K-Homology, Assembly and Rigidity Theorems for Relative Eta Invariants

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Outline

- K-homology, elliptic operators and C*-algebras.
- Geometric K-homology and index theory.
- The assembly map.
- Relative eta invariants.

Joint work with John Roe.

K-Homology and elliptic operators

- M = a smooth, closed manifold.
- D = a linear elliptic operator on M (partial differential, first order).

Atiyah's observation: D determines a class in the K-homology group $K_0(M)$ via the index map

$$\text{Index}\colon K^0(M) \longrightarrow \mathbb{Z}$$
$$[E] \mapsto \text{Index}(D_E)$$

. . . and the families index map

$$\text{Index}\colon K^0(M \times X) \to K^0(X).$$

K-homology and elliptic operators, continued

The point: K-homology conceptualizes many constructions in index theory.

Example: The index is induced from the map $\varepsilon : M \longrightarrow \text{pt}$.

Example: The index of a boundary operator is zero:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_1(W, \partial W) & \longrightarrow & K_0(\partial W) & \longrightarrow & K_0(W) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & K_1(\text{pt}, \text{pt}) & \longrightarrow & K_0(\text{pt}) & \underset{=}{\longrightarrow} & K_0(\text{pt}) & \longrightarrow & \cdots \end{array}$$

Example: From $\alpha \in K_0(\mathbb{R}^2)$ and $\beta \in K^0(\mathbb{R}^2)$ with $\alpha \cap \beta = 1$ one obtains Bott periodicity.

Analytic cycles for K-homology

Atiyah's question: Can K-homology be defined using elliptic operators?

Note: if D is an operator on M and $f\colon M \to N$, then we obtain

$$f_*[D] \in K_0(N)$$

A good definition should naturally account for this ... which the definition

$$K_0(M) = K^0(T^*M)$$

does not.

Analytic cycles for K-homology, continued

Atiyah's suggestion: An analytic cycle for $K_0(X)$ is a bounded *Fredholm* operator F on a Hilbert space H that is equipped with an action of $C(X)$ ... such that F is *pseudolocal:* $\varphi_1 F \varphi_2$ is a *compact operator*, if φ_1, φ_2 have disjoint supports.

The point: (1) For such an (F, H), the families index map

$$\text{Index}\colon K^0(X \times Y) \to K^0(Y).$$

is still defined.

(2) $F = D(I + D^* D)^{-1/2}$ = order zero ψDO is an example.

Analytic K-homology

Analytic cycles for the odd K-homology group

An analytic cycle for $K_1(X)$ is given by a Fredholm operator F, as before, which is now *self-adjoint*.

Compare: $\Big\{\text{Self-adjoint Fredholms}\Big\} \sim \Omega\Big(\text{Bott Spectrum}\Big)$.

Definition

$K_0^{analytic}(X)$ = Grothendieck group of homotopy classes of cycles.

Theorem (Brown, Douglas and Fillmore; Kasparov)

The index mappings identify $K_0^{analytic}(X)$ with the K-homology group $K_0(X)$.

C*-algebra K-theory and duality

Definition (for later use . . .)

$\mathcal{D}(X)$ = commutant modulo compact operators of $C(X)$ (acting on a single fixed Hilbert space H).

Now let $\mathcal{C}(X)$ = compact operators on H.

Theorem (for later use . . .)

$K_n^{analytic}(X) \cong K_{n-1}\big(\mathcal{D}(X)/\mathcal{C}(X)\big)$.

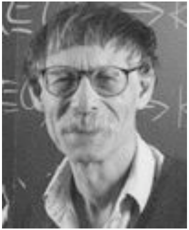
Explanation (n odd)

If P is a projection element in $\mathcal{D}(X)/\mathcal{C}(X)$ then the operator

$$F = 2P - I$$

lifts to an analytic cycle for $K_1^{\text{analytic}}(X)$.

Geometric K-homology



Paul Baum

Geometric cycle for $K_n(X)$: (M, E, f)

- M = spin^c-manifold ($n \equiv \dim(M)$).
- E = complex vector bundle on M.
- $f \colon M \to X$.

