C*-Algebras and Group Representations

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Outline

Summary

Mackey pointed out an analogy between irreducible representations of a semisimple group and irreducible representations of its Cartan motion group.

Can C^*-algebras and noncommutative geometry cast new light on Mackey's analogy?

- Some more or less historical remarks.
- The Connes-Kasparov conjecture.
- The Mackey analogy.
- Results and perspectives.

Groups and Convolution Algebras

- G = a Lie group.
- $C_c^\infty(G)$ = *convolution algebra* of functions on G.

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g)\, dh.$$

Under favorable circumstances a group representation $\pi\colon G \to \operatorname{Aut}(V)$ induces an algebra representation:

$$\pi\colon C_c^\infty(G) \to \operatorname{End}(V)$$

$$\pi(f)v = \int_G f(g)\, \pi(g) v\, dg.$$

Under favorable circumstances the reverse is true as well.

...but why bother with the convolution algebra?

One answer: it is effective for compact groups.

- G = a compact Lie group.
- $\sigma\colon G \to \mathrm{Aut}(V)$ irreducible representation.
- $p(g) = \dim(\sigma)\,\mathrm{Trace}(\sigma(g^{-1}))$.

Theorem (Peter-Weyl)

If G is compact, then $C^\infty(G) \approx \oplus_\sigma \mathrm{End}(\sigma)$.

From this, Weyl was able to determine the irreducible representations of compact groups.

Group C*-Algebras

To study *unitary* representations we might consider this:

Definition

$C^*(G)$ = Completion of $C_c^\infty(G)$ in the norm $\|f\| = \sup_\pi \|\pi(f)\|$.

It is a *C^*-algebra*.

Theorem (Gelfand and Naimark)

If A is a commutative C^*-algebra, then $A \cong C_0(\widehat{A})$.

Corollary

For abelian groups one has $C^*(G) \cong C_0(\hat{G})$.

Topology on the Dual

The unitary dual of G may be topologized using convergence of *matrix coefficient functions:*

$$\phi(g) = \langle v, \pi(g)v \rangle.$$

This turns out to be the same as a *hull-kernel topology* on the dual of $C^*(G)$:

- J = ideal in $C^*(G)$.
- $\{\, \pi \mid \pi[J] \neq 0 \,\}$ = open subset of the dual.

Reduced Group C*-Algebra

We shall need the *reduced group C^*-algebra* later ...

Definition

The *reduced group C^*-algebra* $C^*_\lambda(G)$ is the image of $C^*(G)$ in the algebra of bounded operators on $L^2(G)$ under the left regular representation.

It is a quotient of $C^*(G)$. So its irreducible representations constitute a closed subset of the unitary dual, called the *reduced* or *tempered* dual $\widehat{G}_\lambda$.

The representations in the reduced dual are those that contribute to the regular representation.

The Role of C*-Algebras

- C^*-algebras ought to be useful when the topology of the dual plays an important role.

- There is a natural notion of Morita equivalence, and Morita equivalent C^*-algebras have equivalent categories of representations.
- Example: If G is compact, then

$$C^*(G) \approx \oplus \operatorname{End}(\sigma) \underset{\text{Morita}}{\sim} C_0(\widehat{G}).$$

So Morita equivalence can make clear the parameters describing representations.

Decomposition of Representations

Key Early Problem (from 1950 on): Plancherel Formula

Decompose the regular representation on $L^2(G)$ into irreducible representations.

Examples

- Compact groups: $f(e) = \sum_\sigma \dim(\sigma)\,\mathrm{Trace}(\sigma(f))$.
- Abelian groups: $f(e) = \int_{\widehat{G}} \hat{f}(\xi)\, d\xi$.

Two Problems

1. Is there, in fact, a natural decomposition at all?
2. If so, describe it explicitly.

Admissible Representations

Let K be a maximal compact subgroup of G.

Conjecture of Mautner

If G is a semisimple group (with finite center) then *every K-isotypical subspace of every irreducible unitary representation of G is finite dimensional.* In other words, every irreducible unitary representation of G is *admissible*.

Admissibility makes it possible to apply the direct integral *decomposition theory* of von Neumann, and so obtain an abstract Plancherel formula. This settles Problem 1.

Mautner's conjecture was proved by Harish-Chandra and Godement, in different ways . . .

Gelfand's Trick

The following gives a hint about proving Mautner's conjecture.

- G = semisimple group.
- K = maximal compact subgroup.

Lemma

The algebra generated by the K-bi-invariant functions on G is abelian, and so its irreducible representations are one-dimensional. □

Notice that the C^*-algebra acts on the space of K-fixed vectors in any unitary representation of G . . .

