Non-Commutative Spaces

John Roe
Penn State University
roe@math.psu.edu

June 19, 2000

Abstract

In this lecture we will describe some examples of noncommutative spaces in the sense of Connes.

Commutative geometry

Theorem 1. [R. Descartes] *Euclidean geometry is the study of three functions x, y, z on $\mathbb{R}^3$.*

In other words, one can recover the geometry of the space $\mathbb{R}^3$ by studying functions on it (the coordinates). For another similar example, consider the following simple theorem: a compact space X is connected iff the algebra $C(X)$ of continuous complex-valued functions on it has no non-trivial *idempotent* (an idempotent in an algebra A is an element $a = a^2 \in A$.)

These results are special cases of a general theorem: the commutative algebra $C(X)$ completely determines the compact Hausdorff space X. (The points of X are just the homomorphisms $C(X) \to \mathbb{C}$.) If X is *locally* compact we replace $C(X)$ by $C_0(X)$, the algebra of continuous complex-valued functions that vanish at infinity.

What is noncommutative geometry?

Basic idea of NCG: we want to consider 'spaces' whose 'coordinate functions' *do not commute*. Motivation comes from the observables of quantum theory.

Key features of Connes' NCG are:

- Replace points by functions,
- Matrices,
- Positivity and Hilbert space.

For instance, we might want to consider the 'quotient space' $S^1/\mathbb{Z}$, where $\mathbb{Z}$ acts by an irrational rotation R. Classically this quotient space is structureless because the $\mathbb{Z}$ action is ergodic. But the quotient space is naturally represented by a noncommutative C^*-algebra, which has a rich structure.

Noncommutative Quotient Spaces

Consider a simple example, of two points identified by an equivalence relation.

This corresponds to the matrix algebra $M_2(\mathbb{C})$.

More generally consider an *étale* equivalence relation $\mathcal{E}$ on a compact space X. For example in the case of the irrational rotation algebra, $\mathcal{E} = S^1 \times \mathbb{Z}$ and $r, s \colon \mathcal{E} \to S^1$ are given by $r(x, n) = x$, $s(x, n) = R^n x$. It now makes sense to multiply 'continuous matrices compactly supported on $\mathcal{E}$', getting a noncommutative algebra $C_c\mathcal{E}$.

The irrational rotation algebra

Keep thinking about the example above. Clearly the algebra is generated by functions on S^1 together with an operator V corresponding to the rotation.

But functions on S^1 can be expanded in Fourier series. Using U to denote the generator (i.e. the function z) we get

Example The *irrational rotation algebra* A_α is generated by two unitaries U and V subject to the relation

$$UV = e^{2\pi i\alpha} VU.$$

This is the algebra corresponding to the *transformation groupoid* (see later) $S^1 \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on S^1 via the irrational rotation α.

Note the word 'unitary' — We have implicitly represented the algebra on an appropriate Hilbert space. This allows us to remove the asymmetry between U and V; we complete to a C^**-algebra*. More about this later.

Quotients and Morita equivalence

Let $\mathcal{E}$ be an étale equivalence relation on the locally compact Hausdorff space X. In order that $X/\mathcal{E}$ be a 'good' (i.e. Hausdorff) space, it is sufficient that $\mathcal{E}$ be *proper* (that is, r and s are proper maps).

Theorem 2. (Rieffel) *Let $\mathcal{E}$ be a proper étale equivalence relation on X as above. Then $C^*(\mathcal{E})$ is* Morita equivalent *to the algebra $C_0(X/\mathcal{E})$.*

(Two C^*-algebras are *Morita equivalent* if they are 'the same up to finite matrices', for instance $\mathbb{C}$ and $M_2(\mathbb{C})$ are Morita equivalent. It is a theorem that A and B are Morita equivalent iff $A \otimes \mathfrak{K}$ and $B \otimes \mathfrak{K}$ are isomorphic.)

Summary: $C^*(\mathcal{E})$ is a noncommutative generalization of $C_0(X/\mathcal{E})$. But $C^*(\mathcal{E})$ can have interesting structure even when $C_0(X/\mathcal{E})$ is highly degenerate.

The Powers-Rieffel idempotent

To show the non-triviality of the irrational rotation algebra A_α we will construct an interesting idempotent. Write elements of A_α as $\sum f_n V^n$, where the f_n are functions on the circle $\mathbb{R}/2\pi\mathbb{Z}$ (regard U as the function $e^{i\theta}$).

Choose $[a,b] \subset [0,2\pi]$ disjoint from its $2\pi\alpha$-translate. Then let f and g_-, g_+ be functions whose graphs are illustrated below, with $g_-^2 + g_+^2 = f^2 - f$, $Vg_-V^* = g_+$, and put

$$e = g_-V^* + f + g_+V.$$

This is an idempotent (exercise!).

Measuring the size of idempotents

The linear functional

$$\tau\colon \sum f_n V^n \mapsto \frac{1}{2\pi}\int f_0$$

is a *trace* on A_α, that is, $\tau(aa') = \tau(a'a)$. We use it to measure the size of idempotents.

Clearly $\tau(1) = 1$, $\tau(e) = \alpha$ for the Powers-Rieffel idempotent e.

Theorem 3. *For any idempotent $x \in A_\alpha$ (or in a matrix algebra $M_n(A_\alpha)$) we have $\tau(x) \in \mathbb{Z} + \alpha\mathbb{Z}$.*

Why? A geometric perspective comes through foliation theory.

Groupoids 101

Definition 1. *A* groupoid *is like a group, except that composition is not defined everywhere.*

More precisely: a groupoid is comprised of a set G of arrows, a set G^0 of objects, two maps $s, r\colon G \to G^0$, and an associative composition law which allows one to form $g_1 g_2 \in G$ provided that $s(g_1) = r(g_2)$. One also requires that each object should have an associated identity arrow in G^0, and that each arrow should have a two-sided inverse.

We will usually consider locally compact *topological groupoids* (obvious definition). Such a groupoid is *étale* if r and s are local homeomorphisms, and it is *smooth* if everything is C^∞ and r and s are subimmersions.

Examples of Groupoids

- A topological *space* X can be considered as a groupoid (with only the identity arrows, i.e. $G = G^0 = X$).

- An *equivalence relation* on X defines a groupoid with objects X (there is exactly one arrow between two points of X if they are equivalent, otherwise none).

- A topological *group* can be considered as a groupoid (with only one object).

- Suppose that a group Γ acts on a space X (on the right). The *transformation groupoid* $X \rtimes \Gamma$ associated to this action has objects X, and arrows pairs (x, γ) with $s(x, \gamma) = x\gamma$, $r(x, \gamma) = x$, and

$$(x, \gamma') \cdot (x\gamma', \gamma) = (x, \gamma'\gamma).$$

- A *foliation* gives rise to a smooth groupoid (in simple cases, just the groupoid of the associated equivalence relation).

Groupoids and algebras

Let G be an étale groupoid. The *groupoid algebra* $C_c(G)$ consists of all continuous, compactly supported functions $f\colon G \to \mathbb{C}$, with composition defined by

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

We regard this as an algebra of functions on the 'noncommutative space' defined by the groupoid. Examples: If G comes from a topological space X, we get the algebra $C_c(X)$. If G comes from a discrete group Γ we get the usual group algebra.

If G is a smooth groupoid there is also a version of this construction — one needs to replace summation by integration, but this can be done canonically using smooth densities, cf. the theory of integration on Lie groups.

The Kronecker foliation of a torus

The foliation algebra for this foliation is Morita equivalent to the irrational rotation algebra.

Each transversal to the foliation gives rise to a projection in the foliation algebra. The trace of such a projection turns out to be the *transverse measure* of the transversal.

Thus, the range $\mathbb{Z}+\alpha\mathbb{Z}$ of dimensions for projective modules is revealed as the image of the integration map $H_2(\mathbb{T}^2;\mathbb{Z}) \to \mathbb{R}$ given by the transverse measure.

Levels of smoothness

We have talked about ‘algebras of functions’. But what kinds of functions?

Theory	Type of function	Noncommutative version
Measure Theory	Borel	Von Neumann algebra
Topology	Continuous	C^*-algebra
Differential Topology	C^∞	Holomorphically closed subalgebra of a C^*-algebra
Algebraic Geometry	Rational	$\mathbb{C}G$

We obtain the first three by completing the last with respect to a suitable topology.

Groupoid C^*-algebras

To get $C(S^1)$ from the trigonometric polynomials, let the algebra of polynomials act on $L^2(S^1)$ by multiplication ($=$ on $\ell^2(\mathbb{Z})$ by convolution) and take the closure in the operator norm.

Similarly one obtains a C^*-algebra from $\mathbb{C}G$ by forming the closure in a suitable operator representation on Hilbert space.

Problem. What is suitable?

The regular representation of $\mathbb{Z}$ on $\ell^2(\mathbb{Z})$ enjoys the following property: if f_n is a sequence of trigonometric polynomials, and $\|f_n\| \to 0$ in the regular representation, then $\|f_n\| \to 0$ in any other representation (exercise!). Thus the regular representation is *universal* (and thus uniquely suitable).

Not all groupoids enjoy this property. This causes trouble.

Lecture 2
Noncommutative Measure Theory

Nigel Higson and John Roe
Penn State University

June 20, 2000

Spectral Theory

$T =$ bounded selfadjoint operator on a Hilbert space H (thus $\langle Tu, v\rangle = \langle u, Tv\rangle$).

$\mathrm{Spectrum}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is singular$\} \subseteq \mathbb{R}$.

Some rather amazing facts:

- The map
$$\sum_{i=1}^{n} a_i x^i \mapsto \sum_{i=1}^{n} a_i T^i$$
from polynomials to operators determines an isometric $*$-algebra homomorphism
$$C(\mathrm{Spectrum}(T)) \to \mathcal{B}(H).$$

- The map further determines an isometric $*$-algebra homomorphism from bounded Borel functions to operators. Pointwise convergence of functions corresponds to pointwise convegence of operators.

Measure classes

The image of the *Borel functional calculus homomorphism* is isomorphic to the algebra $L^\infty(X, \mu)$ for some measure μ on $X = \mathrm{Spectrum}(T)$ (assuming the Hilbert space H is separable).

The operator T determines the *measure class* $[\mu]$ (two measures belong to the same class if they have the same nullsets).

Theorem. *The measure class* $[\mu]$ *and a multiplicity function (discussed later) determine* T *up to unitary equivalence (that is, up to change of orthonormal basis in the Hilbert space).*

Example. Associated to $T = i\frac{d}{dx}$ is the Lebesgue measure class on $X = \mathbb{R}$. (This operator is *unbounded*, and requires somehat careful treatment).

Operator Algebras

Definition. Operator algebra = von Neumann algebra = W^*-algebra = unital $*$-algebra of operators on H which is closed under the topology of pointwise convergence.

This definition is not optimal . . . but it is simplest.

Example $L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$.

Example: If π is a unitary representation of a group on a Hilbert space then its *commutant*, the algebra of all intertwining operators, is a von Neumann algebra.

Double Commutant Theorem. *Every von Neumann algebra is the commutant of its commutant.*

Example, continued: Observe that a subspace of H is a subrepresentation of π if and only if the orthogonal projection onto it is an operator in the commutant of π.

Projections in $M \leftrightarrow$ *Subrepresentations of* π

Decomposition of Representations

Algebraic Background:

G = finite group
K = field
π = finite-dimensional representation over K

Theorem. *Commutant of π = Direct sum of central simple algebras.*

Decomposition of Hilbert Space:

(X, μ) = measure space (only $[\mu]$ is relevant)
$\{H_x\}_{x \in X}$ = family of Hilbert spaces
$\mathcal{S}$ = measurable family of sections

$\Rightarrow$ We can form $H = \oplus \int_X H_x \, d\mu(x)$.

Theorem. *If M is a von Neumann algebra on H (separable) then there are canonical decompositions*

$$H = \oplus \int_X H_x \, d\mu(x) \qquad M = \oplus \int_X M_x \, d\mu(x)$$

where the M_x have trivial center.

Factors

Definition. A *factor* is a von Neumann algebra with trivial center.

Finite dimensional case: $M \cong M_n(\mathbb{C})$ = (central) simple algebra.

Message: Algebraically, factors play the role of central simple algebras over $\mathbb{C}$, in infinite dimensions (more on this later).

Definition. A factor M is of *type I* if $M \cong \mathcal{B}(H)$ (algebraically) for some H.

Are there non-type I factors? ... Yes ... Let G be a group and let M be the commutant of the left-regular representation on $\ell^2(G)$.

Theorem. *The commutant of M is the commutant of the* right-*regular representation on $\ell^2(G)$.*

Lemma. $T \mapsto \langle T\delta_e, \delta_e \rangle$ *is a faithful trace on M.*

Corollary. *If G has no finite conjugacy classes then M is not of type I.*

Comparison of Projections

Projections in $M \leftrightarrow$ Subrepresentations of π

$P_1 \sim P_2 \quad \leftrightarrow \quad$ π_1 and π_2 are unitarily equivalent

$[P_1] \leq [P_2] \quad \leftrightarrow \quad$ π_1 is unitarily eq. to a subrepresentation of π_2

Theorem. *This is a partial order on equivalence classes. If M is a factor then this is a* linear *order.*

P_1 is *minimal* $\quad \leftrightarrow \quad$ π_1 is irreducible

P_1 is *infinite* $\quad \leftrightarrow \quad$ π_1 is equiv. to a proper subrepresentation of itself

P_1 is *finite* $\quad \leftrightarrow \quad$ π_1 is *not* equiv. to a proper subrepresentation of itself

Proposition. M *is a factor if and only if π is* isotypic*: every two subrepresentations of π have equivalent subrepresentations.*

Proposition. *A factor has a minimal projection if and only if it is algebraically isomorphic to $\mathcal{B}(H)$ for some H (i.e. iff it is of type I).*

Decomposition of Representations

Theorem. *If M is the commutant of a commutative von Neumann algebra then*

$$M = \oplus \int_X M_x \, d\mu(x),$$

where each M_x is algebraically isomorphic to the full algebra of bounded operators on some Hilbert space

If M is the commutant of a selfadjoint operator T then X above is the spectrum of T, μ is the measure discussed earlier, and $x \mapsto \sqrt{\dim(M_x)}$ is the multiplicity function which, together with $[\mu]$, serves to characterize T.