We can form $f_*([D] \cap [E]) \in K_n(X)$.

$\Rightarrow$ (M, E, f) determines a $K_n(X)$ class.

Equivalence relation on geometric cycles.

- Direct sum/disjoint union and Bordism.
- Bundle Modification. P = principal G-bundle over M. (S^{2k}, F, ε) = G-equivariant cycle over pt, Index = 1.

$$(M, E, f) \sim (M, E, f) \times_P (S^{2k}, F, \varepsilon)$$

The index theorem

$$
\begin{array}{ccc}
K_0^{\text{geom}}(X) & \xrightarrow{(M,E,f)\mapsto f_*[D_E]} & K_0^{\text{analytic}}(X) \\
\downarrow h_*^{\text{geom}} & \boxed{h\colon X \to Y} & \downarrow h_*^{\text{analytic}} \\
K_0^{\text{geom}}(Y) & \xrightarrow[(M,E,f)\mapsto f_*[D_E]]{} & K_0^{\text{analytic}}(Y)
\end{array}
$$

$$\boxed{\varepsilon\colon X \to \text{pt}}$$

- $\varepsilon_*^{\text{geom}} =$ topological index
- $\varepsilon_*^{\text{analytic}} =$ analytic index

Poincaré bundle

- π = (free) abelian group.
- $\widehat{\pi}$ = Pontrjagin dual.
- $f\colon X \to B\pi$.

The *Poincaré line bundle* over $X \times \widehat{\pi}$ is

$$P = \big(\widehat{X} \times \widehat{\pi} \times \mathbb{C}\big)\big/\pi,$$

where the action of π is

$$g \cdot (x, y, z) = (gx, y, \langle g, y\rangle z).$$

We obtain a so-called *dualizing class* $[P] \in K^0(X \times \widehat{\pi})$ and

$$K_n(X) \xrightarrow{\ \otimes [P]\ } K^n(\widehat{\pi}).$$

Mishchenko bundle

- π = any (torsion-free) group.
- $C^*\pi$ = group C^*-algebra.
- $f\colon X \to B\pi$.

The *Mishchenko line bundle* over X is the flat line bundle

$$M = (\widehat{X} \times C^*\pi)/\pi.$$

where the action of π is

$$g \cdot (x, y) = (gx, gy).$$

Its fibers are rank-one free right modules over $C^*\pi$.

We obtain a class $[M] \in K_0(C(X, C^*\pi))$ (C^*-algebra K-theory) and a "dualizing map"

$$K_n(X) \xrightarrow{\otimes [M]} K_n(C^*\pi).$$

The assembly map

Kasparov's *assembly map* is defined to be

$$K_n(B\pi) \xrightarrow{\mu = \otimes[M]} K_n(C^*\pi).$$

Key properties:

- If D = Dirac operator on M and $f\colon M \to B\pi$, and if M has postive scalar curvature, then

$$\textit{Lichnerowicz} \quad \Rightarrow \quad \mu(f_*[D]) = 0.$$

- If D = signature operator on M and $f\colon M \to B\pi$, then

$$\mu(f_*[D]) = \text{oriented homotopy invariant.}$$

The strong Novikov conjecture

Strong Novikov conjecture

The Kasparov assembly map

$$K_n(B\pi) \xrightarrow{\quad\mu\quad} K_n(C^*\pi)$$

is always rationally injective.

Consequences

- Gromov-Lawson-type results ... for example if M is aspherical and spin, it does not admit a p.s.c. metric.
- The Novikov conjecture.

Other assembly conjectures

Integral strong Novikov conjecture

If π is torsion-free, then the Kasparov assembly map is split injective.

(Generally false) isomorphism conjecture

If π is torsion-free, then the Kasparov assembly map is an isomorphism.

Baum-Connes isomorphism conjecture

If π is torsion-free, then the modified Kasparov assembly map

$$K_n(B\pi) \xrightarrow{\quad\mu\quad} K_n(C^*_\lambda \pi).$$

is an isomorphism.

