Almost Homomorphisms and KK-Theory

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0. Introduction

The object of these notes is to introduce a notion of "almost homomorphism" for C^*-algebras, and explore its consequences for K-theory and KK-theory.

Roughly speaking an almost homomorphism from A to B is a continuous family of functions $\varphi_t: A \to B$ which asymptotically obey the axioms for $*$-homomorphisms between C^*-algebras. Our definition has the following important features:

(i) An almost homomorphism $\{\varphi_t\}: A \to B$ induces a homomorphism $\Phi_*: K_*(A) \to K_*(B)$ of K-theory groups.

(ii) Almost homomorphisms can be composed, at the level of homotopy, and the composition is consistent with the action on K-theory.

(iii) The homotopy classes of almost homomorphisms from $C_0(\mathbf{R}) \otimes A$ to $C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}$ form an abelian group, denoted $E(A, B)$. This "E-theory" is a bifunctor, and composition of almost homomorphisms gives rise to a product structure analagous to the Kasparov product in KK-theory. There is in fact a natural transformation $KK(A, B) \to E(A, B)$ preserving the product structure.

(iv) The natural transformation from KK-theory to E-theory is an isomorphism on the class of K-nuclear C^*-algebras introduced by Skandalis. However, arbitrary short exact sequences of C^*-algebras give rise to periodic six-term exact sequences in E-theory, in both variables: no nuclearity hypothesis is required. The new theory is the "universal" refinement of KK-theory with this excision property.

(v) An unbounded Kasparov (A, B)-bimodule induces in a natural way an almost homomorphism from $C_0(\mathbf{R}) \otimes A$ to $C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}$. Furthermore, the "assembly map" $K_*(C_0(TM)) \to K_*(C_r^*(\pi_1 M))$, appearing in the Strong Novikov Conjecture and the

Baum-Connes Conjecture, is induced by an almost homomorphism from $C_0(TM)$ to $C_r^*(\pi_1 M) \otimes \mathcal{K}$. Other examples appear to arise in the theory of almost flat vector bundles.

The present notes are devoted to setting up the general theory of almost homomorphisms: our goal will be the definition of E-theory and the characterization of the theory mentioned in item (iv). The examples mentioned above will be discussed briefly in Section 8, but the possible applications to the Novikov and Baum-Connes Conjectures will be considered elsewhere.

1. Almost homomorphisms

There are a number of variations of the basic definition, but the following seems to be the simplest and most natural.

Definition 1. *Let A and B be C^*-algebras. An* almost homomorphism *from A to B is a family of functions $\{\varphi_t\}_{t\in[1,\infty)}\colon A \to B$ such that:*

(i) the functions $t \mapsto \varphi_t(a)$ are continuous for every $a \in A$; and

(ii) for every $a, a' \in A$ and $\alpha \in \mathbf{C}$,

$$\lim_{t\to\infty} \left(\varphi_t(a) + \alpha\varphi_t(a') - \varphi_t(a + \alpha a')\right) = 0$$

$$\lim_{t\to\infty} \left(\varphi_t(a)\varphi_t(a') - \varphi_t(aa')\right) = 0$$

$$\lim_{t\to\infty} \left(\varphi_t(a)^* - \varphi_t(a^*)\right) = 0.$$

Notice that the continuity condition imposed is very weak (in particular, it is not required that each individual φ_t be continuous). However, we shall show in the next section that it is possible to tighten the continuity requirements without essentially altering the theory. In any case, there is the following "automatic continuity" property.

