Noncommutative Geometry

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Noncommutative Geometry

Alain Connes

What is Noncommutative Geometry?

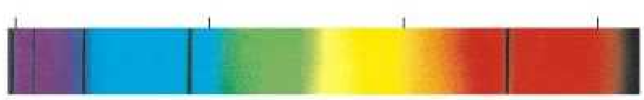
- Geometric spaces approached through their algebras of functions.
- The spaces are often very singular (defined by equivalence relations, or even groupoids).
- The function algebras are typically noncommutative.
- The algebras/spaces are analyzed using Hilbert space tools.
- In particular, spectral properties of algebras, viewed as algebras of operators on Hilbert space, are crucial. One might call the subject spectral geometry.

What are its Origins?

Werner Heisenberg

What Heisenberg understood . . . is that [the] Ritz-Rydberg combination principle actually dictates an algebraic formula for the product of any two observable physical quantities . . .

Emission spectrum of hydrogen



Absorption spectrum of hydrogen

Heisenberg wrote down the formula for the product of two observables;

$$(AB)_{(i,k)} = A_{(i,j)}B_{(j,k)}$$

and he noticed of course that this algebra is no longer commutative,

$$AB \neq BA.$$

...The right way to think about this new phenomenon is to think in terms of a new kind of space in which the coordinates do not commute. The starting point of noncommutative geometry is to take this new notion of space seriously.

Alain Connes
Noncommutative geometry, Year 2000

Commentary of Riemann

. . . it seems that the empirical notions on which the metric determinations of space are based . . . lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena.

Bernhard Riemann
On the Hypotheses which lie at the Foundations of Geometry

What Has Noncommutative Geometry Accomplished?

- Manifold topology (progress on the Novikov conjecture, Gromov-Lawson conjecture, etc).

- Harmonic analysis, especially of discrete groups.

- Models in physics (notably of the quantum Hall effect).

- Foliation theory and Atiyah-Singer index theory, on singular spaces, or parametrized by singular spaces.

- In addition, NCG may offer the prospect for progress in fundamental physics, arithmetic, . . .

Spectral Theory and Hilbert Space

David Hilbert in 1900

In the winter of 1900-1901 the Swedish mathematician Holmgren reported in Hilbert's seminar on Fredholm's first publications on integral equations, and it seems that Hilbert caught fire at once . . .

Hermann Weyl
David Hilbert and his mathematical work

Helmholtz Equation

Hilbert saw two things: (1) after having constructed Green's function K for a given region Ω and for the potential equation $\Delta u = 0$ . . . , the equation

$$\Delta\phi - \lambda\phi = 0$$

for the oscillating membrane changes into a homogeneous integral equation

$$\phi(s) - \lambda \int K(s,t)\phi(t)\,dt = 0$$

with the symmetric K, $K(t,s) = K(s,t)$ . . . ; (2) the problem of ascertaining the "eigen values" λ and "eigen functions" $\phi(s)$ of this integral equation is the analogue for integrals of the transformation of a quadratic form of n variables onto principal axes.

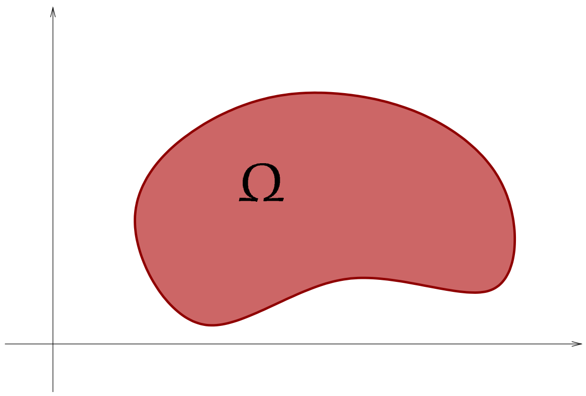
Hermann Weyl
David Hilbert and his mathematical work

Problem of H.A. Lorentz

. . . there is a mathematical problem which will perhaps arouse the interest of mathematicians . . . In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones of an organ pipe . . . there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its area.

H.A. Lorentz
Wolfskehl Lecture, 1910

Reformulation



$$\Delta u_n = \lambda_n u_n$$
$$u_n|_{\partial\Omega} = 0$$

$N(\lambda) = \#\Big\{$ eigenvalues of Δ less than or equal to $\lambda \Big\}$.

