On Connes' Godbillon-Vey Theorem

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Introduction

The following quite spectacular theorem relates the differential topology of a foliation (V, F), in the form of the Godbillon-Vey class, to the measure theory of (V, F), in the form on Connes' foliation von Neumann algebra.

Theorem (Connes, 1984)

Let (V, F) be an oriented and transversally oriented, codimension-one foliation.

If the Godbillon-Vey class of (V, F) is nonzero, then the von Neumann algebra of the foliation admits a nonzero invariant measure on its flow of weights.

In particular, the von Neumann algebra has a Type III component.

Contents

The goal is to prove the theorem using de Rham theory.

- Preliminaries on differential forms
- The von Neumann algebra of a foliation
- Geometry: jet spaces and characteristic classes
- Construction of invariant measures

Homotopy invariance of cohomology will imply the measures are invariant.

Poincaré duality will guarantee that at least one of the measures is nonzero.

Differential Forms

We shall need to work extensively with differential forms since the Godbillon-Vey class is defined using differential forms on V.

Differential forms are used to generalize the fundamental theorem of calculus to higher dimensions, as Stokes theorem.

- $\Omega^* =$ Graded-commutative algebra.
- $\Omega^0 = C^\infty(V)$.
- Ω^1 is dual to the $C^\infty(V)$-module of vector fields on V.
- There exists $d\colon \Omega^p \to \Omega^{p+1}$, obeying Leibniz, such that

$$\langle X, df \rangle = X(f)$$

  on functions.
- Ω^* is contravariantly functorial.

Differential Forms and Cohomology

One has $d^2 = 0$ and the *de Rham cohomology groups* of V are

$$H^p_{\mathrm{dR}}(V) = \mathrm{kernel}\big(d\colon \Omega^p \to \Omega^{p+1}\big)\Big/\mathrm{image}\big(d\colon \Omega^{p-1} \to \Omega^p\big)$$
$$= \text{closed forms}\Big/\text{exact forms.}$$

We shall use the Poincaré Lemma and Poincaré Duality:

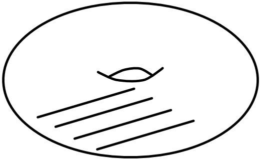
Theorem

If ν is a closed differential form on an oriented manifold V, and if the cohomology class of ν is nonzero, then there is a closed, compactly supported differential form ω such that

$$\int_V \nu \wedge \omega \neq 0.$$

Foliations and Differential Forms

V = Smooth manifold
F = Foliation of V



The *distribution* $F \subseteq TV$ is *integrable*, that is, $[F, F] \subseteq F$, and by Frobenius F determines the leaves of the foliation uniquely.

The foliation is also determined by the ideal of differential forms on V generated by the one-forms that vanish on F. Integrability of F implies that it is a *differential ideal.*

Flat Bundle Construction

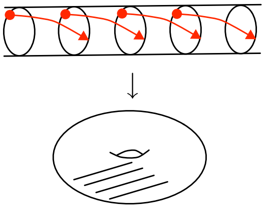
One source of foliations:

$$V = (T \times L)/\Gamma$$

Here:

- Γ acts freely and properly on L
- Γ acts on T in any manner
- F consists of the vectors tangent to L.

For example, the Kronecker foliation is of this type:



$L =$ real line
$T =$ circle
$\Gamma =$ integers

Codimension-One Foliations

We shall be concerned with *codimension-one* foliations of V, whose leaves have dimension one less than the dimension of V. Thus

$$\text{rank}(F) + 1 = \text{rank}(TV).$$

In the flat bundle examples, this means the transversal space T has dimension one.

The foliation is *transversally oriented* if there is a one-form ω on V such that

$$F = \text{kernel}(\omega).$$

The foliation is *oriented* if the leaves are consistently oriented, meaning that F is oriented.

Oriented + transversally oriented $\Rightarrow$ V is oriented.

The Godbillon-Vey Class

$(V, F) =$ Codimension-one, transversally oriented, oriented.
$\omega \quad =$ An associated one-form on V, as above.

Lemma

There is a one-form α on V such that $d\omega = \omega \wedge \alpha$.

This is the differential forms version of the fact that F is integrable.

Theorem (Godbillon-Vey)

The differential form $\alpha \wedge d\alpha$ is closed. Its cohomology class

$$\mathrm{GV} = [\alpha \wedge d\alpha] \in H^3_{\mathrm{dR}}(V)$$

is independent of ω and α.