Theorem. *If the commutant of π decomposes as above then π decomposes as a direct integral of multiples of irreducible representations.*

Remark. Among discrete groups, type I implies virtually abelian.

Types I, II and III

Relative dimension of projections in factors:

For finite projections $P_0, P_1 \in M$ (with $P_1 \neq 0$) form

$$\dim(P_0)/\dim(P_1) \in [0, \infty)$$

using a 'Euclidean algorithm'.

Absolute dimension function:

$$\dim(P) = \begin{cases} 0 & \text{if } P \text{ is zero} \\ \infty & \text{if } P \text{ is infinite} \\ \dim(P)/\dim(P_{\text{ref}}) & \text{if } P \text{ is finite} \end{cases}$$

Proposition. *There is a maximum dimension and range* $(\dim)$ *is arithmetically closed within* $[0, d_{\max}]$*.*

Types of Factors:

Type I	$\{0, 1, 2, \ldots, n\}$ or $\{0, 1, 2, \ldots, \infty\}$
Type II	$[0, 1]$ or $[0, \infty]$
Type III	$\{0, \infty\}$

Representations of $*$-Algebras

GNS construction:

$\mathcal{A}$ = complex $*$-algebra

$\varphi\colon \mathcal{A} \to \mathbb{C}$, $\varphi(a^*a) \geq 0$

$\mathcal{H}_\phi$ = GNS Hilbert space, $\langle a, b\rangle = \varphi(b^*a)$

π_φ = GNS representation, $\pi_\varphi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\varphi)$
(assuming $\varphi(a^*b^*ba) \leq C_b\varphi(a^*a)$)

Example:

$\mathcal{A} = C(X)$

$\varphi(f) = \int_X f(x)\, d\mu(x)$ (Riesz Theorem)

$\mathcal{H}_\varphi = L^2(X, \mu)$

Infinite Tensor Products

Powers factors:

$A = \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})$,   CAR algebra

$\varphi = \bigotimes_{j=1}^{\infty} \varphi_\lambda$,   product state

$$\varphi_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{1+\lambda}(\lambda a + d)$$

R_λ = double commutant of $\pi_\varphi[A]$

Examples:

$\lambda = 0 \quad \Rightarrow$ Type I factor

$\lambda = 1 \quad \Rightarrow$ Type II factor

Theorem. *The R_λ are type III factors.*

Theorem. (R. Powers) *For $\lambda \in (0,1)$ the R_λ are pairwise nonisomorphic type III factors.*

What Type?

Does R_λ have any non-zero, finite projections?

Is there a faithful, normal, semifinite trace on R_λ?

$$\left.\begin{array}{c} T \geq 0 \in R_\lambda \\ X = \mathrm{Spectrum}(T) \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{c} \mu_T(E) = \dim(\chi_E(T)) \\ \mathit{Trace}(T) = \int_X x \, d\mu_T(x) \end{array}\right.$$

Obstacles:

- the trace, should it exist, need not be finite on any element of $\otimes_j M_2(\mathbb{C})$
- $\otimes_j M_2(\mathbb{C})^=$ (the norm closure) is independent of λ
- algebraic structure need not pass to weak closure: for the tracial product states it happens that $[\otimes_j M_2(\mathbb{C})]'' \cong [\otimes_j M_3(\mathbb{C})]''$

Geometric Examples of Factors

Data:

(X, μ), measure space
G, discrete group
$G \times X \to X$, measurable action

Assumptions:

action preserves class of μ
action is free

Example:

$\mathrm{PSL}(2, \mathbb{Z}) \times \mathrm{P}^1(\mathbb{R}) \to \mathrm{P}^1(\mathbb{R})$.

Group-measure space construction:

$\{\ell^2(Gx)\}_{x \in X}$ = measurable field of Hilbert spaces

$M = \big\{$measurable families $\{T_x \in \mathcal{B}(\ell^2(Gx))\}_{x \in X/G}\big\}$

Theorem. M *is a factor iff the action is ergodic.*

Type III Factors

Theorem. *Suppose that the action is ergodic.*

- M *is type I iff there are atoms.*

- M *is type II iff there are no atoms but there is an invariant measure in the class of* μ*.*

- M *is type III iff there are no atoms and no invariant measures in the class of* μ*.*

This applies to the factors R_λ:

$X = \prod_j \mathbb{Z}/2\mathbb{Z}$

$\mu = \prod_j \mu_\lambda$

$G = \oplus_j \mathbb{Z}/2\mathbb{Z}$

Theorem. *If* $\lambda \in (0, 1)$ *then there is no invariant measure in the class of* μ*.*

Dynamics

Prelude: Let G act on X, preserving the class of μ. There is an associated unitary representation on $L^2(X,\mu)$:

$$\pi(g)f = \left[\frac{dg_*\mu}{d\mu}\right]^{\frac{1}{2}} g^*f$$

In fact there is a whole 'principal series' of unitary representations:

$$\pi_t(g)f = \left[\frac{dg_*\mu}{d\mu}\right]^{it} \pi(g)f \qquad (t \in \mathbb{R})$$

Groupoid perspective: Note that

$$\oplus \int_X \ell^2(Gx)\,dx = L^2(X \times G)$$

On $L^2(X \times G)$ define

$$U_t h = \left[\frac{dg_*\mu}{d\mu}\right]^{it} h$$

Then $U_t M U_t^* = M$, where M is the von Neumann algebra associated to the action of G on X.

Tomita's Theorem

Summary: The dynamics $\sigma_t \colon M \to M$ can be obtained directly from the weight on M associated to μ by an algebraic construction.

Theorem. *Let φ be a faithful, normal, semifinite weight on M and let H_φ be the associated GNS Hilbert space. The unbounded operator $S \colon x \mapsto x^*$ on H_φ has a polar decomposition*

$$S = J\Delta^{\frac{1}{2}}, \qquad \Delta = S^*S,$$

and $JMJ = M'$ while $\Delta^{it} M \Delta^{-it} = M$.

The Powers examples: On R_λ we have

$$\sigma_t = \mathrm{Ad}_{U_t} \colon R_\lambda \to R_\lambda$$

$$U_t = A_\lambda^{it} \otimes A_\lambda^{it} \otimes \dots$$

$$A_\lambda = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix} \qquad \text{(note: } \varphi_\lambda(T) = \mathrm{trace}_2(A_\lambda T))$$

Connes' Theorem

What does Tomita's theory do?

Weights $\rightarrow$ *Automorphism groups*

Compare this to:

$T = T^* \quad \rightarrow \quad U = \exp(iT)$

Heisenberg relations $\rightarrow$ *Weyl relations*

In measure theory, equivalent measures are related by Radon-Nikodym derivatives:

$$\int_X f(x)\, d\mu = \int_X f(x)h(x)\, d\nu$$

$$\varphi(T) = \psi(A^*TA)$$

Connes' R-N Theorem. *Let* M *be a von Neumann algebra. For any* φ *and* ψ *there is a unitary 'cocycle'* $\{U_t\} \subseteq M$ *such that*

$$\sigma_t^{\varphi}(T) = \sigma_t^{\psi}(U_t^* T U_t),$$

for all $T \in M$ *and* $t \in \mathbb{R}$.

Invariants in the Type III situation

Definition. $T(M) = \{ t \in \mathbb{R} : \sigma_t^\varphi$ is inner$\}$.

Lemma. $T(M)$ *is independent of* φ *— it is an invariant of* M *alone.*

Recall that for R_λ,

$$\sigma_t(T_1 \otimes T_2 \otimes \dots) = A_\lambda^{it} T_1 A_\lambda^{-it} \otimes A_\lambda^{it} T_2 A_\lambda^{-it} \otimes \dots$$

where $A_\lambda = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}$.

Lemma. *This is inner for a given* $t \in \mathbb{R}$ *iff* $A_\lambda^{it} = I$*. Hence* $T(R_\lambda) = \frac{2\pi}{\log \lambda}\mathbb{Z}$*.*

Hence the invariant $T(M)$ distinguishes the Powers factors from one another.

Non-Commutative Topology

John Roe
Penn State University
roe@math.psu.edu

June 21, 2000

Abstract

C^*-algebras are the noncommutative counterpart of topological spaces. Their algebraic topology can be studied by means of K-theory.

The big picture

Theory	Type of function	Noncommutative version
Measure Theory	Borel	Von Neumann algebra
Topology	Continuous	C^*-algebra
Differential Topology	C^∞	Holomorphically closed subalgebra of a C^*-algebra
Algebraic Geometry	Rational	$\mathbb{C}G$

Today's Reading Assignment: A. Connes, *An analogue of the Thom isomorphism for crossed products. . .* , Advances in Math **39**(1981), 31–55.

C^*-algebras

Definition 1. *A* C^*-algebra *is a Banach* $*$*-algebra which is isomorphic to a subalgebra of* $\mathfrak{B}(H)$*, for some Hilbert space* H*.*

Thus a C^*-algebra is a *norm-closed* subalgebra of $\mathfrak{B}(H)$; contrast the 'pointwise' topology considered in lecture 2. (Since the pointwise topology is weaker than the norm topology, every von Neumann algebra is in particular a C^*-algebra; but it is not usually helpful to think like this.)

Example Let X be a compact metrizable space. Let μ be a suitable measure on X. The representation $\rho: C(X) \to \mathfrak{B}(L^2(X, \mu))$ by multiplication operators shows that $C(X)$ is a C^*-algebra.

Note that in this example we can recover the measure by

$$\int f \, d\mu = \langle \rho(f)\xi, \xi \rangle$$

where $\xi \in L^2(X, \mu)$ is the constant function 1. This points the way to an abstract characterization of C^*-algebras.

More about C^*-algebras

Definition 2. *A C^*-algebra is a Banach $*$-algebra A in which the norm and involution are related by the C^*-identity $\|a^*a\| = \|a\|^2$.*

Theory of positive linear functionals (the GNS construction) connexts this abstract definition with the concrete one; Hahn-Banach arguments produce sufficiently many states to give a concrete representation of every abstract C^*-algebra. The abstract definition is useful e.g. to show that A/I is a C^*-algebra when A is a C^*-algebra and I is a closed ideal.

Commutative C^*-algebras are precisely of the form $C_0(X)$ (Gelfand-Naimark). Consequently: If $a \in A$ is selfadjoint, and f is continuous on the spectrum of a, then $f(a) \in A$ also — the (continuous) *functional calculus*. If f is only (bounded) Borel then $f(a) \in A''$ but usually $f(a) \notin A$.

Group (and Groupoid) C^*-algebras

Let Γ be a discrete group. The group ring $\mathbb{C}\Gamma$ is a complex involutive algebra. We may obtain C^*-algebras by completing $\mathbb{C}\Gamma$ in various norms.

- Define $\tau\colon \mathbb{C}\Gamma \to \mathbb{C}$ by $x = \sum c_g g \mapsto c_1$. Then τ is a positive linear functional (in fact a trace) and the associated GNS representation is the *regular representation* of $\mathbb{C}\Gamma$ on $\ell^2\Gamma$. Completing in this norm gives the *reduced* C^*-algebra, $C^*_{red}(\Gamma)$. Notice that by construction, τ extends to a trace on $C^*_{red}(\Gamma)$.

- Alternatively, for *every* state σ, let $\|x\|_\sigma$ be the norm in the GNS representation associated to σ. The maximal norm on $\mathbb{C}\Gamma$ is $\|x\|_{\max} = \sup_\sigma \|x\|_\sigma$, and completing in this norm gives the *maximal* C^*-algebra $C^*_{max}(\Gamma)$. Note that the trivial representation gives a homomorphism $C^*_{max}(\Gamma) \to \mathbb{C}$.

Theorem 1. (Hulanicki) $C^*_{max}(\Gamma) = C^*_{red}(\Gamma)$ *if and only if* Γ *is amenable.*

Projections in C^*-algebras

C^*-algebras are usually not stuffed full of projections.

Kaplansky Conjecture Let Γ be a torsionfree discrete group. Then $C^*_{red}(\Gamma)$ has *no* nontrivial projections. Consequently, every selfadjoint operator in $C^*_{red}(\Gamma)$ has connected spectrum.

Note that, in physical terms, to study the spectrum of a selfadjoint operator is to study the range of values that an observable can assume.

Atiyah Conjecture (simplified version) Let Γ be as above, let $x \in \mathbb{C}\Gamma$. Then the operator $\lambda(x) \in \mathfrak{B}(\ell^2\Gamma)$ is either zero or injective. (This is really a conjecture about the L^2 Betti numbers.)

The example of $\mathbb{Z}$ shows that the Kaplansky conjecture belongs to (noncommutative) topology, whereas the Atiyah conjecture belongs to (noncommutative) (semi?)-algebraic geometry.

Projections + Matrices = K-Theory

Definition 3. *Let A be a C^*-algebra with unit. $K_0(A)$ is the abelian group with one generator for each isomorphism class of finitely generated projective A-modules, and with the relations*

$$[M] + [N] = [M \oplus N].$$

Lemma 1. *If e, f are projections in a unital C^*-algebra A and $\|e - f\| < 1$, then there is a unitary $u \in A$ with $f = u^*eu$.*

Proof The operator $v = ef + (1 - e)(1 - f)$ has $ev = vf$ and satisfies $\|1 - v\| < 1$, so it is invertible. Use polar decomposition to replace v by its unitary part u. $\square$

Thus one can translate the definition of K-theory as follows: the generators are homotopy classes of projections $e = e^2 = e^*$ in matrix algebras $M_n(A)$. This makes the link with topology. Note that K-theory is Morita invariant.

Where do elements of $K_0(A)$ come from?

- Spectral projections.
- *Indices* of elliptic operators 'over A' (for instance, leafwise elliptic operators on a foliation).

Remark Even though the kernel and cokernel projections of an elliptic operator T over A belong to the von Neumann algebra A'', and not necessarily to A itself, their formal difference does define a K-theory class for A. This is another consequence of the ideas underlying the 'heat equation' method.

For example, the integrality of the L^2 Betti numbers individually (Atiyah conjecture) is a much deeper statement than the integrality of the L^2 Euler characteristic (Atiyah L^2-index theorem).