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{\text{Area}(\Omega)}{\text{constant}}.$$

This is equivalent to the asymptotic relation

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{\text{constant}}{\text{Area}(\Omega)}.$$

The idea was one of many, as they probably come to every young person preoccupied with science but while others soon burst like soap bubbles, this one soon led, as a short inspection showed, to the goal. I was myself rather taken aback by it as I had not believed myself capable of anything like it. Added to that was the fact that the result, although conjectured by physicists some time ago, appeared to most mathematicians as something whose proof was still far in the future.

Hermann Weyl
Gibbs Lecture, 1948

Compact Operators

Definition. A bounded linear operator $T\colon H \to H$ on a Hilbert space is *compact* if it maps the closed unit ball of Hilbert space to a (pre)compact set.

Example. If T is a norm-limit of finite-rank operators then T is compact.

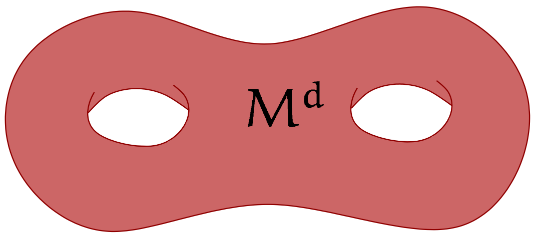
Elementary calculus $\Rightarrow$ the maximum value of the function $\varphi(v) = \|Tv\|^2$ on the closed unit ball of H is an eigenvalue for T^*T.

Theorem (Hilbert et al). *If $T\colon H \to H$ is a compact and selfadjoint operator then there is an orthonormal basis for H comprised of eigenvectors for T. Thus*

$$T \sim \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{pmatrix}. \qquad \square$$

Theorem (Rellich Lemma). *Δ^{-1} is a compact operator.* $\square$

Spectral Theory for the Laplacian



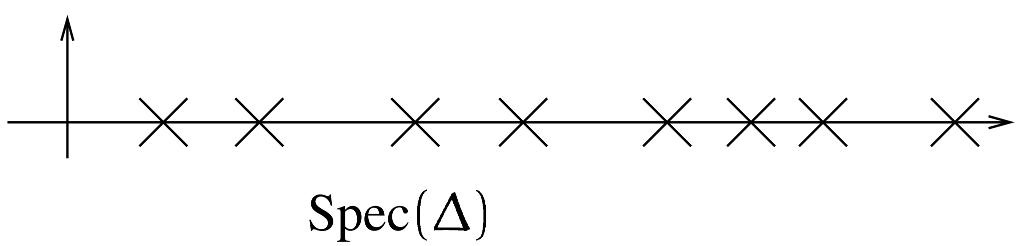
$$\Delta = \nabla^* \nabla$$

The Laplace operator.

Theorem. *There is an orthonormal basis for* $L^2(M)$ *consisting of functions* f_n *for which*

$$\Delta f_n = \lambda_n f_n$$

in the distributional sense. The eigenvalues λ_n *are positive and converge to infinity.* □



Remark. In fact one can show that $f_n \in C^\infty(M)$. This follows from elliptic regularity.

Singular Values

Definition. The *singular values* $\mu_1(T), \mu_2(T), \ldots$ of a bounded operator T are the scalars

$$\boxed{\mu_n(T) = \inf_{\dim(V)=n-1} \sup_{v \perp V} \frac{\|Tv\|}{\|v\|}.}$$

Observe that $\mu_1(T) \geq \mu_2(T) \geq \ldots$ and that

$$\text{T is compact} \quad \Leftrightarrow \quad \lim_{n\to\infty} \mu_n(T) = 0.$$

Now let T be compact, self-adjoint, and positive (meaning $\langle Tv, v \rangle \geq 0$). List the eigenvalues $\lambda_n(T)$ in decreasing order, and with multiplicity.

