Dimensions and C*-Algebras

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Outline

Rings of Operators

John von Neumann

Definition

- $\pi : \Gamma \to U(H)$ a unitary representation of a group.
- Commutant:
  $\mathfrak{M} = \{ T \in B(H) \ : \ T\pi(g) = \pi(g)T \}$.

Commutant $=$ algebra of operators—a von Neumann algebra:

- $T \in \mathfrak{M} \quad \Rightarrow \quad T^* \in \mathfrak{M}$.
- $T_n \in \mathfrak{M} \,\&\, T_n v \to Tv, \forall v \ \Rightarrow \ T \in \mathfrak{M}$.

Projections and Subrepresentations

$$\mathfrak{M} = \{\, T \in B(H) \,:\, T\pi(g) = \pi(g)T, \forall g \in \Gamma \,\}$$

- Subrepresentations of π correspond to projections in $\mathfrak{M}$.
- Equivalence of subrepresentations corresponds to Murray-von Neumann equivalence of projections: $P \sim Q \;\Leftrightarrow\; P = UV,\, Q = VU$.

Definition

Denote by $\mathrm{Dim}(\mathfrak{M})$ the set of equivalence classes of projections—it is a partially ordered set

Factors and Isotypical Representations

Definition

A unitary representation $\pi \colon \Gamma \to U(H)$ is isotypical if any two subrepresentations have equivalent subsubrepresentations.

Theorem

Every representation decomposes canonically into isotypical representations.

Theorem

The commutants of isotypical representations are factors—von Neumann algebras with trivial centers.

Factors and Isotypical Representations

Type I Isotypical Representations

- In simple situations, an isotypical representation is necessarily a multiple of a single irreducible representation.
- The corresponding factor is isomorphic to the algebra of all bounded operators on a Hilbert space K, where the dimension of K is equal to the multiplicity of the irreducible representation.

Factors and Isotypical Representations

In general, isotypical representations can be far more complicated!

Example

- Regular representation of a free group F on $\ell^2(F)$.
- It can be decomposed into irreducible representations in two (indeed many) totally disjoint ways.
- The complete structure of this representation is unknown.

Murray-von Neumann Classification

Theorem

If $\mathfrak{M}$ is a factor, then the partially ordered set $\mathrm{Dim}(\mathfrak{M})$ *of equivalence classes of projections in $\mathfrak{M}$ is isomorphic one of:*

- $\{0, 1, 2, \ldots, N\}$ *Type I*
- $[0, K]$ *Type II*
- $\{0, \infty\}$ *Type III*

Remark

Possibly $N = \infty$ and $K = \infty$.

Examples?

- (X, μ): measure space.
- G: discrete group of transformations of X (preserving the class of μ)

Definition

$\mathfrak{M}(G, X)$ is the von Neumann algebra of operators on $L^2(G \times X)$ generated by:

- Multiplication operators M_f, $f \in L^\infty(X)$
- Unitary translation operators U_g, $g \in G$

such that

$$U_g M_f U_g^* = M_{g(f)}.$$

Group Measure Space Construction

Key Idea:

The unitary operators $U_g \in \mathfrak{M}(G, X)$ implement the action of G on the algebra of measurable functions on X.

Theorem

Assume the action of G on X is free. Then $\mathfrak{M}(G, X)$ is a factor if and only if the action of G on X is ergodic.

Theorem

Assume that the action of G on X is ergodic. Then $\mathfrak{M}(G, X)$ is of type I or II if and only if there is an invariant measure ν in the class of μ, in which case

$$\operatorname{Dim}\big(\mathfrak{M}(G, X)\big) \cong \big\{\nu(E) : E \subseteq X\big\}$$

Examples

$X = \mathbb{Z}$, $G = \mathbb{Z}$, action by translations. Type I.

$X = S^1$, $G = \mathbb{Z}$, action by irrational rotations. Type II.

$X = S^1$, $G = SL(2, \mathbb{Z})$, action by projective transformations of $S^1 \cong \mathbb{R}P^1$. Type III.

C*-Algebras

Definition

A C^*-algebra is a norm-closed $*$-algebra of operators on a Hilbert space.

- C^*-algebra: $T_n \in A \,\&\, \|T_n - T\| \to 0 \Rightarrow T \in A$.
- Von Neumann algebra: $T_n \in \mathfrak{M} \,\&\, T_n v \to Tv, \Rightarrow T \in \mathfrak{M}$.

- Abelian C^*-algebra: $C(X)$.
- Abelian von Neumann algebra: $L^\infty(X)$.

- Abelian group C^*-algebra: $C^*(G) = C(\widehat{G})$.

Crossed Product Algebras

- X: smooth manifold.
- G: discrete group of diffeomorphisms of X.