Traces and K-theory

A *trace* τ on a C^*-algebra A is a state that is invariant under inner automorphisms: $\tau\colon A \to \mathbb{C}$ satisfies $\tau(u^* a u) = \tau(a)$ for unitary u. It will then extend to a trace on $M_n(A)$ by the formula $[a_{ij}] \mapsto \sum_i \tau(a_{ii})$.

A trace τ gives rise to a *dimension function* $\tau_*\colon K_0(A) \to \mathbb{R}$.

Remark In practice we often need to consider 'unbounded traces', e.g. on $\mathfrak{K}(H)$. Then delicate questions of analysis intervene.

Smooth subalgebras

Definition Let A be a C^*-algebra, $\mathcal{A} \subseteq A$ a dense subalgebra. We say that $\mathcal{A}$ is *smooth* in A if the following is true: for every $a \in \mathcal{A}$, and every function f holomorphic on a neighborhood of the spectrum (in A) of a, the element $f(a) \in A$ in fact belongs to $\mathcal{A}$.

Examples (i) $C^\infty(M)$, M a compact smooth manifold, is a smooth subalgebra of $C(M)$; (ii) The subalgebra of A_α consisting of those formal sums $\sum a_{mn} U^m V^n$, with $\{a_{mn}\}$ of rapid decay, is smooth; (iii) Let τ be a (densely defined) unbounded trace on a C^*-algebra A, which is *semicontinuous*; then the domain of τ is a smooth subalgebra. For the standard trace on $\mathfrak{K}(H)$ one obtains in this way the subalgebra $\mathfrak{L}^1(H)$ of *trace class operators*.

(A trace τ on A is semicontinuous if for every $\alpha \in \mathbb{R}$, the set $\{a \in A : a \geq 0,\ \tau(a) \leq \alpha\}$ is closed in A; compare Fatou's Lemma.)

Theorem 2. *Let $\mathcal{A}$ be a smooth subalgebra of A; then the inclusion $\mathcal{A} \to A$ induces an isomorphism on K_0.*

Spectral topology

Using K-theory to investigate the spectrum of a selfadjoin operator T:

1. Embed T in a C^*-algebra A with a *faithful* trace τ, normalized so that $\tau(1) = 1$.

2. Compute $K_0(A)$ and the range of $\tau\colon K_0(A) \to \mathbb{R}$.

3. Each connected component of spectrum(T) gives rise to a projection in A. Use the information about the traces of such projections.

Elementary example A selfadjoint $n \times n$ matrix has no more than n eigenvalues.

K_1 and Bott periodicity

The group $K_0(A)$ has an ‘odd’ counterpart $K_1(A)$, generated by homotopy classes of unitary elements in matrix algebras $M_n(A)$. There is an elementary isomorphism $K_1(A) \equiv K_0(SA)$, where the suspension SA is defined to be $C_0(\mathbb{R}) \otimes A$.

Bott Periodicity Theorem There is also an isomorphism $K_0(A) \equiv K_1(SA)$.

It follows that K_0 and K_1 together comprise a generalized homology theory for C^*-algebras. For instance a short exact sequence

$$0 \to J \to A \to A/J \to 0$$

of C^*-algebras gives rise to a cyclic six-term exact sequence in K-theory.

Remark For a space X we have $K_i(C_0(X)) = K^{-i}(X)$, so that this is a true generalization of Atiyah-Hirzebruch K-theory.

K-Theory for A_α

We ask: What are the K-theory groups for the irrational rotation algebra A_α?

Answer: (Pimsner-Voiculescu) $K_0(A_\alpha) = \mathbb{Z} \oplus \mathbb{Z}$, $K_1(A_\alpha) = \mathbb{Z} \oplus \mathbb{Z}$. Moreover, the dimension function given by the canonical trace on A_α maps $K_0(A_\alpha)$ to the lattice $\mathbb{Z} \oplus \alpha\mathbb{Z} \subseteq \mathbb{R}$.

The Pimsner-Voiculescu calculation proceeds via an exact sequence for the K-theory of a crossed product by $\mathbb{Z}$. More geometrically, work with the (Morita equivalent) foliation algebra for the Kronecker foliation on $\mathbb{T}^2$. This is a transformation groupoid algebra for a flow ($\mathbb{R}$-action). Thus it suffices to compute K-theory for the crossed product C^*-algebra $A \rtimes \mathbb{R}$.

Theorem 3. (Connes) *The leafwise index of id/dx gives a 'Thom isomorphism' $K_i(A) \to K_{i\pm 1}(A \rtimes \mathbb{R})$.*

Note that if the $\mathbb{R}$-action is trivial this is just the Bott periodicity theorem.

The Thom isomorphism

Idea of proof — reduce to the case of a trivial $\mathbb{R}$-action.

- If $A = \mathbb{C}$ we always have trivial action.
- Every K_0-class for A arises from a homomorphism $\mathbb{C} \to A$ (up to finite matrices).
- Can change the $\mathbb{R}$-action to an 'exterior equivalent' one which preserves a given smooth projection e.

Discussion Let the derivation δ generate the $\mathbb{R}$-action. The extent to which the action fails to preserve e is measured by the *bounded* operator $[e, \delta(e)]$. Use this to make a 'small' perturbation of the *unbounded* infinitesimal generator of the $\mathbb{R}$-action, so as to make it commute with e.

Cannot do this for $\mathbb{R}^2$-actions — curvature intervenes.

The range of the trace

Suppose given a flow on a compact space X, and let μ be an invariant measure. One obtains from μ a trace τ on $A = C(X) \rtimes \mathbb{R}$.

Theorem 4. *The range of* $\dim_\tau\colon K_0(A) \to \mathbb{R}$ *is equal to the range of the map* $H^1(X;\mathbb{Z}) \to \mathbb{R}$ *given by first pairing with the vector field which generates the flow, and then integrating with respect to* μ. *(exercise — this is well defined on cohomology.)*

Proof proceeds via the formula

$$\dim_\tau(\phi(u)) = \frac{1}{2\pi i} \int u^{-1}\delta(u)\, d\mu$$

for u unitary over $C(X)$. (ϕ is the Thom isomorphism.)

Corollaries: (i) Computation of the range of the trace on A_α, above; (ii) A simple C^* algebra with no nonzero idempotent — take $X = S^3$ and a minimal diffeomorphism.

The tangent groupoid

Reference: *Noncommutative Geometry* pp102–111

Let M be a compact smooth manifold. Then $M \times M$ is (in a trivial way) a smooth groupoid. Its associated algebra $\mathbb{C}[M \times M]$ is the algebra of smoothing operators on M. The tangent bundle TM is a smooth groupoid too: it is a bundle over M whose fibers are groups. Let us glue these together.

Definition 4. *The* tangent groupoid *to M, $G(M)$, has objects $M \times [0,1]$, and it is equal to TM over $M \times \{0\}$ and to $M \times M$ over $M \times \{t\}$ for $t \neq 0$.*

This gives a smooth groupoid — local coordinates near $(x,0)$ may be given by

$$(x,v,t) \mapsto \begin{cases} (x, \exp_x(tv), t) & (t > 0) \\ (x,v) & (t = 0) \end{cases}$$

Notice that the decomposition of the objects $M \times [0,1] = M \times \{0\} \cup M \times (0,1]$ allows us to write $G(M) = G_0(M) \cup G_+(M)$, the disjoint union of a closed and an open subgroupoid.

The tangent deformation

A decomposition like this gives an exact sequence

$$0 \to C^*(G_+) \to C^*(G) \to C^*(G_0) \to 0.$$

Here $C^*(G_+)$ has trivial K-theory and so $K_*(C^*(G)) \cong K_*(C^*(G_0)) \cong K^*(T^*M)$. Evaluation at some finite value of t (say $t = 1$) gives a homomorphism $C^*(G) \to \mathfrak{K}$, and so on the level of K-theory we obtain a homomorphism

$$K^0(T^*M) \to \mathbb{Z}.$$

Theorem 5. *The above homomorphism is the Atiyah-Singer analytical index map.*

To prove this, construct a linear section of $C^*(G) \to C^*(G_0)$, using the symbol calculus for pseudodifferential operators.

$$k_t(x,y) = (2\pi)^{-n} \int e^{i\xi\cdot(x-y)/t}\,\sigma(x,\xi)\,d\xi.$$

A noncommutative view of the index theorem

One can regard the index theorem as a reduction of the above exact sequence of noncommutative spaces to the level of *commutative* geometry. Suppose M embedded in Euclidean space $\mathbb{R}^N$. From this we obtain a free action of the groupoid G on the space $\mathbb{R}^N$. The transformation groupoid $G \rtimes \mathbb{R}_N$ is then a proper equivalence relation, as are the subgroupoids $G_0 \rtimes \mathbb{R}_N$ and $G_+ \rtimes \mathbb{R}_N$.

By Rieffel's Theorem (lecture 1) these groupoids are Morita equivalent to ordinary 'commutative' spaces.

The index theorem is thus reduced to a version of Bott periodicity relating the K-theory groups of the algebras associated to the groupoids G and $G \rtimes \mathbb{R}_N$ — that is, to the Thom isomorphism theorem.

Introduction to the Baum-Connes conjecture

It is natural to enquire whether *every* groupoid G has the property that its C^*-algebra $C^*(G)$ is 'K-theoretically equivalent' to an ordinary commutative space. In a more precise form, this is the Baum-Connes conjecture: a certain *assembly map*

$$\mu\colon K_*^{top}(G) \to K_*(C^*_{red}(G))$$

is conjectured to be an isomorphism.

The assembly map is a generalization of the Atiyah-Singer index map.

The BC conjecture has been verified in a wide variety of examples. Recent constructions lead to groupoids G for which it fails: these exploit the tension between C^*_{max} and C^*_{red}, together with the fact that a decomposition like $G = G_0 \cup G_+$ above in general gives an exact sequence of C^*-algebras only on the C^*_{max} level.

Hike

For those who will be here on Friday June 23rd, a *hike* is being planned. Would those interested please sign below. Consult Guoliang Yu for details.

Lecture 4
Noncommutative Differential Topology

Nigel Higson and John Roe
Penn State University

June 22, 2000

Characteristic Numbers

M, smooth manifold

$P\colon M \to M_K(\mathbb{C})$, $\begin{cases}\text{projection-valued function}\\ \text{vector bundle}\end{cases}$

$V \subseteq M$, oriented closed submanifold

Definition. $c_V(P) = \int_V \mathrm{trace}(PdPdP \cdots dPdP)$.

Proposition. *Fixing V, the scalar $c_V(P)$ only depends on $[P] \in K^0(M)$.*

- $\mathrm{trace}(PdPdP \cdots dPdP)$ is a closed form.

- Given $P\colon I \times M \to M_n(\mathbb{C})$,

$$\int_{\partial I \times M} \mathrm{trace}(PdPdPdP) = \int_{I \times M} \mathrm{trace}(dPdPdP \cdots dPdP) = 0.$$

Remark: If $\dim(V)$ is odd then $c_V(P) = 0$ (in fact $\mathrm{trace}(PdPdP \cdots dP) = 0$).

Characteristic Numbers in Noncommutative Geometry

A, noncommutative algebra

$P \in M_K(A)$, idempotent over A

Example:

$$A = \mathbb{T}^2_\theta = \{\sum_{m,n} a_{mn} U^m V^n\}$$

$$(UV = e^{2\pi i\theta} VU)$$

$P =$ Powers-Rieffel projection
$= U^{-1}g(V) + f(V) + g(U)U$

Problem: Define characteristic numbers $c_V(P)$ as before.

If $c_V(P) = \int_V \text{trace}(PdPdP \cdots dPdP)$ then

- What is V?
- What is $\int$?
- What is d?

Cycles

Definition. An n-*cycle* over an algebra A is

- a differential graded algebra Ω^* with an algebra map from A into Ω^0, and
- a closed, graded trace on Ω^n.

The trace properties:

$$\int \omega_1\omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)} \int \omega_2\omega_1.$$

$$\int d\omega = 0.$$

Warning: It is not necesarily true that $d1 = 0$ nor that $1 \cdot \omega = \omega$, nor that $\omega_1\omega_2 = \pm\omega_2\omega_1$.

Proposition. *If* $X = (\Omega_X, \int_X)$ *is an* n*-cycle then the characteristic number*

$$c_X(P) = \int_X \operatorname{trace}(PdPdP \cdots dPdP)$$

depends only on $[P] \in K_0(A)$.

As before, if n is odd then $c_X(P)$ is zero.

Traces

$$\{0\text{-cycles over } \mathcal{A}\} = \{\text{traces on } \mathcal{A}\}$$

If P is a path of projections and $\frac{d}{dt}P = \dot{P}$ then

$$P^2 = P \ \Rightarrow\ P\dot{P} = \dot{P}P^{\perp} \ \Rightarrow\ P\dot{P}P = 0 = P^{\perp}\dot{P}P^{\perp},$$

and therefore

$$\tau(\dot{P}) = \tau(P\dot{P}P^{\perp} + P^{\perp}\dot{P}P) = 0.$$

Commutative case:

$\tau\colon C^{\infty}(M) \to \mathbb{C}$, τ = any distribution.

However, $\tau(P) = \mathrm{constant} \cdot \dim(P)$.

Noncommutative examples:

$$\tau\colon \mathbb{T}^2_{\theta} \to \mathbb{C}, \quad \tau(\textstyle\sum a_{mn} U^m V^n) = a_{00}$$

$$\tau_{\lambda}\colon \mathbb{C}[G] \to \mathbb{C}, \quad \tau_{\lambda}(\textstyle\sum a_g [g]) = a_e$$

Topological Invariance

A = Banach or C^*-algebra

$\mathcal{A} \subseteq A$ dense subalgebra of A

Not every cycle X over $\mathcal{A}$ induces a 'topologically invariant' map,

$c_X \colon K_0(A) \to \mathbb{C}$.

Example: Let $\mathcal{A} = \mathbb{C}[G] \subseteq C^*_r(G) = A$. Then for $\tau_0(\sum a_g[g]) = \sum a_g$, no corresponding map

$\tau_0 \colon K_0(A) \to \mathbb{C}$

is known, in general. (Note that τ_0 is even a 'state': $\tau_0(a^*a) \geq 0$.)

Sufficient condition for traces: Semicontinuity.

Sufficient condition for n-cycles:

$$\left|\int (a_1 db_1)(a_2 db_2) \cdots (a_n db_n)\right| \leq C_{b_1,\ldots,b_n} \|a_1\| \cdots \|a_n\|$$

(These are called n-*traces*.)