$G = PSL(2, \mathbb{R})$
$K = PSO(2)$
$P =$ Upper triangular matrices in G

Iwasawa decomposition:

$$G = K \times P = G/P \times G/K = \mathbb{RP}^1 \times \mathbb{H}^2.$$

If $\Gamma =$ surface group, then we can carry out the flat bundle construction

$$V = G/\Gamma = (\mathbb{RP}^1 \times \mathbb{H}^2)/\Gamma.$$

This is a closed, oriented, foliated 3-manifold.

Proposition (Roussarie)

In this example, $\mathrm{GV} \neq 0$ *in* $H^3_{\mathrm{dR}}(V) \cong \mathbb{R}$*.*

The von Neumann Algebra of a Foliation

$(V, F) =$ Foliated manifold

$vN(V/F) =$ Connes' foliation von Neumann algebra

$vN(V/F) =$ algebra of measurable families of bounded operators T_v, parametrized by $v \in V$, where:

- T_v acts on the L^2-space of the leaf through v.
- $T_{v_1} = T_{v_2}$ if v_1 and v_2 lie on the same leaf.

Proposition

If $V = (T \times L)/\Gamma$ (and if Γ acts analytically), then $vN(V/F)$ is equivalent to the crossed product (group measure space construction) $vN(T/\Gamma)$ associated to the action of Γ on T.

KMS and the Flow of Weights

$M =$ von Neumann algebra

Theorem (Tomita, Takesaki, Connes)

The process

$$
\begin{array}{lcl}
\textit{Faithful normal state} & \mapsto & \textit{Modular automorphism group} \\
 & \mapsto & \textit{Crossed product by modular group} \\
 & \mapsto & \textit{Center with dual } \mathbb{R}\textit{-action}
\end{array}
$$

constructs an invariant of M.

The invariant, the *flow of weights*, is a flow on a measurable space.

The theorem is difficult to prove, but the invariant is easy to compute.

Crossed Product von Neumann Algebras

$\Gamma \times T \longrightarrow T$ Action on a measurable space.

μ Fully supported measure on T.

The measure μ gives a faithful normal state on $vN(T/\Gamma)$.

Define an action Γ on $T \times \mathbb{R}^+$ by

$$g\colon (t, x) \mapsto \left(g \cdot t, \frac{dg_*\mu}{d\mu}(t) \cdot x\right).$$

It commutes with the obvious multiplication action of $\mathbb{R}^+$.

Proposition

The flow of weights for the crossed product $vN(T/\Gamma)$ is the multiplication action of $\mathbb{R}^+$ on $L^\infty(T \times \mathbb{R}^+)^\Gamma$.

We identify $\mathbb{R}$ and $\mathbb{R}^+$ via the exponential.

The theorem we are trying to prove boils down to this:

Theorem (Connes)

Assume that Γ *acts on a circle or line* T *by orientation preserving diffeomorphisms.*

Assume also that for some oriented $V = (T \times L)/\Gamma$ *the Godbillon-Vey class* $\mathrm{GV} \in H^3_{\mathrm{dR}}(V)$ *is nonzero.*

Then there is a nonzero flow-invariant finite measure on the measurable space of Γ*-invariant subsets of* $T \times \mathbb{R}^+$*.*

$T =$ Smooth oriented manifold of dimension n

Definition

A *k-jet* at $t \in T$ is an equivalence class of local diffeomorphisms $\mathbb{R}^n \to T$ at t (mapping 0 to t).

Two local diffeos are *equivalent* if they agree to kth order at 0.

Definition

Denote T_k be the smooth bundle over T with fiber over $t \in T$ the k-jets at t.

- The group of orientation-preserving diffeomorphisms of T acts on T_k on the right, by composition.
- The group G_k of k-jets at $0 \in \mathbb{R}^n$ acts on the right, also by composition. This is a *finite-dimensional Lie group*, coordinatized by Taylor coefficients.

Thickenings and Gelfand-Fuks

Let $V = (T \times L)/\Gamma$, as usual.

Define $V_k = (T_k \times L)/\Gamma$.

- There is $V_k \longrightarrow V$. Fibers are copies of G_k.
- When $\dim(T) = 1$ the map is a homotopy equivalence.

Gelfand-Fuks: The process

Closed $\text{Diff}^+(T)$-invariant forms on T_k

$\mapsto$ Closed Γ-invariant forms on $T_k \times L$

$\mapsto$ Closed forms on $V_k = (T_k \times L)/\Gamma$

$\mapsto$ Classes in $H^*_{\text{dR}}(V_k)$

$\mapsto$ Classes in $H^*_{\text{dR}}(V)$

creates characteristic classes for foliations.

Codimension One

Assume $\dim(T) = 1$ and fix a (local) coordinate on T.