Godement's Approach to the Conjecture

- σ = irreducible representation of K.
- $p(k) = \dim(\sigma)\,\mathrm{Trace}(\sigma(k^{-1}))$.

Definition (of the local Hecke algebra associated with σ)

$A^*(G,\sigma) = C^*(G)^K \cap C^*(G)p$.

Lemma

There is an equivalence of categories

$$\left\{\begin{array}{l}\textit{Unitary rep'ns of G generated by}\\ \textit{their } \sigma\textit{-isotypical subspaces}\end{array}\right\} \approx \left\{\textit{Rep'ns of } A^*(G,\sigma)\right\}.$$

By studying finite-dimensional representations, Godement showed that $A^*(G,\sigma)$ satisfies suitable polynomial identities.

- $U(\mathfrak{g})$ = *universal enveloping algebra* of (the complexification of) $\mathfrak{g} = \mathrm{Lie}(G)$.

Definition (Harish-Chandra, Lepowsky, et al)

The *local Hecke algebra* associated to σ is

$$A(\mathfrak{g}, \sigma) = \big[U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathrm{End}(\sigma)\big]^K.$$

It is an algebra with multiplication given by the formula

$$(S_1 \otimes T_1)(S_2 \otimes T_2) = S_1 S_2 \otimes T_2 T_1.$$

Lemma

If G acts on W, then $A(\mathfrak{g}, \sigma)$ acts on $\mathrm{Hom}_K(\sigma, W)$ as follows:

$$(S \otimes T) \cdot L = SLT.$$

Harish-Chandra's Approach, Continued

The functor from representations of G to representations of $A(\mathfrak{g}, \sigma)$ is not an equivalence.But it has natural left and right adjoints and it induces a bijection

$$\left\{\begin{matrix}\text{Irreducible representations of } G \text{ with} \\ \text{non-zero } \sigma\text{-isotypical subspaces}\end{matrix}\right\} \approx \left\{\text{Irreducible representations of } A(\mathfrak{g}, \sigma)\right\}.$$

Theorem (Harish-Chandra)

The algebra

$$A(\mathfrak{g}, \sigma) = \left[U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \operatorname{End}(\sigma)\right]^K$$

is finitely generated as a module over the center of the enveloping algebra.

Infinitesimal *versus* Global

Harish-Chandra has obtained deep properties [of spherical functions] using Lie algebra methods extensively.

The author makes an attempt to derive some properties . . . without using the Lie algebra . . . but only a few of the results of Harish-Chandra are obtained and some of them in a significantly weaker form.

It is a very worthwhile problem to study spherical functions by quite general and purely integral methods, but whether the deeper properties can be so obtained remains an open question.

Mautner
Review of Godement's 1952 paper

Discrete Series

Definition

An irreducible unitary representation of G is a *discrete series representation* if it can be realized as a summand of the regular representation of G on $L^2(G)$.

Work of Harish-Chandra:

Parametrization of the discrete series.

Work of Langlands, Schmid, Parthasarathy, Atiyah

Geometric construction of the discrete series.

Comment:

The discrete series occur as isolated points in the reduced dual, and so should be accessible by C^*-algebra methods.

Let $M = G/K$. Under an orientation hypothesis on M, there is a *Dirac induction map*

$$\text{D-Ind}\colon \langle \text{representations of } K \rangle \longrightarrow \langle \text{representations of } G \rangle.$$

Given a representation of K on V, let

$$E = G \times_K (V \otimes S).$$

Here S is a "spinor representation" of K.

An equivariant Dirac operator D acts on the sections of E, and we define

$$\text{D-Ind}(V) = \text{Kernel}_{L^2}(D).$$

Theorem (Atiyah and Schmid)

The discrete series are parametrized by "nonsingular" irreducible V via Dirac induction.

Dirac induction gives rise to a map

$$\text{D-Ind}\colon R(K) \longrightarrow K(C^*_\lambda(G)).$$

It accounts for the representations of K that are “singular” in the theory of the discrete series . . .

Connes-Kasparov Conjecture

The Dirac induction map in C^*-algebra K-theory is an isomorphism.

This is now viewed as part of the *Baum-Connes conjecture*.

The Connes-Kasparov Conjecture

The Connes-Kasparov conjecture is now proved (twice).

Proof of Wassermann. Via representation theory (explicit computation of the reduced C^*-algebra and Dirac induction).

Proof of Lafforgue. Via KK-theory (using a sophisticated generalization of Bott periodicity).

Lafforgue's argument shows that the discrete series are parametrized by a subset of the nonsingular part of dual of K.

The index theorem and further representation theory show that the subset is all of the nonsingular part.