Cycles for the Irrational Torus

For $A = \mathbb{T}^2_\theta$ with $\theta = 0$ one can take

$$\Omega^0 = A$$

$$\Omega^1 = A \cdot ds \oplus A \cdot dt \qquad \begin{cases} dU = 2\pi i U ds \\ dV = 2\pi i V dt \end{cases}$$

$$\Omega^2 = A \cdot dsdt \qquad dsdt = -dtds.$$

$$\int \sum_{m.n} a_{mn} U^m V^n \, dsdt = a_{00}$$

Lemma. *The same formulas define a 2-cycle on $\mathbb{T}^2_\theta$, for any θ.*

Computation: For the Powers-Rieffel projection,

$$\frac{1}{2\pi i} \int PdPdP = 1$$

(See Connes, *A survey of foliations and operator algebras.*)

Flows and Multiflows

$\alpha\colon \mathbb{R} \times A \to A,$ smooth action

$\delta(a) = \lim_{t\to 0} \frac{\alpha_t(a)-a}{t},$ derivation

$\tau\colon A \to \mathbb{C},$ invariant trace $\begin{cases} \tau(\alpha_t(a)) = \tau(a) \\ \tau(\delta(t)) = 0 \end{cases}$

$\Rightarrow \quad \Omega^0 = A\ ,\ \Omega^1 = A \cdot dt,\ da = \delta(a)dt$

$$\int a\,dt = \tau(a)$$

$$\int a^0 da^1 = \tau(a^0\delta(a^1))$$

Given commuting flows α_1 and α_2, we can form

$da = \delta_1(a)dt_1 + \delta_2(a)dt_2$

and (supposing τ is invariant for both flows)

$$\int a\,dt_1dt_2 = \tau(a)$$

$$\int a^0da^1da^2 = \tau\big(a^0(\delta_1(a^1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))\big)$$

Curvature

Lemma. *The characteristic number*

$$c_\alpha(P) = \tau\big(a^0(\delta_1(a^1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))\big)$$

depends only on the outer equivalence class of the action of $\mathbb{R}^2$ on A.

Proof. Given $\alpha \sim \alpha'$, form the combined action

$$\alpha'' \colon \mathbb{R}^2 \to \mathrm{Aut}(M_2(A)),$$

$$\alpha''_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_t(a) & * \\ * & \alpha'_t(d) \end{pmatrix}$$

Construct from it a 2-cycle over $M_2(A)$. Since

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$$

we get

$$c_\alpha(P) = c_{\alpha''} \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} = c_{\alpha''} \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} = c_{\alpha'}(P)$$

Example: Powers-Rieffel projection (compare Thom isomorphism theorem of Connes).

Cyclic n-Cocycles

$$\varphi(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \cdots da^n$$

- cyclicity:

  $\varphi(a^0, a^1, \ldots, a^n) = (-1)^n \varphi(a^1, \ldots, a^n, a^0)$

- $b\varphi(a^0, \ldots, a^{n+1}) = 0$

  $$\begin{aligned} b\varphi(a^0, \ldots, a^{n+1}) = \varphi(a^0 a^1, \ldots, a^{n+1}) \\ - \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) \\ + \ldots \\ + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n) \end{aligned}$$

Proposition. *Cyclic n-cocyles are precisely the functionals associated to n-cycles $(\Omega_{\mathrm{univ}}, \int)$ on the universal differential graded algebra over A.*

Cyclic Cohomology

Lemma. *Let* φ *be a* cyclic $(n+1)$*-multlinear functional on* A*.*

- $b\varphi$ is cyclic too.
- $b^2\varphi = 0$.

Definition. $HC^n(A)$ = cyclic n-cocyles modulo cyclic coboundaries.

Pairings:

$$\langle x, \varphi \rangle \in \mathbb{C} \qquad \begin{cases} HC^{2n}(A) \otimes K_0(A) \to \mathbb{C} \\ HC^{2n+1}(A) \otimes K_1(A) \to \mathbb{C} \end{cases}$$

The latter is

$$\begin{aligned} \langle [U], \varphi \rangle &= (\varphi \times \text{trace})(U^{-1}, U, U^{-1}, U \ldots, U^{-1}, U) \\ &= \textstyle\int_X \text{trace}[U^{-1} dU]^n, \end{aligned}$$

assuming that $d1 = 0$.

Cyclic Cohomology and Manifolds (First Look)

For $V \subseteq M$ oriented we define

$$\varphi_V(a^0, a^1, \ldots, a^n) = \int_V a^0 da^1 \cdots da^n$$

(de Rham differential)

We obtain:

geometric n-cycles $\rightarrow$ *closed de Rham currents* $\rightarrow$ *cyclic n-cocyles*

However:

- b(manifold n-chain) =manifold $(n-1)$-cycle
- b(cyclic n-cochain) = cyclic $(n+1)$-cocycle
- Note the mismatch — there are some surprises!

Products and Suspension

Lemma. $\mathrm{HC}^n(\mathbb{C}) = \begin{cases} \mathbb{C} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

Product on cycles:

$$(\Omega_{A_1}, \textstyle\int_{A_1}) \times (\Omega_{A_2}, \textstyle\int_{A_2}) = (\Omega_{A_1} \hat{\otimes} \Omega_{A_2}, \textstyle\int_{A_1} \hat{\otimes} \textstyle\int_{A_2})$$

This induces

$$\mathrm{HC}^{n_1}(A_1) \otimes \mathrm{H}^{n_2}(A_2) \to \mathrm{HC}^{n_1+n_2}(A_1 \otimes A_2)$$

Lemma. $\mathrm{HC}^*(\mathbb{C})$ *is a polynomial algebra with degree two generator* $\varphi(1,1,1) = 1$*.*

(Compare with $P^\infty(\mathbb{C})$.)

Suspension:

$$\mathrm{HC}^*(\mathbb{C}) \otimes \mathrm{HC}^*(A) \to \mathrm{HC}^*(A)$$

$$S\colon \mathrm{HC}^*(A) \to \mathrm{HC}^{*+2}(A)$$

from generator of $\mathrm{HC}^2(\mathbb{C})$.

Proposition. $\langle x, \varphi \rangle = \langle x, S\varphi \rangle$

Bordism

Lemma. *If* $V \subseteq M$ *bounds then* $c_V(P) = 0$ *for all* P*.*

Proof. Stokes' Theorem.

Definition. An n-cycle $X = (\Omega_X, \int_X)$ *bounds* if there exists a d.g.a.–trace pair $W = (\Omega_W, \int_W)$, and a surjection $r\colon \Omega_W \to \Omega_X$, such that

$$\int_X r[\omega] = \int_W d\omega \qquad \text{“Stokes' Theorem”}$$

Lemma. *If* $X = (\Omega_X, \int_X)$ *bounds then* $c_X(P) = 0$*, for all projections* P*.*

Remark: This gives a natural context for showing that

$$[P] \mapsto \int_X \operatorname{trace}(P dP dP \cdots dP dP)$$

is well defined (depends only on $[P] \in K_0(A)$).

Algebraic Structure of Bordism and Suspension

$HH^n(A)$ = Hochschild cohomology (drop cyclicity condition on $(n+1)$-linear functionals; keep the same differential b).

Theorem. *A cycle bounds iff its cyclic cohomology class is in the image of the map*

$$B = AB_0 \colon HH^{n+1}(A) \to HC^n(A)$$

$$B_0\varphi(a^0,\dots,a^n) = \varphi(1,a^0,\dots,a^n) \pm \varphi(a^0,\dots,a^n,1)$$

A = cyclic symmetrization

Theorem. *A cycle bounds iff its cyclic class is in the kernel of*

$$S \colon HC^n(A) \to HC^{n+2}(A)$$

Definition.

$HCP^j(A)$ = stable bordism class of cycles

$$= \varinjlim{}_S HC^{j+2k}(A)$$

Example: Bass Conjecture

$$\tau_0 \colon \mathbb{C}[G] \to \mathbb{C}, \qquad \tau_0(\textstyle\sum_g a_g[g]) = \sum_g a_g$$

$$\tau_\lambda \colon \mathbb{C}[G] \to \mathbb{C}, \qquad \tau_\lambda(\textstyle\sum_g a_g[g]) = a_e$$

If G is torsion free, is

$$\tau_0 = \tau_\lambda \colon K_0(\mathbb{C}[G]) \to \mathbb{C}?$$

Stupid question: is $\tau_0 - \tau_\lambda$ a cyclic coboundary? No — there are no coboundaries in degree 0.

But we *can* try to solve

$$S(\tau_0 - \tau_\lambda) = b\psi$$

Example: If $G = \mathbb{Z}$ then take

$$\psi(a_m[m], a_n[n]) = \begin{cases} \frac{m-n}{m+n} a_m a_n & m+n \neq 0 \\ 0 & m+n = 0 \end{cases}$$

Exercise: What to do when $G = F_2$?

Other important examples arise in index theory . . .

Homological Algebra

- From the map $I\colon HC^n(A) \to HH^n(A)$ associated to the inclusion of the cyclic complex into the Hochschild complex we get

$$\xrightarrow{I} HH^{n+1}(A) \xrightarrow{B} HC^n(A) \xrightarrow{S} HC^{n+2}(A)$$
$$\xrightarrow{I} HH^{n+2}(A) \xrightarrow{B}$$

- Consider $IB\colon HH^n(A) \to HH^{n+1}(A)$. One has $(IB)^2 = IBIB = I(BI)B = 0$.

- $HC^*(A) \sim$ cohomology of IB-complex (there is a spectral sequence . . .)

- $bB = -Bb$ and $B^2 = 0$

- $HH^n(A)$ fits into the ordinary framework of homological algebra

Smooth Manifolds

Example: $A = C^{\infty}(M)$ (topological algebra)

- $HH^{p}(A) \cong \mathcal{D}_{p}(M) =$ de Rham currents.
- $IB =$ de Rham boundary
- $HCP^{ev/odd}(A) \cong H^{deRham}_{ev/odd}(M)$

Groups

Group n-cocycles:

$$c\colon G^{n+1} \to \mathbb{C}$$

$$c(gg_0, \ldots gg_n) = c(g_0, \ldots, g_n)$$

$$bc(g_0, \ldots, g_{n+1}) = \sum (-1)^j c(g_0, \ldots, \widehat{g_j}, \ldots, g_{n+1}) = 0$$

Cyclic cocycles from group n-cocycles:

$$\varphi_c(g_0, \ldots, g_n) = \begin{cases} c(1, g_1, g_1 g_2, \ldots) & \text{if } g_0 \cdots g_n = 1 \\ 0 & \text{if } g_0 \cdots g_n \neq 1 \end{cases}$$

Example:

- $f\colon G \to \mathbb{Z}$, group homomorphism
- $c(g_0, g_1) = f(g_1) - f(g_0)$, group cocycle
- $\varphi_c(g_0, g_1) = \begin{cases} f(g_1) & \text{if } g_0 g_1 = 1 \\ 0 & \text{if } g_0 g_1 \neq 1 \end{cases}$

Group Cocycles and Novikov

Example, continued:

- $\varphi_c(x, y) = \frac{1}{2\pi i}\tau_\lambda(x\delta_f(y))$, where δ_f generates the automorphism group $\alpha_t[g] = \exp(2\pi i t f(g))[g]$.

- $\varphi_c(x, y)$ is a 1-trace (it pairs with $K_1(C_r^*(G))$)

- from $c = c_1 \times \cdots \times c_j$, a product of group 1-cocycles, we get a j-trace φ_c and a map φ_{c*} on $K_j(C_r^*(G))$.

- Index formula of Connes and Moscovici: for $\alpha \colon M^n \to BG$ and for D an elliptic operator on M,

  $$\varphi_{c*}(\mu_r[D]) = (-1)^n \operatorname{constant}_j \langle ch(\sigma_D) \cdot \mathcal{T}_M \cdot \alpha^*(c), [T^*M] \rangle$$

- $\Rightarrow$ the Novikov conjecture for products of one dimensional cohomology classes

Elliptic operators, cyclic theory, and zeta functions

John Roe
Penn State University
roe@math.psu.edu

June 25, 2000

Abstract

In this lecture we will show how abstract elliptic operators give rise to cyclic cocycles which have strong integrality properties.

A quick review

- Noncommutative spaces — group and groupoid algebras.
- Noncommutative measure theory — von Neumann algebras — dynamic.
- Noncommutative topology — K-theory — spectral projections, indices.
- Differential topology — characteristic class (Chern-Weil) theory leads to the formalism of cyclic cohomology.

Key examples — $C(X)$, $\mathbb{C}\Gamma$, the *irrational rotation algebra*

$$A_\alpha = \langle U, V | UV = e^{2\pi i\alpha} VU \rangle.$$

Cyclic cohomology

For an algebra A one defines an n-*cycle over* A to be a package

$$A \longrightarrow \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{\int} \mathbb{C}$$

where (Ω, d) is a differential graded algebra (not necessarily commutative) and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace.

The essential information is contained in the multilinear functional $\tau(a_0, \ldots, a_n) = \int a_0 da_1 \cdots da_n$, which is a *cyclic cocycle*, that is

$$\tau \circ \lambda = (-1)^\lambda \tau, \quad b\tau = 0,$$

where λ can be any cyclic permutation of $a_0, \ldots, a_n$, and b is the 'Hochschild coboundary'.

It turns out that b preserves the cyclic symmetry condition (Surprise!) and therefore n-cycles are in fact the (co)cycles for *cyclic cohomology theory* $HC^*(A)$.

Structure on cyclic cohomology

$HC^*(A)$ is a module over $HC^*(\mathbb{C})$, which is a polynomial ring with one 2-dimensional generator. The corresponding operation on $HC^*(A)$ is denoted $S\colon HC^n(A) \to HC^{n+2}(A)$.

One has the *Connes exact sequence*

$$HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} HH^{n+1}(A) \xrightarrow{B} HC^n(A) \ldots$$

where $HH^*(A)$ is the Hochschild cohomology of A with coefficients in A^*.

There are pairings of $HC^n(A)$ with $K_{[n]}(A)$, $[n]$ being n modulo 2, and these pairings are compatible with the S operation. The pairings are given by 'integration of the Chern character'.