Definition

The crossed product algebra $C^*_\lambda(G, X)$ is the C^*-algebra of operators on $L^2(G \times X)$ generated by:

- Multiplication operators M_f, where f is smooth and compactly supported.
- Unitary translation operators U_g, $g \in G$

such that

$$U_g M_f U_g^* = M_{g(f)}.$$

Traces and Dimensions

Lemma

Assume that there is a G-invariant measure ν on X. The formula

$$\tau\colon \sum_{g\in G} M_{f_g} U_g \mapsto \int_X f_e(x)\, d\nu(x)$$

defines a trace on $C^*_\lambda(G,X)$:

$$\tau(xy) = \tau(yx), \quad \forall x, y \in C^*_\lambda(G,X).$$

- If p and q are equivalent projections, then $\tau(p) = \tau(q)$.
- We can ask: what is the set of numerical dimensions

$$\{\, \tau(p) \;:\; p \in C^*_\lambda(G,X) \,\}?$$

Traces and Dimensions

Improved Question

- Trace $\tau : A \to \mathbb{C}$
- What is

$$\mathrm{Dim}_\tau(A) = \left\{ \begin{matrix} \text{abelian group generated by} \\ \tau(p), \text{ where } p \in M_n(A) \end{matrix} \right\} ?$$

The Irrational Rotation C*-Algebra

Example

Let $X = S^1$, and let $G = \mathbb{Z}$ act on the circle by rotations through multiples of the angle $2\pi\theta$, where θ is irrational. The C^*-algebra $C^*_\lambda(G, X)$ is the irrational rotation algebra A_θ.

Theorem

$\mathrm{Dim}_\tau\big(A_\theta\big) = \mathbb{Z} + \theta\mathbb{Z}$.

Theorem (Rieffel)

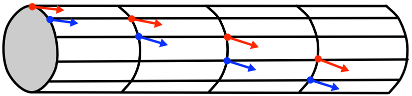
$\big\{\, \tau(p) \,:\, p \in A \,\big\} = \big(\mathbb{Z} + \theta\mathbb{Z}\big) \cap [0, 1]$.

Twisted Product

- M: Connected manifold with fundamental group G.
- X: Manifold with smooth action of G.

Definition

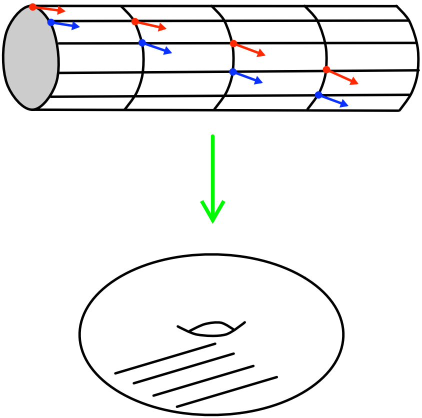
$$M\#X = (\widetilde{M} \times X)/G.$$



$\widetilde{M} \times X$

Twisted Product

$\widetilde{M} \times X$



$M \# X$

Ruelle-Sullivan Current

- M: Connected manifold with fundamental group G.
- X: Manifold with smooth action of G.
- ν: Invariant measure on X

Definition

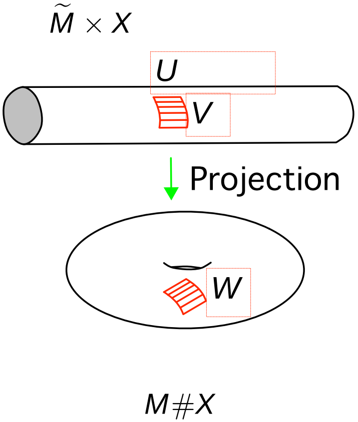
Let $n = \dim(M)$. Define $C_\nu \colon \Omega^n_c(M\#X) \to \mathbb{R}$ as follows:

- If ω is supported in a small set $W \subseteq M\#X$ diffeomorphic to $U \times V \subseteq \widetilde{M} \times X$,

$$C_\nu(\omega) = \int_V \left(\int_{U\times\{x\}} \omega \right) d\nu(x).$$

- For general ω, use a partition of unity.

Ruelle-Sullivan Current



Local Definition of Current

$$C_\nu(\omega) = \int_V \left(\int_{U \times \{x\}} \omega \right) d\nu(x).$$

Conjecture on Dimensions

Theorem

C_ν is closed: $C_\nu(d\eta) = 0$, for all $\eta \in \Omega_c^{n-1}(M \# X)$.

As a result, C_ν defines a map $C_\nu \colon H_c^n(M \# X) \to \mathbb{R}$.
We are interested in values on integral cohomology classes . . .