Theorem. *If* T *is compact, self-adjoint, and positive then* $\mu_n(T) = \lambda_n(T)$*.*

Proof. $$T = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{pmatrix} \qquad \square$$

Trace Class Operators

Lemma.

$$\mu_n(T_1+T_2) \leq \mu_n(T_1)+\mu_n(T_2) \leq \mu_{2n}(T_1+T_2).$$

$$\mu_n(ST), \mu_n(TS) \leq \|S\| \mu_n(T). \qquad \square$$

Definition. The *trace ideal* in $\mathcal{B}(H)$ is

$$\mathcal{L}^1(H) = \left\{ T \,|\, \sum \mu_n(T) < \infty \right\}.$$

Definition. If $T \in \mathcal{L}^1(H)$ then

$$\operatorname{Tr}(T) = \sum_{j=1}^{\infty} \langle v_j, T v_j \rangle.$$

The sum is over an orthonormal basis. Note: if $\{v_1, \ldots, v_N\}$ is an orthonormal set then

$$\sum_{n=1}^{N} |\langle v_n, T v_n \rangle| \leq \sum_{n=1}^{N} \mu_n(T).$$

As with the usual trace,

$$S \in \mathcal{B}(H),\ T \in \mathcal{L}^1(H) \quad \Rightarrow \quad \mathrm{Tr}(ST) = \mathrm{Tr}(TS).$$

Example. If k is smooth on $M \times M$ and if

$$Kf(x) = \int_M k(x,y) f(y)\, dy,$$

then K is a trace-class operator, and

$$\mathrm{Tr}(K) = \int_M k(x,x)\, dx.$$

Dixmier Trace

Definition. $\mathcal{L}^{1,\infty}(H) = \left\{ T \,\middle|\, \sup_n n \cdot \mu_n(T) < \infty \right\}$.

Observe that $\mathcal{L}^1(H) \subseteq \mathcal{L}^{1,\infty}(H) \subseteq \mathcal{B}(H)$.

Definition. If $T \in \mathcal{L}^{1,\infty}(H)$ is positive and LIM_ω is a Banach limit then define

$$\begin{aligned} \mathrm{Tr}_\omega(T) &= \mathrm{LIM}_\omega \frac{1}{\log(N)} \sum_{n \leq N} \mu_n(T) \\ &= \mathrm{LIM}_\omega \frac{1}{\log(N)} \sum_{n \leq N} \lambda_n(T). \end{aligned}$$

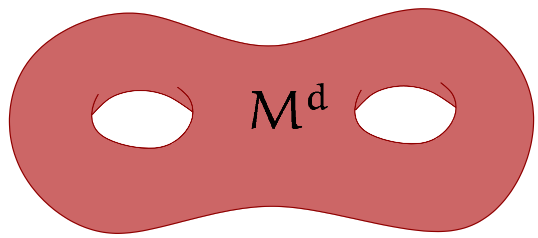
Theorem (Dixmier). *If* LIM_ω *has the property*

$$\mathrm{LIM}_\omega(\sigma_1, \sigma_2, \sigma_3, \ldots) = \mathrm{LIM}_\omega(\sigma_1, \sigma_1, \sigma_2, \sigma_2, \ldots)$$

then $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$. □

Integration

Now back to Weyl's Theorem . . .
$\lambda_n(\Delta) \sim c \cdot \mathrm{vol}(M) n^{\frac{2}{d}}$



Weyl's Theorem shows that

$$\mathrm{Tr}_\omega(\Delta^{-\frac{d}{2}}) = c \cdot \mathrm{vol}(M).$$

More generally, given $f \colon M \to \mathbb{C}$ we get

$$\mathrm{Tr}_\omega(f \cdot \Delta^{-\frac{d}{2}}) = c \cdot \int_M f \, d\mathrm{vol}\,.$$

This suggests that $\Delta^{-\frac{d}{2}}$ is some sort of 'volume element' for the manifold M . . .

Spectral Triples

Definition. A *spectral triple* is a triple $(\mathcal{A}, H, D)$ consisting of a separable Hilbert space H, an algebra $\mathcal{A}$ of bounded operators on H, and a (typically unbounded) selfadjoint operator D on H, for which:

- the operator $(I + D^2)^{-1}$ is compact, and
- if $a \in \mathcal{A}$ then the commutator $[D, a] = Da - aD$ extends to a bounded operator on H.

According to Connes, spectral triples constitute an extension of the notion of Riemannian geometric space which is broadly applicable to problems in fundamental physics, number theory, ...

Remark. If $\mathcal{A}$ has no unit, replace $(I + D^2)^{-1}$ with $a(I + D^2)^{-1}$ in the above.

The Standard Example

Basic ideas:

- Regard D as a 'square root' of Δ.
- Think of $[D, a]$ as a gradient of $a \in A$.

The simplest case is

$$\left(A = C_c^\infty(\mathbb{R}), \quad H = L^2(\mathbb{R}), \quad D = \frac{1}{\sqrt{-1}} \frac{d}{dx}.\right)$$

The theory of Dirac-type operators in geometry provides further 'commutative' examples in the context of Riemannian manifolds.