Elliptic operators

Let D be a first order elliptic operator on a compact n-dimensional manifold M. Example: $M = \mathbb{T}^n$ and D is the operator

$$\gamma_1 \frac{\partial}{\partial x^1} + \cdots + \gamma_n \frac{\partial}{\partial x^n}$$

where the Pauli matrices $\gamma_1, \ldots, \gamma_n$ are generators of a Clifford algebra.

Key properties:

1. D is an unbounded selfadjoint operator on L^2.

2. D has discrete eigenvalues tending to infinity, with finite-dimensional eigenspaces; which is to say the *resolvent operators* $(D \pm i)^{-1}$ are compact.

3. $[D, f]$ is *bounded* for a dense set of functions f on M (for instance the smooth functions).

Note. $\mathbb{Z}/2$-grading in the even-dimensional case.

Index theory and K-homology

Properties 1 and 2 imply that $\ker D$ is a finite-dimensional space. When n is even, $kerD$ is graded and the *index* $\operatorname{Index} D = \dim\ker D^{+} - \dim\ker D^{-}$ is an important invariant.

Property 3 allows us to 'couple' the index to a vector bundle V — there are many choices for such a coupled operator D_V but (3) implies that they all differ by bounded operators and hence have the same index. The assignment $V \mapsto \operatorname{Index} D_V$ gives a homomorphism $K^0(M) \to \mathbb{Z}$.

Atiyah, Brown-Douglas-Fillmore, and Kasparov developed this idea into *analytic K-homology* — 'abstract' elliptic operators (satisfying appropriate versions of 1–3 above) are the cycles for this theory which pairs integrally with K-theory.

The definition of K-homology works even for noncommutative algebras.

Eigenvalues and zeta functions

Let $\mu_1, \mu_2, \ldots$ be the eigenvalues of $|D|$ arranged in increasing order. Property (2) of our list says that $\mu_j \to \infty$ as $j \to \infty$. But one can obtain a more precise quantitative form of this statement.

Theorem 1. (Weyl) *One has $\mu_j \asymp j^{1/n}$, where we recall that n is the dimension of M.*

It follows that the series defining the *zeta function*

$$\zeta_D(s) = \sum_{j=1}^{\infty} \lambda_j^{-s} = \mathrm{Tr}\,|D|^{-s}$$

is convergent for $\Re s > n$. (One must make a special case of the zero eigenvalues, if any.) Note that this *is* the Riemann zeta function when $D = id/dx$ on the circle S^1.

Theorem 2. *The function $\zeta_D(s)$ extends to a meromorphic function on $\mathbb{C}$ with poles at $n, n-1, \ldots$.*

The proofs involve studying the heat equation.

Normalized elliptic operators

Let D be as above. To simplify assume that D is invertible, and let $F = D|D|^{-1}$ be the *phase* of D. It is a bounded operator and the results above translate to:

1. F is selfadjoint and $F^2 = 1$,

2. For all continuous functions f on M, the commutator $[F, f]$ is a *compact* operator on $H = L^2$. In fact for a dense set of functions f on M (for instance the smooth functions) this commutator belongs to the *Schatten ideal* $\mathfrak{L}^p(H)$, for every $p > n$.

Discussion. The Schatten ideal $\mathfrak{L}^p$ is generated by those positive operators whose p'th powers are of trace class — compare the Lebesgue space L^p. Most familiar inequalities between Lebesgue spaces have noncommutative counterparts — see B. Simon, *Trace ideals and their applications*.

Summable Fredholm modules and cycles

A (normalized) *Fredholm module* over a C^*-algebra A is given by a representation ρ of A on a Hilbert space H, together with a selfadjoint operator $F \in \mathfrak{B}(H)$ with $F^2 = 1$ and $[F, \rho(a)] \in \mathfrak{K}(H)$ for all $a \in A$.

(There is a graded variant of the above definition: H is a graded Hilbert space, the representation of A is by even operators, and F itself is odd.)

The Fredholm module (ρ, H, F) is p-*summable* if there is a dense subalgebra $\mathcal{A} \subseteq A$ such that $[F, \rho(a)]$ belongs to the Schatten ideal $\mathfrak{L}^p$ for all $a \in \mathcal{A}$.

Theorem 3. (Connes) *Let* (ρ, H, F) *be a* p*-summable Fredholm module. Then the equalities*

$$d\alpha = [F, \alpha], \qquad \int \alpha = \mathrm{Tr}(\epsilon\alpha)$$

define a p*-cycle over* $\mathcal{A}$*, called the* character *of the Fredholm module. (*ϵ *is the grading operator.)*

One should take $p \equiv 0, 1$ mod 2 according to grading.

Small technical improvement

Observe that the mapping

$$\mathrm{Tr}'(\alpha) = \tfrac{1}{2}\,\mathrm{Tr}(\epsilon F[F,\alpha])$$

agrees with $\mathrm{Tr}(\alpha)$ whenever $\alpha \in \mathfrak{L}^1$, but is defined for all α such that $[F,\alpha] = d\alpha$ is in $\mathfrak{L}^1$. This allows us to relax slightly the order of summability required in the definition of the character. For instance, a 1-summable Fredholm module has a character in $HC^0(\mathcal{A})$ defined by

$$a \mapsto \int' a = \tfrac{1}{2}\,\mathrm{Tr}(\epsilon F[F,\rho(a)]).$$

Exercise. Check the trace property.

Pairing theorem

Let (ρ, H, F) be a Fredholm module over A and let $e \in A$ be a smooth projection. Then $\rho(e)F\rho(e)$ is a graded Fredholm operator on the space $\rho(e)H$ and we can form its index. This gives a map

$$K_0(A) \to \mathbb{Z}, \qquad e \mapsto \operatorname{Index} \rho(e)F\rho(e)$$

corresponding to $V \mapsto \operatorname{Index} D_V$ in the commutative case.

Now suppose that F is p-summable. Then we may pair the cycle corresponding to F with the projection e according to the recipe of lecture 4, that is $\int edede \cdots de$, and we have

Theorem 4. $\operatorname{Index}(eFe) = c_k \int e \underbrace{dede \cdots de}_{k,\ \textit{even}}.$

Proof If P is Fredholm, Q a parametrix modulo $\mathfrak{L}^p$, then

$$\operatorname{Index}(P) = \operatorname{Tr}(1 - QP)^p - \operatorname{Tr}(1 - PQ)^p.$$

Stability theorem

There is an ambiguity in the dimension p of the character — any integer, of appropriate parity, and greater than the degree of summability of the module, will do.

Theorem 5. *Let* (ρ, H, F) *be a* p*-summable Fredholm module and let* τ_p, τ_{p+2} *be its* p*-dimensional and* $(p+2)$*-dimensional characters. Then* $[\tau_{p+2}] = c_p\, S[\tau_p] \in HC^{p+2}(\mathcal{A})$*, where* c_p *is a universal constant.*

Exercise. Do the computation by hand for $p = 0$.

Thus the character is unambiguously defined in the periodic theory $HCP^*(\mathcal{A})$.

Remark. Cycles coming from summable Fredholm modules automatically satsify the summability conditions to be higher traces — that is, they give linear functionals on the K-theory of the full C^*-algebra A.

Example I: ordinary manifolds

Let F be the Fredholm module defined by the Dirac operator D over a spin manifold M^n. It is p-summable for all $p > n$ (as we saw above), so it has a character in $HC^n(C^\infty(M))$.

Recall from lecture 4 that

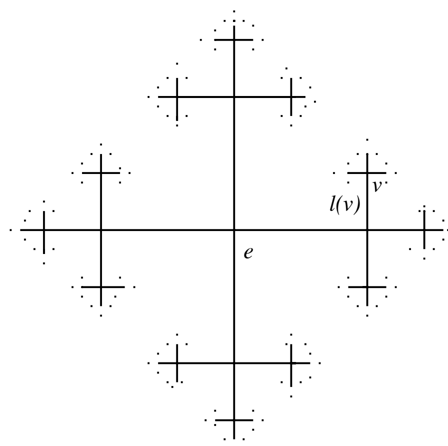
$$HC^n(C^\infty(M)) = Z_n(M) \oplus H_{n-2}(M) \oplus H_{n-4}(M) \oplus \cdots.$$

Theorem 6. *The character of the Dirac module has top-dimensional component equal to the current* $(f_0, \ldots, f_n) = \int_M f_0 df_1 \cdots df_n$*, and the homology components are those of the Poincaré dual of the* $\widehat{\mathcal{A}}$*-genus.*

The proof uses the heat equation.

Example II: the free group

Let $\Gamma = F_2$, $A = C_r^*(\Gamma)$, $\mathcal{A} = \mathbb{C}\Gamma$, and define a Fredholm module as follows:



Graded Hilbert space $H = L^2(V) \oplus L^2(E)$.

$$F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix},$$

where U sends vertex v to edge $\ell(v)$. (1-dimensional correction gets $F^2 = 1$ on the nose.)

This Fredholm module is 1-summable. Similar finitely summable 'dual Dirac' constructions are possible on noncompact symmetric spaces G/K of rank one.

Kaplansky-Kadison for the free group

Lemma 1. *The character in* $HC^0(\mathbb{C}\Gamma)$ *of the Fredholm module defined above is equal to the von Neumann trace* $\tau\colon \mathbb{C}\Gamma \to \mathbb{C}$.

It follows that the map $\dim_\tau\colon K_0(A) \to \mathbb{R}$ is given by pairing with a Fredholm module, and so is *integer valued*. In particular the trace of a projection in A must be either 0 or 1. Since τ is faithful we get

Theorem 7. *The* C^**-algebra* $A = C^*_r(F_2)$ *has no non trivial projections.*

Problem/Conjecture Is it true that for *any* torsionfree group, $S^n\tau$ is the character of a Fredholm module for sufficiently large n? This would imply Kadison-Kaplansky conjecture by arguments as above.

Atiyah conjecture for the free group

Lemma 2. *Let U, T be operators on a Hilbert space H, and suppose that U commutes modulo finite rank operators with T. Then U also commutes modulo finite rank operators with the orthogonal projection operator onto the kernel of T.*

(Connes' terminology: the orthogonal projection is *quasicontinuous*.) Proof uses the fact that the set of operators of rank $\leq r$ is closed in the weak (pointwise) topology. No analog for e.g. trace class operators. Thus this is special to the free group.

Hence find that F gives a Fredholm module even over the algebra $\mathfrak{A}$ obtained by adjoining kernel projections to $\mathcal{A} = \mathbb{C}\Gamma$. ($\mathfrak{A}$ is a subalgebra of the von Neumann algebra A'', but it contains elements not belonging to the C^*-algebra A.) Hence we get:

Theorem 8. (Linnell) *Any nonzero element of $\mathcal{A} = \mathbb{C}\Gamma$ is injective as an operator on $\ell^2(\Gamma)$.*

Example III: the irrational rotation algebra

As described in lecture 4, the algebra

$$A_\alpha = \left\langle U, V | UV = e^{2\pi i\alpha} VU \right\rangle$$

admits a canonical 2-trace

$$\tau_2(a_0, a_1, a_2) = \tau_0\big(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))\big).$$

We observed the *integrality* of $\langle e, \tau_2 \rangle$ for the Powers-Rieffel projection e. Why is this?

Theorem 9. *τ_2 is the character of a Fredholm module.*

Extension of this to quasicontinuous elements related to the quantum Hall effect, see *Noncommutative Geometry* pp355–367.

Seeking a local index formula

The index formulae above all depend ultimately on the formula

$$\mathrm{Index}(P) = \mathrm{Tr}(1 - QP)^p - \mathrm{Tr}(1 - PQ)^p$$

which is *not local*.

A truly *local* formula would *neglect* trace class operators (because, for example, pseudodifferential operators are trace class off the diagonal). Can such an index formula be found in the abstract case?

Connes-Moscovici showed that the answer is yes under a natural hypothesis. Recall that the p-summability hypothesis is equivalent to the convergence of the Dirichlet series for a certain zeta function in a right-hand half plane. It turns out that a fully local index formula can be achieved if we assume in addition that $\zeta(s)$ has an analytic continuation.

See lecture 9.

Lecture 6
Classification of Factors

Nigel Higson and John Roe
Penn State University

June 25, 2000

Von Neumann Algebras

$\mathcal{B}(H)$ = Bounded operators on a separable Hilbert space

$\mathcal{L}^1(H)$ = Trace class operators on H

Proposition. *$\mathcal{L}^1(H)^* \cong \mathcal{B}(H)$ via the pairing $\langle T_1, T_2 \rangle = \mathrm{trace}(T_1 T_2)$.*

Definition. The *ultraweak* (u.w.) topology on $\mathcal{B}(H)$ is the associated weak* topology.

Double Commutant Theorem. *A unital $*$-algebra $M \subseteq \mathcal{B}(H)$ is u.w.-closed iff $M'' = M$.*

Definitions.

A $*$-subalgebra $M \subseteq \mathcal{B}(H)$ is a *von Neumann algebra* iff $M = M''$.

M is a factor iff $M \cap M' = \mathbb{C}I$.

Classification into Types

Suppose M is the *commutant* of a unitary representation π (of some object, e.g. a group). Then recall that:

$$\textit{Projections in } M \quad \leftrightarrow \quad \textit{Subrepresentations of } \pi$$

Moreover there are further correspondences:

Equivalence of projections	$\leftrightarrow$	Equivalence of subrepresentations
Order on equiv alence classes	$\leftrightarrow$	Inclusion of sub-representations

Theorem (Murray and von Neumann). *If M is a factor then the ordering on equivalence classes of projections is one of:*

- $\{0, 1, \ldots, n\}$ $\{0, 1, \ldots, \infty\}$ Type I
- $[0, 1]$ $[0, \infty]$ Type II
- $\{0, \infty\}$ Type III

Algebraic versus Spatial Isomorphism

Obvious: Each von Neumann algebra is a dual space.

Not so obvious: The space $M_* \subseteq M^*$ is unique.

$\Rightarrow$ The u.w. topology is intrinsic to M.

Now suppose given $\left.\begin{array}{l}\pi_1 \colon M \to \mathcal{B}(H_1) \\ \pi_2 \colon M \to \mathcal{B}(H_2)\end{array}\right\}$ u.w. continuous

• The commutant of $\pi_1 \oplus \pi_2$ in $\mathcal{B}(H_1 \oplus H_2)$ is a factor (if M is).