Lemma 1. *Let $\{\varphi_t\}: A \to B$ be an almost homomorphism. Then for every $a \in A$, $\limsup \|\varphi_t(a)\| \le \|a\|$.*

Proof. Adjoin units to A and B and extend φ_t to a function $\varphi_t: \tilde{A} \to \tilde{B}$ by setting $\varphi_t(a + \alpha 1) = \varphi_t(a) + \alpha 1$. If $u \in \tilde{A}$ is unitary then by Definition 1,

$$\lim_{t\to\infty} (\varphi_t(u)\varphi_t(u)^* - 1) = 0,$$

from which we see that $\lim_{t\to\infty} \|\varphi_t(u)\| = 1$. An arbitrary $a \in A$ with $\|a\| < 1$ can be expressed as a convex combination of a finite number of unitaries in $\tilde{A}$, say $a = \sum \alpha_i u_i$, and so

$$\limsup \|\varphi_t(a)\| \le \sum \alpha_i \lim_{t\to\infty} \|\varphi_t(u_i)\| = 1.$$

The result follows from this $\square$

Definition 2. *Let B be a C^*-algebra. Denote by $\mathcal{C}B$ the C^*-algebra of bounded continuous functions from $[1,\infty)$ to B; denote by $\mathcal{C}_0 B$ the ideal in $\mathcal{C}B$ consisting of functions which vanish at ∞; and denote by $\mathcal{Q}B$ the quotient algebra $\mathcal{C}B/\mathcal{C}_0 B$.*

It follows from Lemma 1 that an almost homomorphism $\{\varphi_t\}: A \to B$ determines a function $A \to \mathcal{C}B$, and it follows from the definition of almost homomorphism that by passing to the quotient $\mathcal{C}B/\mathcal{C}_0 B$, we obtain a $*$-homomorphism $\Phi: A \to \mathcal{Q}B$. Conversely, starting off with a $*$-homomorphism $\Phi: A \to \mathcal{Q}B$, by composing with any (set-theoretic) section $\mathcal{Q}B \to \mathcal{C}B$, and then with the evaluation maps $e_t: \mathcal{C}B \to B$ ($t \in [1,\infty)$), we obtain an almost homomorphism $\{\varphi_t\}: A \to B$. This is a one-to-one correspondence, modulo the following equivalence relation on almost homomorphisms.

Definition 3. *Two almost homomorphisms $\{\varphi_t\}$ and $\{\varphi'_t\}$ are* asymptotically equivalent *if $\lim_{t\to\infty} \|\varphi_t(a) - \varphi'_t(a)\| = 0$ for every $a \in A$.*

Often our almost homomorphisms will be defined only up to asymptotic equivalence. For example, suppose that $\mathcal{A}$ is a dense $*$-subalgebra of A and suppose that we are given a family of maps $\{\varphi_t\}_{t\in[1,\infty)}\colon \mathcal{A} \to B$ which satisfies the conditions of Definition 1, as well as the conclusion of Lemma 1. Then the family defines a continuous $*$-homomorphism from $\mathcal{A}$ to $\mathcal{Q}B$. This extends to A and so we obtain an almost homomorphism, up to asymptotic equivalence. In connection with this, the following observation is useful.

Lemma 2. *Let $\{\varphi_t\}$ and $\{\varphi'_t\}$ be two almost homomorphism such that $\lim_{t\to\infty} \|\varphi_t(a) - \varphi'_t(a)\| = 0$ for every element a in a dense subset of A. Then $\{\varphi_t\}$ and $\{\varphi'_t\}$ are asymptotically equivalent.*

Proof. Observe that $\{\varphi_t\}$ and $\{\varphi'_t\}$ determine the same homomorphism from A to $\mathcal{Q}B$ (by continuity of $*$-homomorphisms), and hence are asymptotically equivalent. □

2. The Homotopy Category of Almost Homomorphisms

Definition 1. *Two almost homomorphisms, $\{\varphi^i_t\}\colon A \to B$ $(i = 0, 1)$, are said to be* homotopic *if there exists an almost homomorphism $\{\varphi_t\}\colon A \to B[0,1]$ from which $\{\varphi^0_t\}$ and $\{\varphi^1_t\}$ are obtained by composition with evaluation at 0 and 1.*

Remark. In this definition, $B[0,1]$ denotes the continuous functions from $[0,1]$ to B. We note that $\mathcal{Q}(B[0,1])$ is *not* isomorphic to $(\mathcal{Q}B)[0,1]$, and so a homotopy of almost homomorphisms is not the same thing as a homotopy of the corresponding $*$-homomorphisms from A to $\mathcal{Q}B$. When we write expressions such as "$\mathcal{Q}B[0,1]$" we shall mean "$\mathcal{Q}(B[0,1])$," and so on.