Contraction of a Lie Group

- G = Lie group.
- H = closed subgroup.

Definition

The *contraction of G along H* is the Lie group $H \ltimes Lie(G)/Lie(H)$.

The contraction of G along H is a first-order, or linear, approximation of G in a neighborhood of H.

Example

- $G = SO(3)$ (the rotation group of a sphere)
- $H = SO(2)$.

The contraction of G along H is the group $SO(2) \ltimes \mathbb{R}^2$ of rigid motions of the plane.

Example

- X = hyperbolic 3-space.
- G = (orientation preserving) isometries of $X = PSL(2, \mathbb{C})$.
- K = isotropy group of a point = $PSU(2) = SO(3)$.

At small scales the space X is nearly Euclidean.

The contraction of G along K is the group of rigid motions of 3-dimensional Euclidean space.

George Mackey, 1916-2006

Mackey Analogy

- G = semisimple Lie group (with finite center).
- K = maximal compact subgroup.
- G_0 = contraction of G along K.

“. . . the physical interpretation suggests that there ought to exist a ‘natural’ one to one correspondence between almost all of the unitary representations of G_0 and almost all the unitary representations of G—in spite of the rather different algebraic structures of these groups.”

George Mackey, 1975

Mackey Analogy, Continued

Mackey's description of the dual of $K \ltimes V$

$$\varphi \in \widehat{V},\, \tau \in \widehat{K}_\varphi \quad \longrightarrow \quad \mathrm{Ind}_{K_\varphi \ltimes V}^{K \ltimes V}(\tau \times \varphi) \in \widehat{K \ltimes V}$$

$$\widehat{K \ltimes V} \cong \Big(\bigsqcup_{\varphi \in \widehat{V}} \widehat{K}_\varphi \Big) / K$$

An example of the analogy

G = complex semsimple, $G = KB$, $B = MAN$.

$$\varphi \in \widehat{A},\, \sigma \in \widehat{M} \quad \longrightarrow \quad \mathrm{Ind}_B^G(\sigma \times \varphi) \in \widehat{G}_\lambda.$$

Typically $K_\varphi = M$. Note that $M \ltimes V$ is to B as $K \ltimes V$ is to G. Also

$$\mathrm{Ind}_B^G(\sigma \times \varphi) = \mathrm{Ind}_{M \ltimes V}^{K \ltimes V}(\sigma \times \varphi),$$

as representations of K.

Complex semisimple groups: conclusion

"Typical" tempered representations of G and of the contracted group $G_0 = K \ltimes V$ agree, both in parametrization and in basic form.

> *We have not yet ventured to formulate a precise conjecture . . . However we feel sure that some such result exists . . .*

Mackey, 1975

Smooth Family of Lie Groups

There is a smooth family of Lie groups interpolating between the groups G_0 and G.

- $G_t = G$ for all $t \neq 0$.
- If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then for any $k \in K$ and $X \in \mathfrak{p}$, the family

$$\begin{cases} 0 \mapsto (k, X) \\ t \mapsto k \exp(tX) \end{cases}$$

  is a smooth section.

Associated to the smooth family of Lie groups $\{G_t\}_{t\in\mathbb{R}}$ is a continuous field of group C^*-algebras $\{C^*_\lambda(G_t)\}_{t\in\mathbb{R}}$.

The Connes-Kasparov isomorphism can be reformulated using this continuous field as follows:

Theorem (Connes and Higson)

The continuous field of group C^*-algebras $\{C^*_\lambda(G_t)\}_{t\in\mathbb{R}}$ has constant K-theory.

In some sense, this makes the Mackey analogy precise.

Can this precise cohomological statement be reconciled with Mackey's representation-by-representation analogy?

Parameters for Representations

- G = complex semisimple group.
- $\widehat{G}_\lambda = (\widehat{M} \times \widehat{A}) / W$ (principal series construction).

- G_0 = contracted group.
- $\widehat{G}_0 = \left(\bigsqcup_{\varphi \in \widehat{V}} \widehat{K}_\varphi \right) / K$ (Mackey machine).

- Up to conjugacy one has $\varphi \in \widehat{A}$, and then

$$M \subseteq K_\phi \subseteq K.$$

Moreover K_φ is *connected*.

- By the Cartan-Weyl highest weight theory,

$$\widehat{G}_0 = \Big(\bigsqcup_{\varphi \in \widehat{V}} \widehat{K}_\varphi \Big) / K = \Big(\bigsqcup_{\varphi \in \widehat{A}} \widehat{M} / W_\varphi \Big) / W = (\widehat{M} \times \widehat{A}) / W = \widehat{G}_\lambda .$$

Topological Structure of the Duals

The bijection is not a homeomorphism and does not by itself explain the Connes-Kasparov isomorphism. However ...