• Subreps of $\pi_1 \oplus \pi_2$ (like π_1 and π_2) are classified by dimension of projections:

$$\pi \mapsto \dim(P_\pi) \in \begin{cases} \{0, \ldots, d_{max}\} \\ [0, d_{max}] \\ \{0, \infty\} \end{cases}$$

• Hence the representation theory of a given M is essentially trivial. (Especially in the type III case!)

Brauer Theory

K = Field

M = finite-dim'l central simple algebra over K

Theorem. $M \cong M_j(D)$*, for some central division algebra* D*.*

Definition. $\mathrm{Br}(K)$ = isomorphism classes of central division algebras over K.

Theorem. $\mathrm{Br}(K)$ *is an abelian group with the operation*

$$[D_1] + [D_2] = [D], \qquad D_1 \otimes_K D_2 = M_j(D).$$

For $K = \mathbb{C}$ the group $\mathrm{Br}(K)$ is trivial, but e.g. for $\mathbb{Q}_p$ there is an isomorphism $\mathrm{Br}(K) \cong \mathbb{Q}/\mathbb{Z}$.

Example: For $a, b \in K^\times$ set

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

If $\mathrm{char}(K) \neq 2$ then

$$D \cong \begin{cases} M_2(K) & \text{if } aX^2 + bY^2 = 1 \text{ has a sol'n} \\ \text{Div. Ring} & \text{if } aX^2 + bY^2 = 1 \text{ has no sol'n} \end{cases}$$

AFD Factors

Definition. A von Neumann algebra M is *approximately finite-dimensional* if it is generated by an increasing family of f.d. $*$-subalgebras.

Example: R_λ = Powers factor from Lecture 2

= Double commutant of $\otimes M_2(\mathbb{C})$ in a GNS representation

Theorem (Murray and von Neumann). *There is a unique AFD factor of type II_1.*

Compare: Group von Neumann algebras. For non-amenable G, classification is largely unknown.

Proof of Theorem. Uses the trace

$$\begin{cases} \mu_t(E) = \dim(\chi_E(T)) \\ \tau(T) = \displaystyle\int_{\text{Spectrum}(T)} \lambda \, d\mu_T(\lambda) \end{cases}$$

and associated norm $\|S\|_2^2 = \tau(S^*S)$ to compute estimates—does not generalize even to II_∞ (since possibly $\|S\|_2 = \infty$ for all S in approximating family).

ITPFI Factors

$$S_j = \begin{pmatrix} \lambda_1^{(j)} & & \\ & \ddots & \\ & & \lambda_{n_j}^{(j)} \end{pmatrix} \qquad \varphi_j(T) = \mathrm{trace}(S_j T)$$

$$\Lambda = \left\{ \lambda_k^{(j)} : 1 \le k \le n_j, \quad j = 1, 2, \ldots \right\}$$

$R_\Lambda =$ Double commutant of $\otimes M_{n_j}(\mathbb{C})$ in GNS representation for $\otimes \varphi_j$

Recall:

Theorem (Powers). *For* $S_j \equiv \begin{pmatrix} \frac{1}{1+\lambda} & \\ & \frac{\lambda}{1+\lambda} \end{pmatrix}$ *and for the values* $0 < \lambda < 1$ *the ITPFI factors obtained are pairwise non-isomorphic.*

A negative indication for the classification problem:

Theorem (Woods). *The moduli space of ITPFI factors is not countably separated.*

Tomita's Modular Theory

- $\varphi\colon M \to \mathbb{C}$, faithful normal state (or weight):

  $$\varphi(T^*T) \geq 0$$

  $$\varphi(T^*T) = 0 \quad \Leftrightarrow \quad T = 0$$

  φ u.w. continuous

- H_φ, GNS Hilbert space.
- $S\colon H_\varphi \to H_\varphi, \quad Sx = x^*$.

Theorem. *There is a polar decomposition* $S = J\Delta^{\frac{1}{2}}$ *(isometry*$\times$*positive operator) and*

$$JMJ = M'$$

$$\Delta^{it} M \Delta^{-it} = M.$$

Recall: This is a difficult theorem in analysis which serves to reduce unbounded operator theory to bounded operator theory (c.f. Weyl's form of Heisenberg's relations).

Connes' Theorem

Theorem. *Any two modular flows σ^{φ} and σ^{ψ} are related by a unitary cocyle in M:*

$$\sigma_t^{\psi}(T) = \sigma_t^{\varphi}(U_t T U_t^*).$$

In other words one has

$$\sigma \colon \mathbb{R} \times M_2(M) \to M_2(M)$$

$$\sigma_t \begin{pmatrix} a & \\ & d \end{pmatrix} = \begin{pmatrix} \sigma_t^{\varphi}(a) & \\ & \sigma_t^{\psi}(d) \end{pmatrix}$$

It follows that associated to M there is a *canonical* homomorphism

$$\mathbb{R} \longrightarrow \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$$

We are eventually going to distill from this the basic classification invariant for AFD factors.

KMS Condition

KMS = Kubo, Martin and Schwinger

Example:

$$\left.\begin{aligned} M &= M_n(\mathbb{C}) \\ \varphi(T) &= \mathrm{trace}(e^{-H}T) \\ H_\varphi &\cong M \end{aligned}\right\} \qquad \left\{\begin{aligned} J\colon x &\mapsto e^{-\frac{1}{2}H}x^* e^{\frac{1}{2}H} \\ \Delta\colon x &\mapsto e^{-H}xe^{H} \\ \sigma_z\colon T &\mapsto e^{izH}Te^{-izH} \end{aligned}\right.$$

Computation:

$$\begin{aligned} \varphi(\sigma_z(T_1)T_2) &= \mathrm{trace}(e^{i(z+i)H})T_1 e^{-izH}T_2) \\ &= \mathrm{trace}(e^{-H}T_2 e^{i(z+i)H}T_1 e^{-i(z+i)H}) \\ &= \varphi(T_2\sigma_{z+i}(T_1)) \end{aligned}$$

KMS Condition:

For every T_1 and T_2 there is a continuous, bounded function F on the strip $0 \leq \mathrm{Im}(z) \leq 1$, holomorphic in the interior, with

$$\left\{\begin{aligned} F(t) &= \varphi(T_2\sigma_t(T_1)) \\ F(t+i) &= \varphi(\sigma_t(T_1)T_2) \end{aligned}\right.$$

KMS Condition, Continued

Theorem. *If σ is Tomita's modular flow associated to a normal state φ then the KMS condition is satisfied.*

Theorem. *Given a normal state φ there is a* unique *flow σ (namely Tomita's) for which the KMS condition holds.*

Example: For the Powers factors, one can check that if

$$S = \begin{pmatrix} \frac{1}{1+\lambda} & \\ & \frac{\lambda}{1+\lambda} \end{pmatrix}$$

then

$$\sigma_t(T_1 \otimes T_2 \otimes \cdots) = S^{it} T_1 S^{-it} \otimes S^{it} T_2 S^{-it} \otimes \cdots$$

satisifies the KMS condition.

The Invariant $S(M)$

G = locally compact abelian group.

Definition. The *spectrum* of an action $\alpha\colon G \times M \to M$ is the complement of the largest open set U in $\widehat{G}$ such that

$$\operatorname{Supp}(\widehat{f}) \subseteq U \quad \Rightarrow \quad \int_G f(g)\alpha(g)\, dg = 0$$

(here $f \in L^1(G)$).

Definition. The *Connes spectrum* of an action $\alpha\colon G \times M \to M$ is

$$S(\alpha) = \cap_p \operatorname{Spectrum}(\alpha\colon G \times pMp \to pMp),$$

where the $p \in M$ are nonzero α-fixed projections.

Proposition. *The Connes spectrum is a closed subgroup of $\widehat{G}$, and is invariant under exterior equivalence of actions.*

Definition. The closed subgroup $S(M) \subseteq \mathbb{R}_+$ is the Connes spectrum of the modular flow (we identify $\mathbb{R}_+$ and $\widehat{\mathbb{R}}$ via $\langle x, y\rangle = x^{iy}$).

A Finer Classification of Factors

- $\lambda = 0$ $\quad$ M is Type III_0 if $S(M) = \{1\}$.
- $0 < \lambda < 1$ $\quad$ M is Type III_λ if $S(M) = \lambda^{\mathbb{Z}}$.
- $\lambda = 1$ $\quad$ M is Type III_1 if $S(M) = \mathbb{R}_+$.

Connes' discrete decomposition:

Let $0 < \lambda < 1$. The analysis of III_λ factors is reduced to II_∞ factors and their automorphisms by a *crossed product* decomposition

$$\mathrm{III}_\lambda = \mathrm{II}_\infty \times_\alpha \mathbb{Z}$$

associated to an automorphism α of a II_∞ factor such that

$$\mathrm{trace}(\alpha(T)) = \lambda\,\mathrm{trace}(T)$$

for all $T \geq 0$. These ideas are developed to their fullest in Takesaki's theory of crossed products and duality, to be discussed next . . .

Crossed Products

$\alpha\colon G \times M \to M$, action of a locally compact abelian group on a von Neumann algebra $M \subseteq \mathcal{B}(H)$.

Definition. The *crossed product* $M \times_\alpha G \subseteq \mathcal{B}(L^2(G, H))$ is the von Neumann algebra generated by

$$\begin{cases} (Tf)(h) = \alpha_h(T)f(h), & T \in M \\ (U_g h)(h) = f(g-h), & g \in G, \end{cases}$$

or equivalently by $L^1(G, M)$ with the twisted multiplication

$$F_1 \star F_2(g) = \int_G F_1(h)\alpha_h(F_2(g-h))\, dh.$$

Definition. The *dual action* of $\hat{G}$ on $M \times_\alpha G$ is:

$$\begin{cases} \gamma(T) = T, & T \in M \\ \gamma(U_g) = \langle \gamma, g \rangle \cdot U_g, & g \in G, \end{cases}$$

Takesaki's Duality Theorem.

- $\left(M \times \alpha G\right)^{\hat{G}} \cong M$
- $M \times_\alpha G \times_{\hat{\alpha}} \hat{G} \cong M \otimes \mathcal{B}(L^2(G))$

Duality and Modular Theory

- $\sigma\colon \mathbb{R} \times M \to M$, modular group
- $N = M \times_\sigma \mathbb{R}$, crossed product
- $\hat{\sigma}\colon \mathbb{R}_+ \times N \to N$, dual action

Proposition. *The kernel of the restricted action* $\hat{\sigma}$ *on the center of* N *is precisely* $S(M)$*.*

Proposition. *The von Neumann algebra* N *has a faithful and semifinite trace* τ*, and moreover*

$$\tau(\hat{\sigma}_\lambda(T)) = \lambda\tau(T),$$

for all $T \geq 0$ *and all* $\lambda \in \mathbb{R}_+$*.*

Proof: Recall that N is generated by L^1-functions $F\colon \mathbb{R} \to M$. If F has compactly supported Fourier transform (so that F extends to an analytic function on $\mathbb{C}$) then define

$$\tau(F) = \varphi(F(i)),$$

where φ is the weight generating the modular flow σ. By the KMS condition, this is a trace.

The Flow of Weights $\mathrm{Mod}(M)$

The story so far . . .

$$M \rightsquigarrow (M, \varphi) \rightsquigarrow (M, \varphi, \sigma) \rightsquigarrow (N, \tau, \hat{\sigma})$$

From M, by choosing a faithful normal semifinite weight φ we have ultimately obtained a semifinite N, with a flow $\hat{\sigma}$ scaling the trace τ.

Crucial point: The data $(N, \tau, \hat{\sigma})$ is determined by M alone.

Lemma. *If M is a factor then the restriction of $\hat{\sigma}$ to the center of N is* ergodic *(it fixes no nontrivial element).*

Definition. The *flow of weights* or *module* for M is the restriction of $\hat{\sigma}$ to the center of N.

This is a refinement of $S(M)$, which is the kernel of this action. Notice that if the action is *transitive* then it is determined by its kernel. The intrasitive actions are (by definition) *virtual subgroups* of $\mathbb{R}_+$

Construction of Factors

Let us begin with:

N, an approximately finite dimensional II_∞ factor

$\alpha\colon \mathbb{R}_+ \times N \to N$, an automorphism group which scales the trace on N.

$S \subseteq \mathbb{R}_+$ a closed (virtual) subgroup

For a true subgroup S define

$$M = N^S = \{\, T \in N \,:\, \alpha_s(T) = T,\ \forall s \in S \,\}.$$

For a virtual subgroup $\mathbb{R}_+ \times A \to A$ define

$$M = N^S = \{\, T \in M \otimes A \,:\, \alpha_s(T) = T,\ \forall s \in \mathbb{R}_+ \,\}.$$

It will emerge that these are AFD factors with given module S, and that these are *all* the infinite AFD factors (excluding the trivial $M = \mathcal{B}(H)$). *Note the analogy with Galois theory.*

Injective Factors

Definition. $\mathcal{M} \subseteq \mathcal{B}(H)$ is *injective* if there is a completely positive splitting $\mathcal{B}(H) \to \mathcal{M}$.

Fairly easy: every AFD von Neumann algebra is injective.

Stability properties: crossed products by abelian groups, increasing unions (hence AFD algebras), decreasing intersections, direct integrals.

Observe that if $\mathcal{M}$ is injective and $P \in \mathcal{M}$ is a projection then $P\mathcal{M}P$ is injective. If $\mathcal{M}$ is II_∞ and P is finite then $P\mathcal{M}P$ is II_1.

Theorem (Connes). *There is a* unique *injective* II_1 *factor.*

Corollary. *There is a unique injective (and hence AFD)* II_∞ *factor.*

Since injective type III factors are ‘reduced’ to II_∞ algebras and automorphisms by modular theory, a classification comes into view ...

Classification of Injective Factors

Theorem. *A factor is AFD if and only if it is injective.*

Theorem. *Type* III *factors which are injective/AFD are classified by their modules* $\mathrm{Mod}(M)$.

This is largely due to Connes, but with essential contributions from Krieger (type III_0) and Haagerup (type III_1). Of course, the modular theory of Tomita and Takesaki is also crucial.

The key methods, beyond those already sketched, build on ideas in ergodic theory to classify the automorphisms of type II algebras.