A consequence of Baum-Connes

Baum-Connes for π (torsion-free) $\Rightarrow$ Spectrum(x) is connected, for every $x \in C^*_\lambda(\pi)$

$\Rightarrow$ Spectra of elliptic operators on π-covering spaces have discrete component structures

Why? (1) Components in spectra determine projection matrices over $C^*_\lambda(\pi)$ (functional calculus).

(2) Integrality of the Fredholm index.

The fiber of the assembly map

One can construct "structure groups" $S_n(\pi)$ ...

$$\cdots \longrightarrow S_1(\pi) \longrightarrow K_1(B\pi) \xrightarrow{\mu} K_1(C^*\pi) \longrightarrow S_0(\pi) \longrightarrow \cdots$$

- Close connections to surgery exact sequence.
- Obviously BC $\Leftrightarrow S_*(\pi) \equiv 0$.
- Some (limited) use in strong Novikov.
- Vanishing theorems in $K_n(C^*\pi)$ made concrete by constructions of elements in $S_n(\pi)$.

The analytic surgery exact sequence

Recall we previously defined:

- $H = L^2(X)$, say.
- $\mathcal{D}(X)$ = C^*-algebra of “pseudolocal operators” on H.

Now define, given $f\colon X \to B\pi$:

- $H_\pi = L^2(X, M)$
  $=$ sections of the Mishchenko line bundle
  $=$ a Hilbert $C^*(\pi)$-module
  $\cong L^2(U) \otimes C^*(\pi)$, locally.
- $\mathcal{D}_\pi(X) = C^*$-algebra of operators on H_π that are locally $T \otimes I$, with $T \in \mathcal{D}(X)$, modulo compact operators.

We get :

$$0 \longrightarrow \mathcal{C}_\pi(X) \longrightarrow \mathcal{D}_\pi(X) \longrightarrow \mathcal{D}(X)/\mathcal{C}(X) \longrightarrow 0$$

The analytic surgery exact sequence, continued

From

$$0 \longrightarrow \mathcal{C}_\pi(X) \longrightarrow \mathcal{D}_\pi(X) \longrightarrow \mathcal{D}(X)/\mathcal{C}(X) \longrightarrow 0$$

we get:

$$\begin{array}{ccccccccc} \longrightarrow & K_0(\mathcal{D}_\pi(X)) & \longrightarrow & K_0(\mathcal{D}(X)/\mathcal{C}(X)) & \longrightarrow & K_1(\mathcal{C}_\pi(X)) & \longrightarrow \\ & \Big\downarrow \overset{=}{def} & & \Big\downarrow \overset{=}{def} & & \Big\downarrow \cong & \\ \longrightarrow & S_1(\pi) & \longrightarrow & K_1(X) & \xrightarrow[\mu]{} & K_1(C^*(\pi)) & \longrightarrow \end{array}$$

Example of an analytic structure

Analytic structure = Element of $S_n(\pi)$.

- $M =$ spin-manifold (odd-dimensional) with p.s.c. metric g
- $f\colon M \to B\pi$.
- $D_\pi =$ Dirac coupled to the (flat) Mishchenko line bundle.

Lichnerowicz	$\Rightarrow$	$D_\pi^2 \gg 0$
	$\Rightarrow$	Spectrum(D_π) has gap at $\lambda = 0$.
	$\Rightarrow$	positive spectral projection P_π of D_π lies in $\mathcal{D}^*_\pi(X)$

Definition

$$[M, g] = [P_\pi] \in \mathcal{S}_1(\pi)$$

Eta invariants

- $D =$ self-adjoint elliptic operator on M (odd-dimensional).
- The *eta-function* of D is

$$\eta_D(s) = \sum_{\text{Spectrum of } D} \text{sign}(\lambda)|\lambda|^{-s}$$
$$= \Gamma\left(\tfrac{s+1}{2}\right)^{-1} \int_0^\infty \text{Trace}\left(D\, e^{-tD^2}\right) t^{\frac{s-1}{2}}\, dt.$$

Theorem

It is meromorphic in $s \in \mathbb{C}$ and regular at $s = 0$.

Definition

$\eta(D) = \eta_D(0) =$ "Trace(sign(D))".