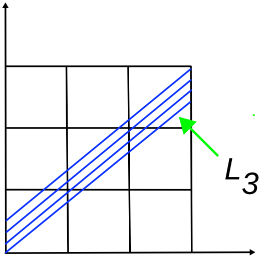
Conjecture (Approximate Form)

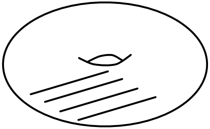
$$\mathrm{Dim}_\tau\big(C^*_\lambda(G, X)\big) = \{\, C_\nu(\omega) \; : \; \omega \text{ closed and integral} \,\}.$$

Irrational Rotations

Example — Irrational Rotation

$$\{ C_\nu(\omega) \ : \ \omega \text{ closed, integral} \} = \mathbb{Z} + \theta\mathbb{Z}$$





Alternate Definition of Current

$$C_\nu(\omega) = \lim_{N\to\infty} \frac{1}{N} \int_{L_N} \omega$$

Some Consequences

Corollary (of the Conjecture)

Take $X = pt$, so that $C^*_\lambda(G, X) = C^*_\lambda(G)$. Then $\mathrm{Dim}_\tau\big(C^*_\lambda(G)\big) = \mathbb{Z}$. As a result, $C^*_\lambda(G)$ has no nontrivial projections.

Corollary (of the Conjecture, due to Connes)

If the group $G = \mathbb{Z}$ acts on $X = S^3$ via a minimal diffeomorphism, then $\mathrm{Dim}_\tau\big(C^*_\lambda(G, X)\big) = \mathbb{Z}$. As a result, $C^*_\lambda(G, X)$ has no nontrivial projections.

Von Neumann Algebras, Revisited

Assume G acts on (X, μ). Define an action on $(X \times \mathbb{R}, \mu \times \text{Lebesgue})$ by

$$g\colon (x,t) \mapsto \Big(g \cdot x, \log \frac{dg_*\mu}{d\mu}(x) + t\Big).$$

Definition

The module of the given action of G on X is the pair $\operatorname{Mod}(G,X) = (\mathfrak{A}, \alpha)$, where

- $\mathfrak{A} =$ abelian von Neumann algebra $= L^\infty(X \times \mathbb{R})^G$
- $\alpha =$ action of $\mathbb{R} =$ translation action.

Theorem

The module is an invariant of $\mathfrak{M}(G,X)$.

Type III Factors

Assuming the action of G on X is ergodic, the module, $\mathrm{Mod}(G, X)$, is an ergodic flow.

μ G-invariant (of $\exists$ G-invariant measure in the class of μ) $\Rightarrow$ Action of G on $X \times \mathbb{R}$ is

$$g\colon (x, t) \mapsto (g \cdot x, t).$$

Conclusion: Invariant measure $\Rightarrow$ $\mathrm{Mod}(G, X)$ is $L^\infty(\mathbb{R})$, equipped with the standard translation action of $\mathbb{R}$.

Theorem

If $\mathrm{Mod}(G, X)$ may be equipped with a finite, $\mathbb{R}$-invariant measure, then $\mathfrak{M}(G, X)$ is a type III factor.

Finite Invariant Measure

- Z: a manifold with commuting smooth G and $\mathbb{R}$ actions.
- ν: a $G \times \mathbb{R}$-invariant measure on Z (typically infinite).
- $E \subseteq Z$: a G-invariant subset.
- $\nu_E(S) = \nu(E \cap S)$: a G-invariant measure on Z.
- $C_{\nu_E} : H^n_c(M \# Z) \to \mathbb{R}$: Ruelle-Sullivan current.

Lemma

Let $\omega \in H^n_c(M \# Z)$. The formula

$$\nu_\omega : E \mapsto C_{\nu_E}(\omega)$$

defines a finite, $\mathbb{R}$-invariant measure on $L^\infty(M \# Z)^G$.

Finite Invariant Measure

Lemma

Let $\omega \in H^n_c(M\#Z)$. *The formula*

$$\nu_\omega : E \mapsto C_{\nu_E}(\omega)$$

defines a finite, $\mathbb{R}$*-invariant measure on* $L^\infty(M\#Z)^G$.

Proof.

Since ν is $\mathbb{R}$-invariant,

$$\nu_\omega(t \cdot E) = C_{\nu_E}(t^*\omega)$$

By the homotopy invariance of cohomology, $\omega = t^*\omega$ in $H^n_c(M\#Z)$. Now use the fact that C_{ν_E} is a closed current. □

Godbillon-Vey Class

Theorem (Connes)

Assume that G acts ergodically on $X = S^1$ by orientation preserving diffeomorphisms. If the Godbillon-Vey class

$$GV \in H^3(M\#X)$$

is nonzero, then $\mathrm{Mod}(G, X)$ *has a finite* $\mathbb{R}$*-invariant measure.*

Proof.

The hypothesis $GV \neq 0$ implies, via Poincaré duality, that a suitable measure ν_ω is not only finite but nonzero. □

Remark

We take Z to be the space of 2-jets over X.