Type II Differential Topology

Nigel Higson and John Roe
Penn State University
higson@math.psu.edu, roe@math.psu.edu

June 26, 2000

Abstract

In this lecture we showcase an early application of noncommutative geometry — the proof of the Novikov Conjecture for Gromov-hyperbolic groups. (Connes-Moscovici, *Cyclic cohomology, the Novikov conjecture, and hyperbolic groups*, Topology **29**(1990), 345–388).

The index theorem

Theorem 1. (Atiyah-Singer) *Let D be the Dirac operator on a compact even-dimensional spin manifold M; then* $\operatorname{Index}(D) = \int_M \widehat{\mathcal{A}}(M)$.

Heat equation proof. Let K_t^+ and K_t^- be the $\pm$-graded parts of the Heat operator e^{-tD^2}. Then

$$\operatorname{Index}(D) = \operatorname{Tr} K_t^+ - \operatorname{Tr} K_t^-$$

for all values of time t. On the other hand as $t \downarrow 0$ the heat kernels $K_t^\pm$ localize near the diagonal in $M \times M$ and there are asymptotic expansions of the form

$$\operatorname{tr} K_t(x,x) \sim a_m(x) t^{-m} + a_{m-1}(x) t^{-m+1} + \cdots$$

where $m = \frac{1}{2}\dim M$. The existence of this asymptotic expansion is what is responsible for the analytic continuation of the ζ function.

Atiyah-Patodi-Singer identified the difference of the terms $a_0^+ - a_0^-$ with the $\widehat{\mathcal{A}}$ form. The proof was later simplified by Getzler using a Clifford symbolic calculus for pseudodifferential operators on spinors.

Local index theorem

One can regard the heat equation formula as follows: the index of D defines an element of $K_0(\mathcal{R})$, where $\mathcal{R}$ is the algebra of smoothing operators; and we are computing the index by pairing with $[\mathrm{Tr}] \in HC^0(\mathcal{R})$. Are there other interesting cyclic cocyles for $\mathcal{R}$?

Answer: No. ($\mathcal{R}$ is essentially a big matrix algebra.)

However, suppose that we pass to the sub 'algebra' $\mathcal{R}_\epsilon$ of operators of propagation $< \epsilon$. Every Alexander-Spanier cocycle $c(x_0, \ldots, x_q)$ on M defines a cyclic cocyle τ_c on $\mathcal{R}_\epsilon$ by sending $(K^0, \ldots, K^q)$ to

$$\int\int c(x_0, \ldots, x_q) K^0(x_0, x_1) K^1(x_1, x_2) \cdots K^q(x_q, x_0).$$

Theorem 2. (Connes-Moscovici) *One has*

$$\tau_{c*}(\mathrm{Index}\, D) = \mathsf{const} \cdot \int_M \widehat{\mathcal{A}}(M) \smile c.$$

Proof is an application of the Getzler calculus.

Elliptic families

Let B be a space and let $E \to B$ be a fiber bundle with manifold fibers. A family of elliptic operators along the fibers has an index which belongs to $K^0(B)$.

Example (Lusztig) Let M be a torus, $\mathbb{R}^n/\mathbb{Z}^n$, n even. The characters χ of $\pi_1 M$ form a group B (which is also a torus!) Every such character corresponds to a flat line bundle. Let D be the signature operator of M; then for each character $\chi \in B$ we may form the twisted signature operator D_χ, and in this way we obtain a family $\tilde{D}$ of elliptic operators parameterized by χ. Thus we get a 'higher signature' $\mathrm{Index}(\tilde{D}) \in K^0(B)$.

Theorem 3. *The higher signature* $\mathrm{Index}(\tilde{D})$ *is an invariant of the oriented homotopy type of* M*.*

Compare the homotopy invariance of the ordinary signature. . .

Noncommutative families

For a different view of Lusztig's construction, consider the lift $\tilde{D}$ of D to the universal covering space $\tilde{M}$ of M. Let $\Gamma = \pi_1(M)$ ($= \mathbb{Z}^m$ in this example.) Let $\widetilde{\mathcal{R}}(M)$ be the algebra of smoothing operators on $\tilde{M}$ which are *of finite propagation* and also Γ-*invariant* (this is the groupoid algebra of the fundamental groupoid of M). Then $\tilde{D}$ is invertible modulo $\widetilde{\mathcal{R}}(M)$. Consequently there is defined an index $\mathrm{Index}(\tilde{D}) \in K_0(\widetilde{\mathcal{R}}(M))$.

Lemma 1. *The C^*-algebra completion of $\widetilde{\mathcal{R}}(M)$, in the norm topology of $\mathfrak{B}(L^2(\tilde{M}))$, is Morita equivalent to $C^*_r(\Gamma)$.*

In the case of the torus, $C^*_r(\Gamma) = C(B)$ and the index of $\tilde{D}$ defined by operator algebras equals the higher index in $K^0(B)$ defined on the previous slide.

Theorem 4. *The higher signature* $\mathrm{Index}(\tilde{D}) \in K_0(C^*_r(\Gamma))$ *is an invariant of the oriented homotopy type of M for every fundamental group Γ.*

The Novikov Conjecture

- Belongs to the realm of *surgery theory*, in which the fundamental question is: What is a manifold?

- Conjecture (and its relatives) reduce the classification of manifolds with given fundamental group Γ to a question of relative homology.

The Novikov Conjecture is this: Let M be a compact manifold with fundamental group Γ and let $c \in H^k(\Gamma; \mathbb{R})$ be a group cohomology class. Then the *Novikov higher signature*

$$\langle L(M) \smile f^*(c), [M] \rangle, \qquad f\colon M \to B\Gamma$$

is an invariant of oriented homotopy type.

Program of proof. Obtain the Novikov higher signatures from $\operatorname{Index}(\widetilde{D}) \in K_0(C_r^*(\Gamma))$.

Application of the local index theorem

Let $c \in H^k(\Gamma; \mathbb{R})$. Then $f^*(c)$ may be represented as an Alexander-Spanier cohomology class on M and therefore defines a cyclic 'cocycle' ϕ_c on $\mathcal{R}_\epsilon(M)$ for sufficiently small ϵ.

Lemma 2. *For sufficiently small ϵ there is a $1:1$ correspondence between $\mathcal{R}_\epsilon(M)$ and $\widetilde{\mathcal{R}}_\epsilon(M)$. Moreover, under this $1:1$ correspondence, ϕ_c corresponds to a genuine cyclic cocyle τ_c on the whole algebra $\widetilde{\mathcal{R}}(M)$.*

Hence

Theorem 5. *For each group cocycle c for M there is a cyclic cocycle τ_c on $\widetilde{R}(M)$ such that $\tau_{c*}(\mathrm{Index}(\tilde{D}))$ is the Novikov higher signature $\langle L(M) \smile f^*(c), [M] \rangle$.*

Almost a proof of Novikov

We have nearly proved the Novikov conjecture.

'All' we need to do is to fill in the dotted arrow in the diagram below

$$\begin{array}{ccc} K_0(\widetilde{\mathcal{R}}(M)) & \longrightarrow & K_0(C_r^*(\Gamma)) \\ & \searrow^{\tau_{c*}} & \vdots \\ & & \mathbb{R} \end{array}$$

Problem: The cocycle τ_c need not, a priori, be a k-trace.

Rapid decay

We ask: When is a sequence $\{c_g\}_{g\in\Gamma}$ the sequence of Fourier coefficients of some element of $C_r^*(\Gamma)$?

For groups Γ of polynomial growth it is clear that a sufficient condition is $|c_g| = O(|g|^{-N})$ for N large enough. But such groups are not very interesting.

Haagerup (*An example of a non nuclear C^*-algebra which has the metric approximation property*, Inventiones **50**(1979) 279–293) proved: Let Γ be a free group. Then there is an $N > 0$ such that if the function $g \mapsto |c_g||g|^N$ belongs to $\ell^2(\Gamma)$, then $\sum c_g g$ belongs to $C_r^*(\Gamma)$; in fact he estimated the $C_r^*(\Gamma)$ norm of the sum in terms of the ℓ^2 norm of $g \mapsto |c_g|(1+|g|)^N$.

A group enjoying this property is said to be of *rapid decay* (RD).

Proof of RD

The key lemma in Haagerup's proof of RD for the free group is the following:

Lemma 3. *Let* δ *and* ℓ *be given. There exists a constant* $C > 0$ *such that, for any* $g \in \Gamma$*, there are at most* C *ways of decomposing* $g = g_1 g_2$ *with* $|g_1| = \ell$ *and* $|g_1| + |g_2| \leq |g| + \delta$*.*

Exercise: Prove this.

Gromov defined a *hyperbolic* group to be one in which all geodesic triangles are thin. Hyperbolic groups have the large scale qualitative features of free groups. Jolissaint extended Haagerup's calculation to prove

Theorem 6. *Hyperbolic groups enjoy property RD.*

Remark: Not all groups have RD — e.g. no exponential growth solvable group has RD.

Novikov for hyperbolic groups

Lemma 4. (Gromov) *Every cohomology class for a hyperbolic group may be represented by a bounded cocycle.*

Choose such a representative for $c \in H^*(\Gamma; \mathbb{R})$. Then we have

Lemma 5. *The homomorphism* $\tau_{c*}: K_0(\widetilde{\mathcal{R}}) \to \mathbb{R}$ *extends to a homomorphism* $K_0(C_r^*(\Gamma)) \to \mathbb{R}$.

The proof uses Haagerup estimates to extend the domain of τ_c to a dense subalgebra of $C_r^*(\Gamma)$ which is smooth (closed under holomorphic calculus). Consequently

Theorem 7. *The Novikov conjecture holds for hyperbolic groups.*

Discussion

This proof is a classic application of cyclic theory to obtain maps which did not (a priori) come from K-homology.

There now exist K-theoretic proofs of NC for hyperbolic groups, using hypereuclideanness, contractibility of the Gromov compactification, amenability of the boundary action, . . . But these seem to use rather different features of hyperbolic geometry — it is an open question to relate the key features of these proofs.

V Lafforgue has shown that uniform lattices in $SL(3,\mathbb{R})$ enjoy RD, and this is a key step in his proof of the Baum-Connes conjecture for such lattices. On the other hand $SL(3,\mathbb{Z})$ does not enjoy RD since it contains an exponential growth solvable subgroup.

Conjecture (Valette) All uniform lattices in $SL(n,\mathbb{R})$ enjoy RD.

Type III Differential Topology

Nigel Higson and John Roe
Penn State University
higson@math.psu.edu, roe@math.psu.edu

June 28, 2000

Abstract

This lecture is an exposition of the work of Connes et al on the transverse fundamental class. Basic reference: *Cyclic cohomology and the transverse fundamental class of a foliation*, in 'Geometric methods in operator algebras', Pitman Research Notes volume 123.

Topology of foliations

A foliation on a manifold M can be completely described by the subbundle $F \leq TM$ of vectors that are tangent to the leaves. However, not all subbundles can occur, but only those that are *integrable* in on of the following equivalent senses:

- The sections of F form a Lie algebra of vector fields;
- The 1-forms annihilating F form a differential ideal.

Bott (around 1970) asked: Are there topological obstructions to integrability?

He proved the following theorem: If $Q = TM/F$ is the *normal bundle* to a foliation of codimension q, then the Pontrjagin ring of Q vanishes in degrees $k > 2q$.

This is carried out by means of Chern-Weil theory applied to a special connection — a *basic connection* — on q.

Secondary characteristic classes

Bott vanishing allows the construction of secondary characteristic classes of a foliation. Namely, one can compare the representatives for primary characteristic classes of Q built from a Riemannian connection, on the one hand, and a basic connection, on the other. Using Bott vanishing one obtains canonical closed forms which implement homotopies between these representatives — secondary characteristic classes.

Example Let F be a foliation of codimension one. Then it is defined by a single 1-form ω. Frobenius' theorem gives $d\omega = \omega \wedge \theta$ for some 1-form θ (which may be interpreted as the connection form of a basic connection). Then $\theta \wedge d\theta$ is a closed form (its exterior derivative $(d\theta)^2$ is a Chern-Weil representative of p_1 relative to a basic connection, hence vanishes.) Its cohomology class is independent of the choices and is called the *Godbillon-Vey* invariant $GV \in H^3(M;\mathbb{R})$.

(Thurston) GV measures 'helical wobble'. It can vary continuously.

The transverse fundamental class problem

Let M be a compact manifold. One can define an 'integration map' in K-theory by

$$x \mapsto \int_M \operatorname{ch} x, \quad K^*(M) \to \mathbb{R}.$$

This integration map is defined cohomologically. One could also find a family of integration maps reflecting the various characteristic classes of TM.

Alternatively one could use index theory (K-homology) to produce an integration map.

Investigate a foliated manifold (M, F), or alternatively an action of a group Γ on V. In either case we would like to produce similar 'integration maps' from $K_*(C_r^*(M, F))$ to $\mathbb{C}$.

About the solution

This problem is solved using cyclic cohomology in the transverse fundamental class paper. Moreover one can reflect the secondary characteristic classes (like GV) of the transverse space in the integration map. The continuous variation of GV shows that these can't arise from K-homology.

The problem is entirely one of *analysis*.

The natural approach would involve a 'transverse Dirac operator', but this makes no sense without an invariant (transverse) Riemannian structure. As we have seen, the secondary classes (like GV) measure the obstruction to the existence of such a structure.

We are in a type III situation.

What use is the transverse fundamental class?

One needs (a) things to pair with it, and (b) methods to evaluate the result.

One pairs the transverse fundamental class with indices of elliptic operators (this can be systematized by using the Baum-Connes assembly map). And one evaluates the result by an appropriate index theorem.

Example Let M be a manifold provided with a spin foliation. Then there is a transverse fundamental class functional $\phi: K_*(C^*(M,F)) \to \mathbb{R}$ such that $\phi(\mathrm{Index}\, D_F) = \widehat{\mathcal{A}}(M)$, where D_F is the leafwise Dirac operator. (Of course, the right hand side need not be an integer.)

Corollary If M has nonzero $\widehat{\mathcal{A}}$ genus, then it admits no spin foliation whose leaves are of positive scalar curvature.

Challenge: Give an elementary proof.