$D = d + d^*$	$D^2 = \nabla^* \nabla$	on forms
$D =$ Dirac Operator	$D^2 = \nabla^* \nabla$	on spinors

The Operator F

Write

$$\boxed{\Delta = D^2 \quad \text{and} \quad D = F\Delta^{\frac{1}{2}},}$$

so that

$$F = F^* \quad \text{and} \quad F^2 \sim I.$$

In the simplest case this is the Hilbert transform:

$$Ff(x) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy.$$

The operator F is important!

Roughly speaking the distinction between D and $\Delta^{\frac{1}{2}}$ corresponds to the distinction between densities and differential forms (on manifolds).

Groupoids and Quotients

Let $\mathbb{G}$ be a smooth étale groupoid:

$$\begin{array}{c} \mathbb{G} \\ \text{range} \downarrow\downarrow \text{source} \\ \mathrm{Obj}(\mathbb{G}) \end{array}$$

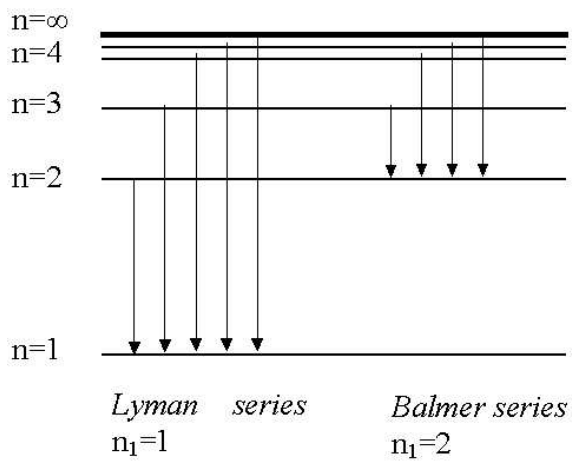
(source and range are local diffeomorphisms).

Let $A = C_c^\infty(\mathbb{G})$ and define

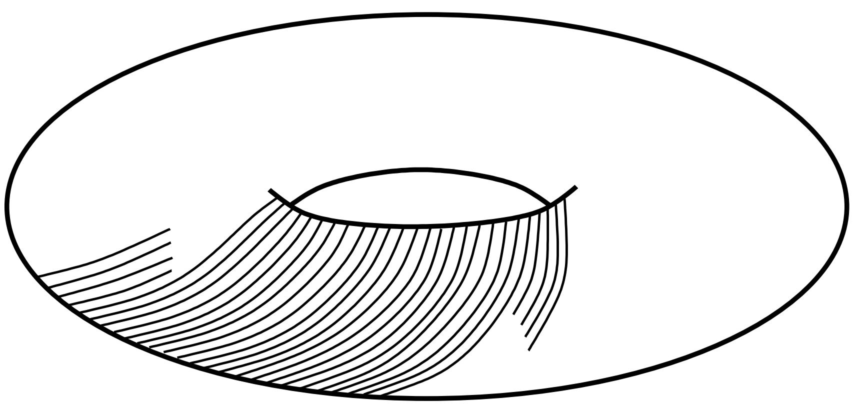
$$f_1 \star f_2(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f_1(\gamma_1)f_2(\gamma_2).$$

We obtain the *convolution algebra* of $\mathbb{G}$.

Examples



Heisenberg example, $A = M_\infty(\mathbb{C})$



Kronecker foliation, $A = \langle U, V | UV = e^{2\pi i\theta} VU \rangle$

Infinitesimals and Differentials

Definition. An infinitesimal of order k is a compact operator T for which $\mu_n(T) = O(n^{-k})$.

Definition. For $a \in A$ define $da = [F, a]$.

Example. In the basic case,

$$df = \text{operator with integral kernel } \frac{f(x) - f(y)}{x - y},$$

give or take a factor of πi.

If we grade differentials $a^0[F, a^1] \dots [F, a^p]$ according to degree and use the graded commutator then

$$d^2\omega \equiv 0,$$

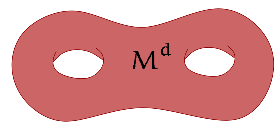
as in de Rham theory.