Then

$$T_1 = T \times G_1 = T \times \mathbb{R}^+$$

and $\mathrm{Diff}^+(T)$ acts by $\phi \cdot (t, x) = (\phi(t), \phi'(t)x)$. In addition

$$T_2 = T \times G_2,$$

where

$$G_2 = \left\{ \begin{pmatrix} x & y \\ 0 & x^2 \end{pmatrix} : x > o \right\}$$

and $\mathrm{Diff}^+(T)$ acts by

$$\phi \cdot \left(t, \begin{pmatrix} x & y \\ 0 & x^2 \end{pmatrix}\right) = \left(\phi(t), \begin{pmatrix} \phi'(t) & \phi''(t) \\ 0 & \phi'(t)^2 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & x^2 \end{pmatrix}\right).$$

Lemma

The differential form

$$\eta = \frac{1}{x^3} dt \wedge dx \wedge dy$$

on T_2 is closed and $\mathrm{Diff}^+(T)$-invariant.

The associated characteristic class is the Godbillon-Vey class

The Essential Point:

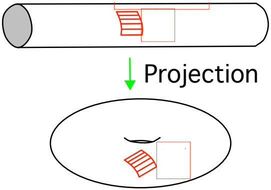
The Godbillon-Vey class corresponds to an invariant 3-form on the manifold T_2, and this manifold happens to be 3-dimensional.

As a result GV corresponds to an invariant measure.

Ruelle-Sullivan Current

$V = (T \times L)/\Gamma$ Oriented foliation (flat bundle construction).
$\boldsymbol{\nu} = \Gamma$-invariant measure on T.

Ruelle-Sullivan current: $C_{\boldsymbol{\nu}} \colon \Omega_c^{\dim(L)}(V) \longrightarrow \mathbb{R}$



Lift ω, supported on the chart in V, to $\widetilde{\omega}$ supported on the chart above.

- If ω is supported in a small chart in V, then

$$C_{\boldsymbol{\nu}}(\omega) = \int_T \left(\int_{\{t\}\times L} \widetilde{\omega} \right) d\nu(t).$$

- In general, use partitions of unity.

Cohomological Properties

Lemma

The Ruelle-Sullivan current

$$C_{\boldsymbol{\nu}} \colon \Omega_c^{\dim(L)}(V) \longrightarrow \mathbb{R}$$

is closed: $C_{\boldsymbol{\nu}}(d\eta) = 0$ *for all* η.

As a result, $C_{\boldsymbol{\nu}}$ defines a map

$$C_{\boldsymbol{\nu}} \colon H_{\mathrm{dR},c}^{\dim(L)}(V) \longrightarrow \mathbb{R}.$$

Example

If the invariant measure $\boldsymbol{\nu}$ is associated to a volume form ν on T, then

$$C_{\boldsymbol{\nu}}(\omega) = \int_V \nu \wedge \omega.$$

Some Simple Invariance Properties

Functoriality

$\boldsymbol{\nu}$ Γ-invariant measure on T, as before.

$f\colon T \to T$ Γ-equivariant diffeomorphism.

$$C_{f_*\boldsymbol{\nu}}(\omega) = C_{\boldsymbol{\nu}}(f^*\omega)$$

Restriction to an Invariant Set

E Γ-invariant subset of T.

$\boldsymbol{\nu}_E(S) := \boldsymbol{\nu}(E \cap S)$.

If $f_*\boldsymbol{\nu} = \boldsymbol{\nu}$, then

$$C_{\boldsymbol{\nu}_{f_*E}}(\omega) = C_{f_*\boldsymbol{\nu}_E}(\omega) = C_{\boldsymbol{\nu}_E}(f^*\omega).$$

Construction of Finite Invariant Measures

Ingredients

- The manifold T_2.
- The $\mathrm{Diff}^+(T)$-invariant volume form ν.
- A closed compactly supported form on $V_2 = \big(T_2 \times L\big)/\Gamma$.

Recipe

We want a measure on the Γ-invariant subsets of $T \times \mathbb{R}^+$.

Recall that $T \times \mathbb{R}^+ = T_1$.

Do this:

Γ-invariant set in $T_1 \quad \mapsto \quad$ Inverse image E in $T_2 \quad \mapsto \quad C_{\nu_E}(\omega)$

This is flow-invariant, by the homotopy invariance of cohomology.

Non-Vanishing and Poincaré Duality

Summary

For each closed, compactly supported ω on V_2, there is an invariant measure on the flow of weights T_1.

But are any of them non-zero?

The total measure of the space T_1 is $C_{\boldsymbol{\nu}}(\omega)$, and as we noted,

$$C_{\boldsymbol{\nu}}(\omega) = \int_{V_2} \nu \wedge \omega$$

But the cohomology class of ν is nonzero, so, according to Poincaré duality, the integral is nonzero for *some* ω.

This completes the proof of Connes' theorem.