C*-Algebra K-Theory

Back to C^*-algebras . . .

A program which has been proposed several times suggests itself: since, in principle, a commutative C-algebra contains all possible information concerning its related compact Hausdorff space, it ought to be possible to extract topological information ring-theoretically. Nothing has yet come of this. Possibly the trouble is that the requisite constructions and calculations are beyond the resources of present-day ring theory.*

Irving Kaplansky, 1958

K-Theory

Definition

If A is a ring (with unit), then $K(A)$ is the abelian group generated by classes $[P]$, where P is any projection in a matrix algebra over A, subject to the relations

- $[P] = [Q]$ if P and Q are equivalent ($P = UV$ and $Q = VU$).
- $[P_1] + [P_2] = [\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}]$.

Thus $K(A)$ is the universal dimension group for projections in matrix algebras over A.

Bott Periodicity

Theorem (Bott)

If A is any C^*-algebra, then

$$\pi_j(GL_\infty(A)) = \pi_{j+2}(GL_\infty(A)),$$

for all j.

Notation

Denote by $Z \times A$ the C^*-algebra of continuous functions from Z into the C^*-algebra A, vanishing at infinity.

Equivalent Formulation

$$K(\mathbb{R}^{2k} \times A) \cong K(A).$$

Assembly

Notation

We shall write $K(Z)$ in place of $K(C_0(Z))$. This is Atiyah-Hirzebruch K-theory. Thanks to Bott Periodicity it is a generalized cohomology theory.

Assembly

Assume M has fundamental group G. There is a natural map

$$\mu\colon K(TM\#X) \to K(C^*_\lambda(G,X))$$

defined using Bott periodicity ...

Assembly

More Notation

If A is a C^*-algebra on which G acts via C^*-algebra automorphisms, let $M\#A$ be the C^*-algebra of continuous functions $f\colon \widetilde{M} \to A$ (vanishing at infinity in M) such that $f(g^{-1}\cdot x) = g \cdot f(x)$.

Example

- The algebra of functions on $M\#X$ is $M\#C_0(X)$.

Remarks

- If the action of G on A is trivial, then $M\#A = M \times A$.
- If the action is inner, then $K(M\#A) = K(M \times A)$.

Assembly

Definition

Assume M is an open subset of $\mathbb{R}^{2k}$. The assembly map

$$\mu\colon K(M\#X) \to K(C^*_\lambda(G,X))$$

is the composition

$$K\left(M\#X\right) \to K\left(M\#C^*_\lambda(G,X)\right) \to K\left(M \times C^*_\lambda(G,X)\right) \\ \to K\left(\mathbb{R}^{2k} \times C^*_\lambda(G,X)\right) \to K\left(C^*_\lambda(G,X)\right).$$

Using periodicity again one can construct the general assembly map

$$\mu\colon K(TM\#X) \to K(C^*_\lambda(G,X))$$

defined for any M (not necessarily an open subset of $\mathbb{R}^{2k}$).

Baum-Connes Conjecture

Baum and Connes

Conjecture

Assume that the universal cover of M is contractible. The assembly map

$$\mu \colon K(TM\#X) \to K(C^*_\lambda(G,X))$$

is an isomorphism of abelian groups.

Baum-Connes Conjecture - Status

Theorem (Higson and Kasparov)

The Baum-Connes conjecture holds for groups which admit a proper, isometric action on Hilbert space.

Corollary (Bekka, Cherix, Valette)

In particular, the conjecture holds for amenable groups.

Theorem (Yu, Tu)

The Baum-Connes assembly map is (split) injective for groups which uniformly embed into Hilbert space.

Corollary (Guentner, Higson, Weinberger)

In particular, the assembly map is injective for linear groups.

Baum-Connes Conjecture - Dimensions

Theorem

The diagram

$$\begin{array}{ccc} K(TM\#X) & \xrightarrow{\mu} & K(C^*_\lambda(G,X)) \\ \downarrow{\scriptstyle \mathrm{Ind}_\nu} & & \downarrow{\scriptstyle \tau_\nu} \\ \mathbb{R} & \xrightarrow[=]{} & \mathbb{R} \end{array},$$

where $\mathrm{Ind}_\nu(x) = C_\nu\big(\mathrm{ch}(x)\,\mathrm{Todd}(T_{\mathbb{C}}M)\big)$*, is commutative.*

Corollary (of the Baum-Connes Conjecture)

$$\mathrm{Dim}_\tau\big(C^*_\lambda(G,X)\big) = \big\{\, C_\nu\big(\mathrm{ch}(x)\,\mathrm{Todd}(T_{\mathbb{C}}M)\big) \;:\; x \in K(TM\#X) \,\big\}$$