Connes' dictionary

Geometric space	*Spectral triple* (A, H, D)
Complex variable	*Operator in* A
Real variable	*Selfadjoint operator in* A
Infinitesimal	*Compact operator*
Differential da	*Commutator* $[F, a]$
Integral $\int a$	*Dixmier trace* $\mathrm{Tr}_\omega(a)$
⋮	⋮
⋮	⋮

Zeta Functions

Theorem. *If* $s > \frac{d}{2}$ *then* Δ^{-s} *is a trace-class operator.* □

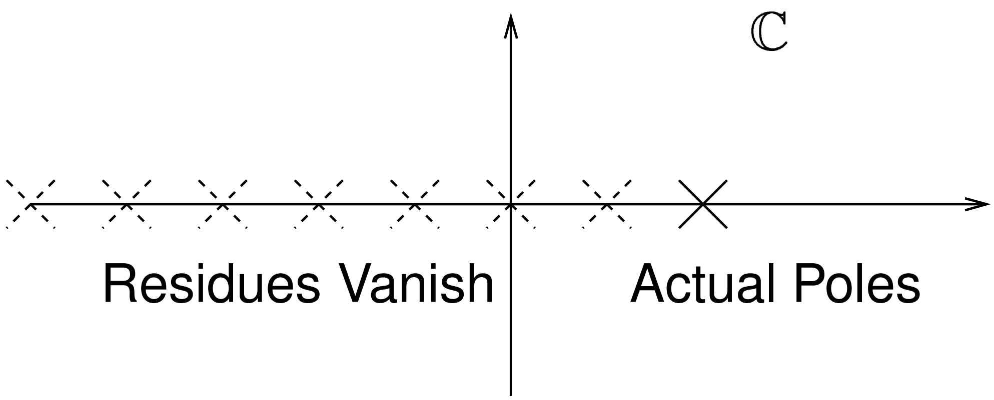


Proof. Follows from Rellich Lemma. □

Theorem (Minakshisundaram and Pleijel). *The zeta function*

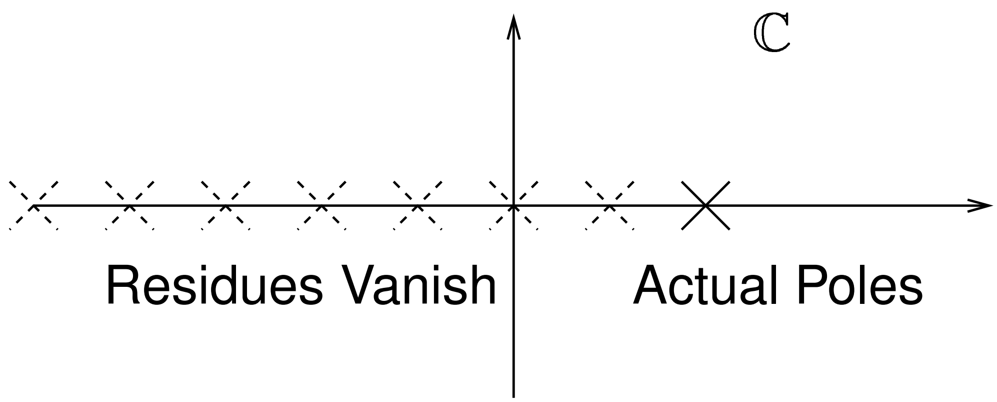
$$\zeta(s) = \mathrm{Tr}(\Delta^{-\frac{s}{2}})$$

is meromorphic on $\mathbb{C}$ *with only simple poles.* □



Singularities of $\zeta(s)$ for a closed surface.

Weyl's Theorem



Abelian-Tauberian Theorem. *Let* T *be a positive, invertible operator and assume that* T^{-s} *is trace-class for all* $s > 1$*. Then*

$$\lim_{\lambda\to\infty} \frac{N_T(\lambda)}{\lambda} = C \quad \Leftrightarrow \quad \lim_{s\searrow 1}\Big((s-1)\,\mathrm{Tr}(T^{-s}) \Big) = C.$$

See Hardy, *Divergent Series*. The theorem says

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \sum_{\lambda_n \le \lambda} 1 = C \quad \Leftrightarrow \quad \lim_{s\searrow 1}\Big((s-1) \sum_n \lambda_n^{-s} \Big) = C.$$