Relative eta invariants

- $D =$ self-adjoint elliptic operator on M.
- $f\colon M \to B\pi$
- $\sigma_1, \sigma_2\colon \pi \to U(N)$.

We can form twisted operators D_1 and D_2 using σ_1 and σ_2.

Definition (Slightly simplified)

$$\begin{aligned}\rho(D, f, \sigma_1, \sigma_2) &= \text{“}\tfrac{1}{2}\operatorname{Tr}(\operatorname{sign}(D_1)) - \tfrac{1}{2}\operatorname{Tr}(\operatorname{sign}(D_2))\text{”} \\ &= \tfrac{1}{2}\eta(D_1) - \tfrac{1}{2}\eta(D_2) \\ &= \tfrac{1}{2}\Gamma\big(\tfrac{1}{2}\big)^{-1}\int_0^\infty \Big(\operatorname{Tr}\big(D_1\, e^{-tD_1^2}\big) - \operatorname{Tr}\big(D_2\, e^{-tD_2^2}\big)\Big)\, t^{-\frac{1}{2}}\, dt\end{aligned}$$

Notation

With σ_1 and σ_2 fixed, we'll write $\rho(D, f)$.

Atiyah-Patodi-Singer Theorem

Theorem (Atiyah-Patodi-Singer)

If (M, D) is the boundary of (W, Q), then

$$\text{APS-Index}(Q) = \int_W \text{local expression} \; - \; \tfrac{1}{2}\eta(D).$$

Corollary

If (M, D, f) is a boundary, then $\rho(D, f) \in \mathbb{Z}$.

Rigidity theorems for the relative eta invariant

- $M =$ closed, odd-dimensional spin-manifold with p.s.c. metric.
- $D =$ Dirac operator.
- $f\colon M \to B\pi$ and $\sigma_1, \sigma_2\colon \pi \to U(N)$.

Theorem (Keswani)

If π is torsion-free and the Kasparov assembly map is an isomorphism, then the relative eta invariants $\rho(D, f)$ vanish.

Similarly, the relative eta invariants for the signature operators on homotopy equivalent oriented manifolds are equal.

Remark

Related results have been investigated by Mathai, Weinberger, Chang, Schick, Piazza, . . .

A theorem of Weinberger

Here is a warm-up theorem, interesting in its own right.

- $M =$ closed, odd-dimensional spin-manifold with p.s.c. metric.
- $D =$ Dirac operator.
- $f\colon M \to B\pi$ and $\sigma_1, \sigma_2\colon \pi \to U(N)$.

Theorem (Weinberger)

The relative eta invariants for Dirac operators on p.s.c. spin manifolds are rational numbers.

Remark

There is a similar theorem for signature operators.

Proof using geometric K-homology

(1) For a geometric cycle (M, E, f) define $\rho(M, E, f) = \rho(D_E, f)$, for some choice of Dirac operator.

(2) By the APS theorem, different choices of Dirac, or indeed bordant cycles, will give the *same* relative eta invariant, modulo an integer.

(3) By an explicit calculation cycles equivalent via bundle modification have the *same* relative eta invariant (for suitable choices of Dirac operators).

(4) We obtain therefore an $\mathbb{R}/\mathbb{Z}$-index map

$$\text{Index}_{\sigma_1,\sigma_2} \colon K_1(B\pi) \longrightarrow \mathbb{R}/\mathbb{Z}.$$

Proof, continued

(5) If the assembly map

$$\mu\colon K_1(B\pi) \longrightarrow K_1(C^*(\pi))$$

was rationally injective, then it would follow that $[D]$ is torsion in the group $K_1(B\pi)$ — and hence that $\text{Index}_{\sigma_1,\sigma_2}[D]$ is torsion in the group $\mathbb{R}/\mathbb{Z}$.

(6) But we can replace π by any quotient group through which σ_1 and σ_2 factor — for instance by a *linear* group.

(7) The assembly map *is* rationally injective for a linear group.