Here are some examples of homotopies.

(i) Any two asymptotically equivalent almost homomorphisms are homotopic via the straight line path between them.

(ii) In particular, if $\lim_{t\to\infty} \|\varphi_t(a)\| = 0$ for every $a \in A$ then $\{\varphi_t\}$ is homotopic to the zero almost homomorphism.

(iii) If $r\colon [1,\infty) \to [1,\infty)$ is a continuous function such that $\lim_{t\to\infty} r(t) = \infty$ then $\{\varphi_t\}$ is homotopic to $\{\varphi_{r(t)}\}$.

Homotopy is an equivalence relation on the set of all almost homomorphisms A to B; we shall denote the set of equivalence classes by $[\![A, B]\!]$. Our goal in this section is to define composition of almost homomorphisms at the level of homotopy, and so organize the sets $[\![A, B]\!]$ into a category. The idea is simple enough: given $\{\varphi_t\}: A \to B$ and $\{\psi_t\}: B \to C$ we wish to reparameterize $\{\psi_t\}$ to obtain a composition of the form $\{\psi_{r(t)} \circ \varphi_t\}$ which satisfies the axioms for an almost homomorphism. However, there are one or two subtleties here (connected with proving the associativity of composition), and so we shall proceed with care.

For the rest of this article, all C^*-algebras A, B, C, etc, will be assumed to be separable.

Definition 2. *A* generating system *for a C^*-algebra A is an increasing family of compact sets $A_1 \subset A_2 \subset \cdots$ such that $A_k A_k \subset A_{k+1}$, $A_k + A_k \subset A_{k+1}$, $A_k^* \subset A_{k+1}$, and $\alpha A_k \subset A_{k+1}$ for $|\alpha| \leq 1$, and such that $\bigcup A_k$ is dense in A.*

Note that $\bigcup A_k$ is a $*$-subalgebra of A. It will be useful to have a notion of almost homomorphism defined on generating systems.

Definition 3. *Let $\{A_k\}$ be a generating system for A and let B be a C^*-algebra. A* uniform almost homomorphism *from $\{A_k\}$ to B (abbreviated* u.a.h.*) is a family of functions $\{\bar\varphi_t\}_{t\in[1,\infty)}: \bigcup A_k \to B$ such that:*

(i) for every $a \in \bigcup A_k$ the map $t \mapsto \bar\varphi_t(a)$ is continuous;

(ii) for every $k \in \mathbf{N}$ and every $\epsilon > 0$ there exists $T \in [1, \infty)$ such that

$$\|\bar\varphi_t(a)\bar\varphi_t(a') - \bar\varphi_t(aa')\| < \epsilon$$

$$\|\bar\varphi_t(a) + \alpha\bar\varphi_t(a') - \bar\varphi_t(a + \alpha a')\| < \epsilon$$

$$\|\bar\varphi_t(a)^* - \bar\varphi_t(a^*)\| < \epsilon$$

$$\|\bar\varphi_t(a)\| < \|a\| + \epsilon$$

for all $t > T$, $a, a' \in A_k$, and $|\alpha| \leq 1$; and *(iii) for every $k \in \mathbf{N}$, $\bar{\varphi}_t(a)$ is jointly continuous in $t \in [1, \infty)$ and $a \in A_k$.*

