The Baum-Connes Conjecture and Parametrization of Representations

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General problem: Obtain a description of the dual (the set of irreducible representations, up to equivalence) of a semisimple group, e.g. $SL(n, \mathbb{R})$.

(Typically *infinite-dimensional* representations, either tempered or admissible, up to infinitesimal equivalence.)

Various solutions: Gelfand, Harish-Chandra, Langlands, Knapp, Zuckermann, Vogan, Beilinson, Bernstein, ...

The Baum-Connes conjecture suggests an approach that is

- very naive
- apparently unnoticed before
- apparently viable

At the moment, the approach strays a bit far from the Baum-Connes point of view. Can it be framed as a refined version of Baum-Connes?

Baum-Connes Assembly Map

Baum-Connes map for a locally compact group:

$$\mu\colon K_*^{\text{top}}(G) \longrightarrow K_*\big(C^*_\lambda(G)\big)$$

Inspired by:

- foliations (Connes)
- discrete series (Parthasarathy, Atiyah-Schmid, Connes-Moscovici)
- geometric K-homology (Atiyah, Baum-Douglas)

We should think of the RHS as the Atiyah-Hirzebruch K-theory of the (reduced) unitary dual of G. In our cases this is (almost) exactly what it is.

The Equivariant Index

Let D be an equivariant Dirac-type operator on a G-compact proper G-manifold M, acting on sections of some $\boldsymbol{S}$.

Its index in $K(C^*_\lambda(G))$ is obtained as follows:

- Manufacture from $C^\infty_c(M, \boldsymbol{S})$ (by a completion operation) a (Hilbert) $C^*_\lambda(G)$-module.
- Manufacture from D (by completing its graph) an unbounded self-adjoint operator on this Hilbert module with compact resolvent.
- Take the index of this Fredholm operator.

Baum-Connes for Lie Groups

Here the conjecture is proved ...

$G =$ Connected semisimple group
$K =$ Maximal compact subgroup

Assume $K\backslash G$ has a G-equivariant Spin^c-structure, given by a K-equivariant Clifford algebra representation $c\colon \mathrm{Cliff}(\mathfrak{p}) \longrightarrow \mathrm{End}(S)$ (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$). Then

$$\boxed{K_*^{\mathrm{top}}(G) = R(G)}$$

and to a representation $\tau\colon K \to U(V)$ corresponds the Dirac operator on $K\backslash G$ with coefficients in V.

Index of the Dirac Operator on K\G

$G =$ Semisimple group
$K =$ Maximal compact subgroup

Let $D =$ Dirac with coefficients in an (irreducible) K-module V.

- $C_c^\infty(K\backslash G, \boldsymbol{S} \otimes \boldsymbol{V})$ is $\big(C_c^\infty(G) \otimes S \otimes V\big)^K$, and the Hilbert $C^*_\lambda(G)$-module completion is

$$\mathcal{H} = \big(C^*_\lambda(G) \otimes S \otimes V\big)^K$$

- This we can *localize* at any $[\pi] \in \widehat{G}_\lambda$:

$$\mathcal{H} \otimes_{C^*_\lambda(G)} H_\pi = \big(H_\pi \otimes S \otimes V\big)^K$$

  We obtain a *finite-dimensional* Hilbert space.

Index of the Dirac Operator on K\G

- The Dirac operator is

$$D = \sum_{\text{o.n.b. for } \mathfrak{p}} X_i \otimes c(X_i) \otimes I$$

- It too localizes to each $(H_\pi \otimes S \otimes V)^K$ (where $[\pi] \in \widehat{G}_\lambda$).

- We obtain

$$\left\{ D_\pi = \sum \pi(X_i) \otimes c(X_i) \otimes I \colon (H_\pi \otimes S \otimes V)^K \longrightarrow (H_\pi \otimes S \otimes V)^K \right\}$$

- Compare Parthasarathy-Atiyah-Schmid, except at issue here is the *topology* of the dual, not measure theory.

Reduced C*-Algebra of a Semisimple Group

Consider for example $G = SL(n, \mathbb{C})$, after Gelfand & Naimark (or any complex semisimple group).

Let B = MAN = Borel subgroup of upper-triangular matrices.

- For $(\sigma, \varphi) \in \widehat{M} \times \widehat{A}$, $\pi_{(\sigma,\varphi)} := \mathrm{Ind}_B^G(\sigma \otimes \varphi \otimes 1)$ is irreducible.
- $C^*_\lambda(G)$ is represented as compact operators in each $H_{(\sigma,\varphi)}$.
- Corresponding to permutations $w \in W$ are intertwiners

$$I_w \colon H_{(\sigma,\varphi)} \to H_{(w\sigma,w\varphi)}.$$

- There results an isomorphism

$$C^*_\lambda(G) \xrightarrow{\cong} C_0\big(\widehat{M} \times \widehat{A}, \mathcal{K}(H)\big)^W$$

  and a Morita equivalence with $C_0(\widehat{M} \times \widehat{A})^W$.

