.

The vanishing theorem now applies to $M_\pm$. A bundle modification argument shows [Q,R]=0.

Thank You!