The proof of meromorphicity uses *pseudodifferential operators*

$$S \sim T\Delta^{-\frac{s+n}{2}}, \qquad T \text{ differential of order } n.$$

Lemma (Guillemin). *Suppose that for every holomorphic family* S_s *there are pseudodifferential* $U_1^{(i)}$*,* $V_s^{(i)}$ *and* R_{s-1} *such that*

$$(d+s)S_s = \sum_i [U_1^{(i)}, V_s^{(i)}] + R_{s-1}.$$

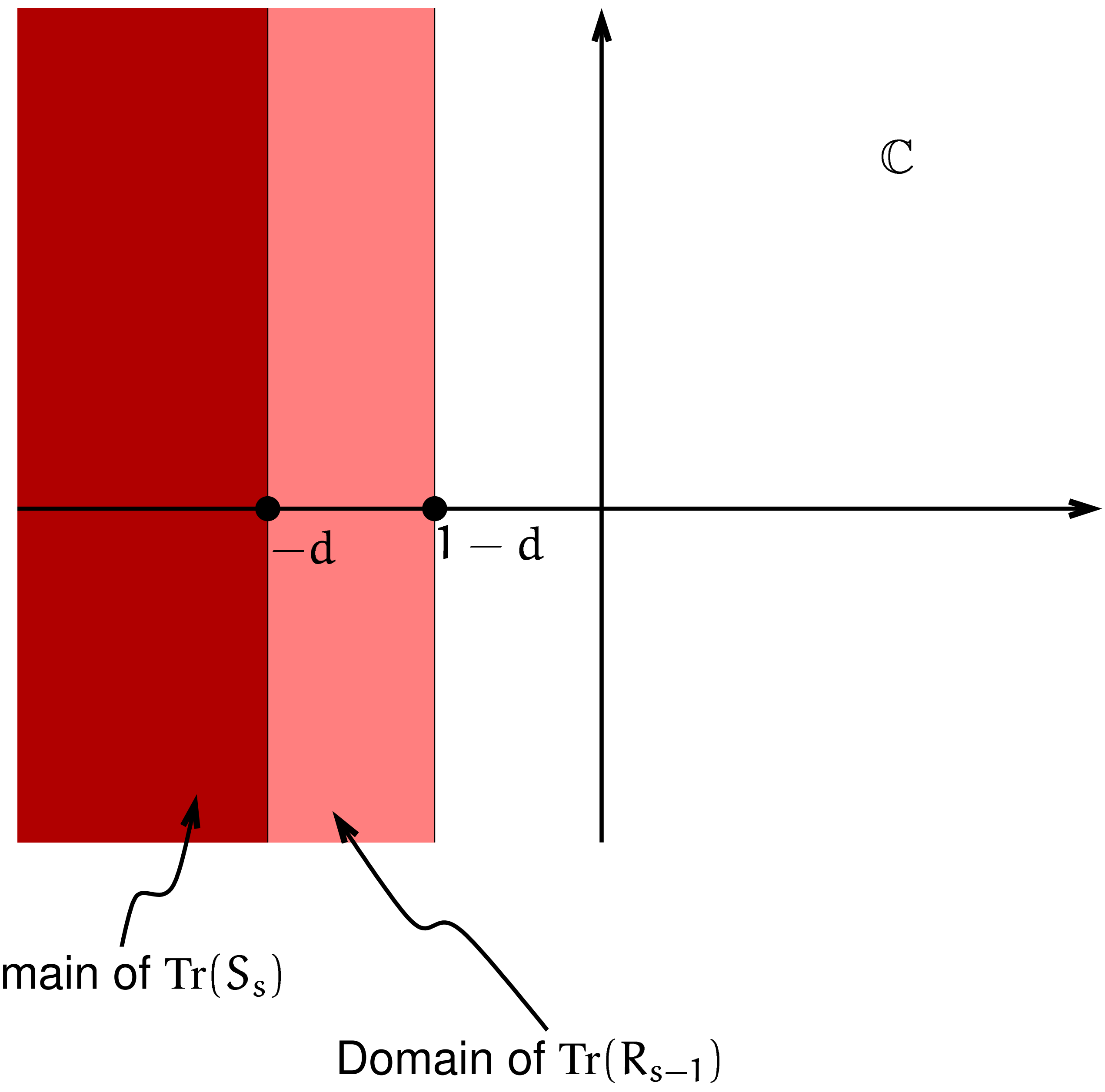
Then $\mathrm{Tr}(S_s)$ *is meromorphic, with simple poles.*