Computation of the Dirac Index

$G =$ Complex semisimple, so that $\widehat{G}_\lambda = (\widehat{M} \times \widehat{A})/W$

$$\left\{ D_{(\sigma,\varphi)} \colon \big(H_{(\sigma,\varphi)} \otimes S \otimes V\big)^K \longrightarrow \big(H_{(\sigma,\varphi)} \otimes S \otimes V\big)^K \right\}$$

This is a cycle for $K^*\big((\widehat{M} \times \widehat{A})/W\big)$. One can calculate:

$$D^2 = \Omega_G \otimes I \otimes I - I \otimes \Omega_K \otimes I + I \otimes I \otimes \Omega_K,$$

where $\Omega =$ Casimir. After some more calculation,

$$D^2 = \|\varphi^2\| - \|\sigma\|^2 + \|\tau + \rho\|^2.$$

Still further calculation shows that:

Theorem (Penington and Plymen)

Index(D) *is supported on the component of* $(\widehat{M} \times \widehat{A})/W$ *labelled by* ρ *plus the highest weight* τ *of* V *and is the push-forward of the base representation into this euclidean space.*

Some Questions

It would be interesting to calculate D more carefully on other than the $\rho + \tau$ component. (Connections to the Dirac family of Freed-Hopkins-Teleman and the orbit method?)

What about the push-forwards of other representations, aside from those with real infinitesimal character? Is there a cycle group more refined than $R(K)$ that more fully accounts for the dual as a geometric space?

Contraction of G to a subgroup K

Examined by Wigner *et al* in the 1950's, 1960's ...

$G =$ Lie group
$K =$ Lie subgroup

The *contraction of G along K* is

$$\boxed{G_0 = K \ltimes \big(\mathrm{Lie}(G)/\,\mathrm{Lie}(K)\big)}$$

Notice that: $G_0 =$ normal bundle for the inclusion of K into G.

If G acts isometrically on X, and if $Y \subseteq X$ is K-invariant, then G_0 acts on the normal bundle of Y in X (affine-isometrically on the fibers).

Cartan Motion Group

$G =$ Semisimple group
$K =$ Maximal compact subgroup

If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then

$$G_0 = K \ltimes \mathfrak{p}$$

and one can form G_t with Lie algebra $\mathfrak{g}_t$:

$$[X, Y]_t = \begin{cases} [X, Y] & \text{if } X \text{ or } Y \in \mathfrak{k} \\ t[X, Y] & \text{if } X \text{ and } Y \in \mathfrak{p}. \end{cases}$$

Baum-Connes Conjecture, Again

The family $\{G_t\}$ determines a specific map

$$K_*(C^*_\lambda(G_0)) \longrightarrow K_*(C^*_\lambda(G))$$

and the Baum-Connes conjecture is equivalent to the assertion that *this map is an isomorphism.*

(Why? Roughly speaking because the Baum-Connes left-hand sides for G and G_0 are the same.)

Equivalent (slightly loose) formulation of the assertion: *the continuous field $\{C^*_\lambda(G_t)\}$ is K-theoretically indistinguishable from a constant field.*

What does this version of Baum-Connes look like, from a representation-theoretic point of view?

Mackey's Analogy

Mackey (early 1970's): "*... there ought to exist a 'natural' one-to-one correspondence between almost all of the unitary representations of G_0 and almost all the unitary representations of G ...*"

By *almost all* he probably meant *with respect to Plancherel measure*.

Given Mackey's suggestion (and supporting computations), there is an interesting tension between measure theory (in the form of his analogy) and cohomology (in the form of Baum-Connes) ...

The Reduced Duals of G_0 and G

$$G_0 = \text{Compact} \ltimes \text{Vector Group} = K \ltimes V$$

By Fourier and Peter-Weyl,

$$C^*(G_0) \cong K \ltimes C^*(V) \cong K \ltimes C_0(\widehat{V}) \cong C_0\left(\widehat{V}, \mathcal{K}(L^2(K))\right)^K.$$

This gives

$$\boxed{\widehat{G_0} = \bigsqcup_{\phi \in \widehat{V}} \widehat{K_\phi} \Big/ K,}$$

while on the other hand

$$\boxed{\widehat{G}_\lambda = (\widehat{M} \times \widehat{A})/W}$$

for complex semisimple G.

A Mackey Bijection

$$\boxed{\bigsqcup_{\varphi \in \hat{V}} \widehat{K}_\varphi \Big/ K} \cong \boxed{\widehat{M} \times \widehat{A} \Big/ W}$$

- $\widehat{V}/K = \widehat{A}/W$
- For all $\varphi \in \widehat{A}$, K_φ is connected with maximal torus M.

$\Rightarrow$ $\widehat{K}_\varphi = \widehat{M}/W_\varphi$ by Cartan-Weyl.

$$\Rightarrow \quad \begin{aligned} \widehat{G}_0 = \bigsqcup_{\varphi \in \hat{V}} \widehat{K}_\varphi \Big/ K &= \bigsqcup_{\varphi \in \widehat{A}} \widehat{K}_\varphi \Big/ W \\ &= \bigsqcup_{\varphi \in \widehat{A}} \widehat{M}/W_\varphi \Big/ W \\ &= (\widehat{M} \times \widehat{A})/W \\ &= \widehat{G}_\lambda \end{aligned}$$

More Calculations

Some work of students . . .