The cocycle

It is easy to produce a cocyle on the smooth groupoid algebra which represents the transverse fundamental class. In fact, it is just given by an equivariant version of the usual de Rham formula for the fundamental class:

$$(f_0, \ldots, f_q) \mapsto \int_V f_0 df_1 \cdots df_q.$$

Here we must take the f's to be elements of the groupoid algebra $C^\infty(V) \rtimes \Gamma$; the operations take place in the algebra $\Omega^*(M) \rtimes \Gamma$ whose elements are formal sums $\sum \omega_g U_g$, the ω being differential forms: and the algebraic rules are

- $(\sum \omega_g U_g)(\sum \alpha_h U_h) = \sum \omega_g \wedge g^*(\alpha_h) U_{gh}$;
- $d(\sum \omega_g U_g) = \sum (d\omega_g) U_g$;
- $\int (\sum \omega_g U_g) = \int \omega_e$.

The extension problem

This cocycle gives a homomorphism $K_*(C_c^\infty(\mathcal{G})) \to \mathbb{C}$, we need to extend it to a homomorphism $K_*(C_r^*(\mathcal{G})) \to \mathbb{R}$. (Compare lecture 7.)

First attempt is the theory of q-traces (lecture 4). These are cocycles for which one has an estimate like

$$|\tau(a_1 dx_1 \cdot a_2 dx_2 \cdots a_q dx_q)| \leq C_{x_1,\ldots,x_q} \|a_1\| \cdots \|a_q\|.$$

(the left side can be interpreted in terms of the multilinear functional τ).

Theorem 1. *A q-trace on a dense subalgebra of a Banach algebra extends to define a homomorphism from the K-theory of the whole algebra to $\mathbb{R}$.*

This theory solves the problem for $q = 1$.

An awkward example

Let $\Gamma = \mathbb{Z}$ acting on $V = \mathbb{T}^2$ via the hyperbolic matrix $\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. (Notice that the crossed product is then the C^*-algebra of an exponential solvable group $\mathbb{Z}^2 \rtimes \mathbb{Z}$.)

Then the transverse cocycle is *not* a 2-trace. This results from the exponential growth of the powers of α which are introduced when one tries to commute x_1 and a_2 in the expression $\tau(a_1 dx_1 a_2 dx_2)$.

Thus one cannot expect that the theory of q-traces will be adequate by itself.

Triangular structures

It is easy to deal with the case when Γ acts by isometries (invariant Riemannian structure).

Next simplest case — Γ acts by triangular matrices $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, where A and D are orthogonal transformations. We refer to a *triangular structure*. (May also assume the A direction is integrable. . .)

Theorem 2. *The transverse fundamental class can be defined in the case that Γ preserves a triangular structure.*

Even in this case the proof is highly non-trivial. It uses the theory of q-traces, applied not to the C^*-algebra but to a Banach subalgebra B defined as the domain of a certain unbounded operator (generalized modular operator); and B is shown to be smooth because of the existence of the triangular structure.

Some K-theoretic technology

Recall the Thom isomorphism in K-theory: if W is, say, a complex vector bundle over B then there is an isomorphism $K^*(B) \cong K^*(W)$.

Proof: The isomorphism is given by taking the product with the 'Bott element' $\beta \in K^0(\mathbb{C}^n)$, and the inverse isomorphism by taking the product with the 'Dirac operator' $D \in K_0(\mathbb{C}^n)$.

In his work on the Novikov Conjecture, Kasparov gave an abstract theory of such 'Dirac' and 'dual Dirac' elements. One need not restrict attention to *vector* bundles; one can build them for any bundle whose fibers are contractible nonpositively curved spaces on which the structure group acts by orientation-preserving isometries.

Transfer to the bundle of metrics

Let Γ act on V as usual, and let W be the bundle over V whose fiber at x is the space of positive definite quadratic forms on T_xV. The fiber of W is a nonpositively curved symmetric space $GL^+(q)/SO(q)$, and Γ acts on it. Kasparov theory then provides a Thom map

$$K_*(C(V) \rtimes \Gamma) \to K_*(C_0(W) \rtimes \Gamma).$$

Moreover

Lemma 1. *The action of Γ on W preserves a triangular structure.*

Now we may define the fundamental class in general by first using the Thom map to reduce to the case where a triangular structure is preserved, and then using Theorem 1.

Remarks: (1) The construction is inspired by the modular theory of type III von Neumann algebras; (2) There is also an analogy to the 'transfer to the disc bundle' used by Farrell-Jones.

Hypoelliptic theory

There is a way to define the transverse fundamental class as a K-homology cycle, using the theory of *hypo*elliptic operators (example: the heat operator). This approach *also* proceeds first by using the transfer to the bundle of metrics to show that we may assume an invariant triangular structure.

Basic idea: as well as the familiar 'first order signature operator' $d + d^*$ on a manifold M, there is also a 'second order signature operator' $dd^* - d^*d$ which has the same index.

If M is a triangularized manifold we may now define a 'mixed signature operator' which is equal to the second order operator in the A directions plus the first order operator in the D directions. Then the deviations from Γ invariance become lower order terms, and we can define a K-homology class.

Ref: Hilsum and Skandalis, *Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov*, Ann Sc Ec Norm Sup **20**(1987), 325–390.

The Godbillon-Vey Class

Our construction of the fundamental class started with a Γ-invariant closed current on V.

Such a current is a particular example of a cocyle in the groupoid de Rham cohomology of $\mathcal{G} = V \rtimes \Gamma$. All such cocyles give cyclic classes for the groupoid algebra, and a particular example is the 'Jacobian cocycle' (we assume codimension one)

$$\ell(\gamma) = \log\Big(\text{Transverse distortion}\Big).$$

This cocycle gives rise to a 'Godbillon-Vey transverse fundamental class'.

Theorem 3. *The GV transverse fundamental class can be reconstructed from the action of the modular group on the standard transverse fundamental class.*

Corollary 1. *Suppose $GV \neq 0$. Then the von Neumann algebra of the foliation is of type III (in fact one can prove it is of type III_0).*

Metric geometry and local index formulae

Nigel Higson and John Roe
Penn State University
higson@math.psu.edu, roe@math.psu.edu

June 29, 2000

Abstract

In this last talk we'll summarize some of Connes' ideas about the metric aspect of noncommutative geometry and associated local index formulae. Main Reference: Connes and Moscovici, *The local index formula in noncommutative geometry*.

Metrics and noncommutative geometry

What does it mean to give a metric on a noncommutative space described by an algebra A?

According to Connes, this data is given by a *spectral triple* (a.k.a. unbounded Fredholm module), that is, a representation $\rho\colon A \to \mathfrak{B}(H)$ together with an unbounded selfadjoint operator D on H, with compact resolvent, such that $[D, \rho(a)]$ is bounded for a dense set of $a \in A$.

Why? Consider the example of a Riemannian manifold M. Then one can recover the distance between two points x, y of M as

$$\sup\{|f(x) - f(y)| : \|[D, \rho(f)]\| \leq 1\}.$$

Passing from D to $F = D|D|^{-1}$ should be thought of as passing from metric to conformal structure.

Summability

An unbounded Fredholm module is p-*summable* if the resolvents $(D \pm i)^{-1}$ belong to the Schatten ideal $\mathfrak{L}^p$. (This implies the corresponding condition for F.)
Exercise: The unbounded Fredholm module defined by the length function on a group is finitely summable if and only if the group has polynomial growth.

We will focus attention on p-summable modules. But one should not think that this condition is universal. In fact one has:

Theorem 1. *If Γ is a nonamenable group then there are no finitely summable* unbounded *Fredholm modules on $C_r^*(\Gamma)$.*

There do exist bounded p-summable Fredholm modules on $C_r^*(\Gamma)$ even for some property T groups Γ.
Question: Are there examples in rank > 1?

Constructing the character

For a finitely summable *bounded* Fredholm module we have seen how to construct a cyclic character:

$$\tau_q(a_0, a_1, \ldots, a_q) = \mathrm{Tr}'(a_0[F, a_1] \cdots [F, a_q]).$$

Recall that this construction produces many characters for the same Fredholm module, which are related by the periodicity operator $S\colon HC^q(A) \to HC^{q+2}(A)$. Ask: What is the *lowest* dimension in which the cyclic character exists?

By Connes' exact sequence to ask whether τ_q can be 'desuspended' is the same as to ask whether the image $I(\tau_q)$ in Hochschild cohomology $HH^q(A)$ is zero. How to calculate this image?

The answer involves the *Dixmier trace.*

The Dixmier trace

Let H be a Hilbert space. The *Macaev ideal* $\mathfrak{M}$ is the collection of all compact operators T on H whose characteristic values μ_i (the eigenvalues of $|T|$) satisfy

$$\sup_N \frac{1}{\log N} \sum_{i=1}^{N} \mu_i < \infty.$$

Define the *Dixmier trace*

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{i=1}^{N} \lambda_i$$

where ω is a (suitable?) ultrafilter on $\mathbb{N}$. This formula defines a *linear* (Surprise!) functional on $\mathfrak{M}$, which is clearly a trace. Moreover it annihilates all operators in the ordinary trace class $\mathfrak{L}^1$.

(Connes denotes $\mathfrak{M}$ by $\mathfrak{L}^{1,\infty}$.)

Wodzicki residue

One can show that a pseudodifferential operator of order $-n$ (for instance $(1 + D^2)^{-n/2}$) on an n-dimensional manifold belongs to the ideal $\mathfrak{M}$. (Look at the circle for an example.)

Theorem 2. *Let P be such a pseudodifferential operator with principal symbol $\sigma(x, \xi)$ (a homogeneous function of degree $-n$ on T^*M.) Then (for any choice of the ultrafilter ω),*

$$\mathrm{Tr}_\omega(P) = c_n \int_{S^*M} \sigma dv.$$

(In other words, the trace depends only on the principal symbol. The right hand side is the *Wodzicki residue*; Wodzicki extended it to a trace on the algebra of *all* pseudodifferential operators. It is the residue at $s = 0$ of the meromorphic function $s \mapsto \mathrm{Tr}(P\Delta^{-s/2})$.)

Proof Vanishing on $\mathfrak{L}^1$ and symmetry.

Key Point: The symbol is a local invariant.

Classical Limit v. Dixmier Trace

Let F be the Fredholm module defined by the Dirac operator D over a spin manifold M^n. It is p-summable for all $p > n$ (as we saw above), so it has a character in $HC^n(C^\infty(M))$.

Recall from lecture 4 that

$$HC^n(C^\infty(M)) = Z_n(M) \oplus H_{n-2}(M) \oplus H_{n-4}(M) \oplus \cdots .$$

Theorem 3. *The character of the Dirac module has top-dimensional component equal to the current* $(f_0, \ldots, f_n) = \int_M f_0 df_1 \cdots df_n$*, and the homology components are those of the Poincaré dual of the* $\widehat{\mathcal{A}}$*-genus.*

The proof uses the classical limit. Connes' idea: Replace this by the Dixmier trace.

The residue theorem for the cyclic character

Let (ρ, H, D) define an unbounded Fredholm module over A, and suppose that this module is $\mathfrak{L}^{n,infty}$ summable (in the obvious sense).

Then for $a_0, \ldots, a_n \in \mathcal{A}$ the operator $a_0[D, a_1] \cdots [D, a_n]|D|^{-n}$ belongs to the Macaev ideal $\mathfrak{M}$, and

$$\phi_\omega(a_0, \ldots, a_n) = \mathrm{Tr}_\omega\big(a_0[D, a_1] \cdots [D, a_n]|D|^{-n}\big)$$

defines a Hochschild cocycle on $\mathcal{A}$.

Theorem 4. *This Hochschild cocycle 'is' the image of the n-dimensional cyclic character τ_n under the map $I\colon HC^n(\mathcal{A}) \to HH^n(\mathcal{A})$. Consequently, if the cohomology class of ϕ_ω is not zero, then τ_n cannot be desuspended — it does not belong to the image of S.*

Proof Exercise?! An easier exercise — compute in the manifold example.

Dimension spectrum

To get a more refined formula one needs to incorporate lower order terms in asymptotic expansion.

Definition 1. *The* dimension spectrum *of a spectral triple (if it exists) is a discrete subset* $\Sigma \subseteq \mathbb{C}$ *with the property that all the zeta functions*

$$\zeta_b(s) = \mathrm{Tr}(b|D|^{-s}),$$

where b *is a useful operator, continue analytically to* $\mathbb{C} \setminus \Sigma$*.*

(The 'useful' operators are just all the things like $[|D|, a]$ which are going to appear in the formulae.) We'll assume that the continuations have simple poles on Σ; there is a more elaborate version of the theory which allows for multiple poles.

Example: The Hilsum-Skandalis hypoelliptic operator corresponding to a triangular structure.

Extended Dixmier trace

One can define a notion of *abstract pseudodifferential operator*, given by expansions like

$$b_q|D|^q + b_{q-1}|D|^{q-1} + \cdots$$

where the b's are useful operators. For any such pseudodifferential operator P, the function $\zeta_P(s) = \mathrm{Tr}(P|D|^{-s})$ is meromorphic on $\mathbb{C}$, and the residue $\chi_0(P)$ of $\zeta_P(s)$ at zero is an extension of the Dixmier trace on $\mathfrak{M}$ to *all* pseudodifferential operators — exactly as in Wodzicki's work.

Remark: The above is really true only in the case of simple poles. For multiple poles there is a more elaborate theory involving all the negative coefficients $\chi_{-i-1}(P)$ in the Laurent series; they are related by a bunch of combinatorial identities of which only the highest one is the trace property.

Local index formula

Theorem 5. *In the situation above the following formulae define a cocycle in the (b, B)-bicomplex of $\mathcal{A}$. Moreover the cyclic class of this cocycle coincides with the cyclic character of D.*

In degree n we have up to a universal constant

$$\sum_{q \geq 0, k_j \geq 0} c_{k,q} \chi_q \big(a_0 \nabla^{k_1}(da_1) \cdots \nabla^{k_n}(da_n) |D|^{-(n+2\sum k_j)} \big),$$

where $da = [D, a]$ and $\nabla a = [D^2, a]$.

According to Connes-Moscovici it is an easy calculation using the Getzler calculus to show that this formula defines the $\widehat{\mathcal{A}}$ genus in the ordinary manifold case. Here the terms with $k > 0$ do not contribute.