- $A^*_\lambda(G,\sigma) \subseteq C^*_\lambda(G)$, local Hecke algebra.
- $\widehat{K}$ is ordered by highest weights.
- $J^*_\lambda(G,\sigma)$ = ideal in $C^*_\lambda(G)$ corresponding to representations with an isotypical summand indexed by some $\sigma' < \sigma$.

Theorem

The continuous field $\big\{ A^*_\lambda(G_t,\sigma) \big/ \big(A^*_\lambda(G_t,\sigma) \cap J^*_\lambda(G_t,\sigma) \big) \big\}$ is isomorphic to a constant field of commutative C^*-algebras.

Topological Structure: Interpretation

Theorem (again)

The continuous field $\left\{A^*_\lambda(G_t,\sigma)\big/\big(A^*_\lambda(G_t,\sigma)\cap J^*_\lambda(G_t,\sigma)\big)\right\}$ *is isomorphic to a constant field of commutative* C^**-algebras.*

The C^*-algebra $A^*_\lambda(G,\sigma)$ is Morita equivalent to the ideal in $C^*_\lambda(G)$ associated to representations that have a non-zero σ-isotypical summand.

Conclusions

1. The field $\{C^*_\lambda(G_t)\}$ is assembled from constant fields of commutative C^*-algebras by Morita equivalences, extensions, and direct limits.

2. The duals of G, complex semisimple, and its contraction G_0 are assembled from the same constituent pieces, consisting of representations with a given lowest K-type.

A second look at the algebraic local Hecke algebra:

$$A(\mathfrak{g}, \sigma) = \left[U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathrm{End}(\sigma) \right]^K.$$

This algebra is filtered by degree in $U(\mathfrak{g})$ and it is easy to compute the associated graded algebra:

$$\mathrm{gr}\, A(\mathfrak{g}, \sigma) = \left[S(\mathfrak{p}) \otimes \mathrm{End}(\sigma^{\diamond}) \right]^K.$$

This is precisely the local Hecke algebra $A(\mathfrak{g}_0, \sigma)$ for the contracted group G_0. It follows for example that

$$HP_*(A(\mathfrak{g}_0, \sigma)) \cong HP_*(A(\mathfrak{g}, \sigma)),$$

which is a sort of (not so interesting) algebraic version of the Connes-Kasparov isomorphism.

Complex Semisimple Groups

If G is complex semisimple, then

$$\mathfrak{g}_{\mathbb{C}} \cong \overline{\mathfrak{g}} \oplus \mathfrak{g} \cong \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}.$$

As a result, we get a simplified formula for the local Hecke algebra:

$$A(\mathfrak{g}, \sigma) = \left[U(\mathfrak{k}) \otimes \mathrm{End}(\sigma^{\diamond}) \right]^{K}.$$

Similarly, for the contracted group G_0 we get

$$A(\mathfrak{g}_0, \sigma) = \left[S(\mathfrak{k}) \otimes \mathrm{End}(\sigma^{\diamond}) \right]^{K}.$$

These are Kirillov's quantum and classical family algebras.

Langlands Classification

For each $\nu \in \mathrm{Hom}(MA, \mathbb{C}^\times)$ there is a not-necessarily unitary principal series representation

$$\mathrm{Ind}_B^G \nu$$

of the complex semisimple group G.

The lowest K-type of $\mathrm{Ind}_B^G \nu$ is the representation with highest weight $\nu|_M$. It has multiplicity one and therefore it determines a unique irreducible subquotient

$$\Lambda(\nu) \ll \mathrm{Ind}_B^G \nu.$$

Theorem (Zhelobenko)

The correspondence $\nu \leftrightarrow \Lambda(\nu)$ determines a bijection

$$\mathrm{Hom}(MA, \mathbb{C}^\times)\big/W \cong \big\langle \textit{nonunitary dual of } G \big\rangle.$$

Mackey Analogy for Nonunitary Representations

Let G be complex semisimple and let $G_0 = K \ltimes V$ be its contraction.

The parameters for the irreducible, not necessarily unitary representations of any semidirect product $K \ltimes V$ may determined using local Hecke algebras:

Theorem (Rader)

$$\langle \textit{nonunitary dual of } K \ltimes V \rangle \cong \Big(\bigsqcup_{\varphi \in \mathrm{Hom}(V, \mathbb{C}^\times)} \widehat{K}_\varphi \Big) \Big/ K.$$

Therefore as before, we obtain a Mackey bijection

$$\langle \text{nonunitary dual of } G_0 \rangle \cong \langle \text{nonunitary dual of } G \rangle.$$