Chris George: Similar bijection for $SL(n, \mathbb{R})$ (and similar structure theorem for the C^*-algebra—next slide—in progress).

John Skukalek: Similar bijection and structure theorem for almost-connected groups with connected component of identity complex semisimple.

Some help from the experts . . .

With the ample assistance of *David Vogan*, the general semisimple (or reductive) case seems well within reach.

Minimal K-Types

What is the connection to Baum-Connes?

Label each irrep of G by its *minimal K-type(s)* (for G complex semisimple, the set $\widehat{K}$ is partially ordered by highest weights and there is a unique minimal K-type in each irrep).

Theorem: *The bijection preserves minimal K-types (which always have multiplicity one).*

Theorem: *The labels determine an increasing filtration of $\{C^*_\lambda(G_t)\}$ by continuous fields of ideals. The corresponding fields of subquotients are Morita-equivalent to constant fields (of commutative C^*-algebras).*

This is for *complex G*, so far.

Infinitesimal Representations

The *Hecke algebra* of G is the convolution algebra $H(G, K)$ of distributions on G supported on K.

Lemma

Nondegenerate $H(G, K)$*-modules* $\equiv$ $(\mathfrak{g}, K)$*-modules*

The Hecke algebra is *filtered* by distribution order.

Lemma

The associated graded algebra for $H(G, K)$ *is the Hecke algebra* $H(G_0, K)$ *for the contracted group.*

There is therefore a natural deformation from $H(G_0, K)$ into $H(G, K)$ ...

Fix $\tau \in \widehat{K}$ and form the *local Hecke algebra*

$$H(G,\tau) = p_\tau H(G)^K p_\tau$$

Lemma

Irreducible $H(G,\tau)$-modules $\cong$ *Irreducible $(\mathfrak{g}, K)$-modules with nonzero τ-component*

As before, $H(G_0,\tau)$ is the associated graded algebra of $H(G,\tau)$

Both are finite type algebras (Harish-Chandra).

Lemma

For G complex semisimple

$$H(G,1) \cong Z(U(\mathfrak{g}))$$

(consider $\mathfrak{g}$ as a complex Lie algebra) and more generally

$$H(G,\tau) \cong \big(U(\mathfrak{g}) \otimes \mathrm{End}(\tau^*)\big)^K$$

So $H(G_0,1) \cong S(\mathfrak{g})^K$ and $H(G_0,\tau) \cong \big(S(\mathfrak{g}) \otimes \mathrm{End}(\tau^*)\big)^K$.

Harish-Chandra defined

$$Z(U(\mathfrak{g})) \longrightarrow U(\mathfrak{a})$$

and proved it to be an isomorphism onto $U(\mathfrak{a})^W$ as follows:

- The image is in $U(\mathfrak{a})^W$ (intertwiners)
- The associated graded map is bijective onto $S(\mathfrak{a})^W$ (Chevalley).

Theorem (Harish-Chandra/Chevalley Isomorphism)

$$H(G,1) \cong S(\mathfrak{a})^W \cong H(G_0,1)$$

This identifies the spherical representations of G and G_0.

There are now several proofs (including an amazing Dirac operator-based proof of Alekseev-Meinrenken ...)

The theorem identifies the spherical duals of G and G_0. Does it extend to other parts of the dual?

Classification of Irreducible Representations

Complex G: Classification begun by Berezin in 1950's (at Gelfand's suggestion), completed 20 years later (Zhelobenko, Duflo, ...).

Let's continue to focus on these.

Harish-Chandra's Subquotient Theorem: *All irreps are subquotients of the principal series.*

Vogan's Refinement: *All irreps with minimal K-type σ occur in certain specific principal series.*

By Vogan's theorem, the lowest τ-K-type representations factor through the image of a generalized Harish-Chandra homomorphism

$$H(G, \tau) \longrightarrow U(\mathfrak{a}).$$

Theorem (A Higher Harish-Chandra Isomorphism)

The images of

$$H(G, \tau) \longrightarrow U(\mathfrak{a})$$

and the associated graded map

$$H(G_0, \tau) \longrightarrow S(\mathfrak{a})$$

are equal.

Proof.

One can use Harish-Chandra's method: apply a Chevalley-type restriction theorem and an intertwiner argument (Weyl character formula). □

Real Groups

(Work in progress with David Vogan.) The correct Harish-Chandra maps have noncommutative target, in general (related to non-uniqueness of minimal K-types). However . . .

Theorem (or "Theorem")

For each $\tau \in \widehat{K}$ *there exists*

$$H(G, \tau) \longrightarrow U(\mathfrak{a}_\tau)$$

(defined by Vogan) that classifies τ*-minimal* K*-type reps.*

The associated graded map classifies τ*-minimal* K*-type representations of* G_0*, and the images of these two maps are equal.*

The correspondence between irreducible representations of G *and* G_0 *so-determined is a well-defined bijection between the duals of* G *and* G_0*.*