Proof. If $\mathrm{Re}(s) \ll 0$ then

$$\begin{aligned}\mathrm{Tr}(S_s) &= \frac{1}{d+s}\Big(\mathrm{Tr}(\textstyle\sum_i [U_1^{(i)}, V_s^{(i)}]) + \mathrm{Tr}(R_{s-1})\Big)\\ &= \frac{1}{d+s}\mathrm{Tr}(R_{s-1})\end{aligned}$$

Hence $\mathrm{Tr}(S_s) = (d+s)^{-1}\,\mathrm{Tr}(R_{s-1})$. □

The poles of $\mathrm{Tr}(S_s)$ are located at $-d, 1-d, 2-d, \ldots$.



Lemma. *If* D *is (pseudo)differential of order* n *then*

$$\sum_{i=1}^{d} [D, x_i] \frac{\partial}{\partial x_i} = nD - R,$$

where R *has order* $n - 1$*, and hence*

$$(d + n)D = \sum_{i=1}^{d} \left[D, x_i \frac{\partial}{\partial x_i} \right] - \sum_{i=1}^{d} \left[x_i D, \frac{\partial}{\partial x_i} \right] + R.$$

Proof. This follows from the Heisenberg relations

$$\left[\frac{\partial}{\partial x_i}, x_j \right] = \delta_{ij} I.$$

□

As a result, Weyl's Theorem follows from Guillemin's Lemma.

Cyclic Cocycles

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple.

Proposition. *The formula*

$$\varphi(a^0, a^1, \ldots, a^n) = \mathrm{Tr}(\varepsilon a^0 [F, a^1] \cdots [F, a^n])$$

defines a multlinear functional on $\mathcal{A}$ with the following properties:

- $\varphi(a^0, a^1, \ldots, a^n) = (-1)^n \varphi(a^1, \ldots, a^n, a^0)$
- $b\varphi(a^0, \ldots, a^{n+1}) = 0$, *where*

$$\begin{aligned} b\varphi(a^0, \ldots, a^{n+1}) = {} & \varphi(a^0 a^1, \ldots, a^{n+1}) \\ & - \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) \\ & + \ldots \\ & + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n). \end{aligned}$$

Cyclic Cohomology

Lemma. *Let φ be a cyclic n-linear functional. Then*

- $b\varphi$ *is a cyclic $(n+1)$-linear functional, and*
- $b^2\varphi = 0$. □

Definition. Let A be an algebra. The *nth cyclic cohomology group of A* is

$$HC^n(A) = \left\{ \begin{array}{l} \text{cyclic } n\text{-cocycles} \\ \quad \text{modulo cyclic coboundaries.} \end{array} \right\}$$

The cocycle φ is a sort of 'fundamental class' for the spectral triple (A, H, D). It reflects information from index theory:

$$\text{Index}(pFp) = (-1)^{\frac{n}{2}} \varphi(p, p, \ldots, p).$$

Continuation of the Dictionary

⋮	⋮
Differential da	*Commutator* $[F, a]$
de Rham theory	*Cyclic cohomology*
⋮	⋮

Local Index Formula

Theorem (Connes and Moscovici). *Let* (A, H, D) *be a spectral triple with* simple dimension spectrum*. The* local formula

$$\psi_p(a^0, \ldots, a^p) =$$

$$\sum_{k \geq 0} c_{p,k} \operatorname{Res}_{s=0} \operatorname{Tr}\left(\varepsilon a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} \Delta^{-\frac{p}{2} - |k| - s}\right),$$

where

$$c_{p,k} = \frac{(-1)^{|k|}}{k_1! \ldots k_p!} \cdot \frac{\Gamma(|k| + \frac{p}{2})}{(k_1 + 1)(k_2 + 2) \cdots (k_p + p)}$$

defines a cocycle which is cohomologous to the fundamental cocycle φ*.*

Notation. $T^{(k)} = \left[D^2, \left[D^2, \ldots [D^2, T] \ldots\right]\right]$.

Comments

- The formula is a starting point for Atiyah-Singer index theory in noncommutative geometry.
- In the classical case (Riemmanian manifolds) the residues are computable from the coefficients of $\mathcal{D}$ (Seeley, Wodzicki, et al).
- Small (smoothing operator) changes in $\mathcal{D}$ leave the index formula invariant.
- In the noncommutative world, *local* means *concentrated at ∞ in momentum space* (*c.f.* Fourier theory).