QED

Traces and the surgery exact sequence

Recall the surgery exact sequence is constructed from:

$$0 \longrightarrow \mathcal{C}_\pi(X) \longrightarrow \mathcal{D}_\pi(X) \longrightarrow \mathcal{D}(X)/\mathcal{C}(X) \longrightarrow 0.$$

We want to complete the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_0(C^*\pi) & \longrightarrow & S_1(\pi) & \longrightarrow & K_1(B\pi) & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle \mathrm{Tr}_{\sigma_1} - \mathrm{Tr}_{\sigma_2}} & & \vdots\, {\color{red}?\ ?} & & \Big\downarrow {\scriptstyle \mathrm{Index}_{\sigma_1,\sigma_2}} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

$$\boxed{\mathrm{Tr}_{\sigma_1,\sigma_2} \colon S_1(\pi) \longrightarrow \mathbb{R}}$$

The relative trace map via analysis

- $S_1(\pi) = K_0(\mathcal{D}_\pi(X))$
- $T \in \mathcal{D}_\pi(X) \quad \Leftrightarrow \quad T = S \otimes I + K$ locally
- In more detail:
  (1) $T \colon L^2(X, M) \to L^2(X, M)$
  (2) $L^2(U, M) \cong L^2(U) \otimes C^*(\pi)$ for small $U \subseteq X$
  (3) $\phi S \psi$ compact if $\mathrm{Supp}(\phi) \cap \mathrm{Supp}(\psi) = \emptyset$.
  Also, K is compact.

Lemma

Replacing "compact" with "trace class" we obtain a dense subalgebra $\mathcal{D}_\pi^{tracial}(X)$ on which the formula

$$\mathrm{Tr}_{\sigma_1,\sigma_2}(T) = \mathrm{Tr}_{\sigma_1}(K) - \mathrm{Tr}_{\sigma_2}(K)$$

defines a trace, and hence a map

$$\mathrm{Tr}_{\sigma_1,\sigma_2} \colon K_0(\mathcal{D}_\pi^{tracial}(X)) \longrightarrow \mathbb{C}.$$

The relative trace map, continued

After (unfortunately considerable) additional work, we obtain from

$$\mathrm{Tr}_{\sigma_1,\sigma_2} \colon K_0(\mathcal{D}_\pi^{\text{tracial}}(X)) \longrightarrow \mathbb{C}$$

the required relative trace map on $K_0(\mathcal{D}_\pi(X))$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_0(C^*\pi) & \longrightarrow & S_1(\pi) & \longrightarrow & K_1(B\pi) & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle \mathrm{Tr}_{\sigma_1} - \mathrm{Tr}_{\sigma_2}} & & \Big\downarrow {\scriptstyle \mathrm{Tr}_{\sigma_1,\sigma_2}} & & \Big\downarrow {\scriptstyle \mathrm{Index}_{\sigma_1,\sigma_2}} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Evaluation of the relative trace on a structure

Lemma

Applied to the structure class of a p.s.c. spin manifold we recover the relative eta invariant of the Dirac operator:

$$\boxed{\operatorname{Tr}_{\sigma_1,\sigma_2}\big([M,g]\big) = \rho(D,f).}$$

If P_π = positive spectral projection of D_π, then

$$P_\pi = \tfrac{1}{2}(\operatorname{sign}(D_\pi) + I).$$

Therefore

$$\begin{aligned} \operatorname{Tr}_{\sigma_1,\sigma_2}(P) &= \text{“} \operatorname{Tr}_{\sigma_1}(P) - \operatorname{Tr}_{\sigma_1}(P) \\ &= \tfrac{1}{2}\operatorname{Tr}_{\sigma_1}(D_\pi) - \tfrac{1}{2}\operatorname{Tr}_{\sigma_2}(D_\pi)\text{”} = \rho(D,f). \end{aligned}$$

as required.

Proof of the main theorem, completed

- $D =$ Dirac on p.s.c. spin-manifold.
- $f\colon M \to B\pi$.

Theorem (Keswani)

If the Kasparov assembly map is an isomorphism, then the relative eta invariants of D are zero.

For if the assembly map is an isomorphism, then the structure group $S_1(\pi)$ is zero.