A uniform almost homomorphism $\{\bar{\varphi}_t\}: \{A_k\} \to B$ gives rise to a $*$-homomorphism from $\bigcup A_k$ to $\mathcal{Q}B$ which is continuous, and hence extends to a $*$-homomorphism $\Phi: A \to \mathcal{Q}B$. Therefore we obtain from $\{\bar{\varphi}_t\}$ an almost homomorphism $\{\varphi_t\}: A \to B$. We shall refer to this as the *underlying* almost homomorphism of C^*-algebras (strictly speaking, we should refer to the *underlying asymptotic equivalence class* determined by $\{\bar{\varphi}_t\}$). Conversely, we shall say that $\{\bar{\varphi}_t\}$ *represents* $\{\varphi_t\}$.

Lemma 1. *Let $\{\varphi_t\}: A \to B$ be an almost homomorphism and let $\{A_k\}$ be a generating system for A. There exists a u.a.h. $\{\bar{\varphi}_t\}: \{A_k\} \to B$ which represents $\{\varphi_t\}$. In fact, there exists an almost homomorphism $\{\bar{\varphi}_t\}: A \to B$ in the asymptotic equivalence class of $\{\varphi_t\}$ which consists of an equicontinuous family of functions.*

Proof. Pass from $\{\varphi_t\}$ to the corresponding $*$-homomorphism $\Phi: A \to \mathcal{Q}B$, and compose this with a continuous section $\mathcal{Q}B \to \mathcal{C}B$ (such a section exists by the Bartle-Graves Theorem). We obtain a map from A to $\mathcal{C}B$, and composing with the evaluation maps $e_t: \mathcal{C}B \to B$, $(t \in [1, \infty))$, we obtain an almost homomorphism $\{\bar{\varphi}_t\}$ which is asymptotically equivalent to $\{\varphi_t\}$, and which consists of an equicontinuous family of maps. This will restrict to a u.a.h. on any generating system. □

Definition 4. *Let $\{A_k\}$ and $\{B_n\}$ be generating systems for A and B. Two uniform almost homomorphisms $\{\bar{\varphi}_t\}: \{A_k\} \to B$ and $\{\bar{\psi}_t\}: \{B_n\} \to C$ are* composable *if for every $T \in [1, \infty)$ and every $k \in \mathbf{N}$ the subset $\{\bar{\varphi}_t(a) : a \in A_k \quad t \leq T\}$ of B is contained in some B_n.*

Lemma 2. *Suppose that $\{\bar{\varphi}_t\}$ and $\{\bar{\psi}_t\}$ are composable. There exists a continuous, increasing function $r: [1, \infty) \to [1, \infty)$ such that $\{\bar{\psi}_{s(t)} \circ \bar{\varphi}_t\}: \{A_k\} \to C$ is a u.a.h. for every continuous, increasing function $s \geq r$, and such that the estimates in Definition 3 hold uniformly for all families of the form $\{\bar{\psi}_{s(t)} \circ \bar{\varphi}_t\}$, where $s \geq r$.*

Proof. Choose an increasing sequence $1 < t_1 < t_2 < \cdots$, converging to infinity, such that if $t \geq t_k$ then

$$\|\bar{\varphi}_t(a)\bar{\varphi}_t(a') - \bar{\varphi}_t(aa')\| < 1/k$$

$$\|\bar{\varphi}_t(a) + \alpha\bar{\varphi}_t(a') - \bar{\varphi}_t(a + \alpha a')\| < 1/k$$

$$\|\bar{\varphi}_t(a)^* - \bar{\varphi}_t(a^*)\| < 1/k$$

$$\|\bar{\varphi}_t(a)\| < \|a\| + 1/k$$

for all $a, a' \in A_k$ and $|\alpha| \leq 1$. Choose $n_1 < n_2 < \ldots$ such that say $\{\varphi_t(a) : a \in A_{k+100}, \quad t \leq t_{k+1}\} \subset B_{n_k}$ (putting in the term "100" gives us a large margin of safety). Finally, choose $r_1 < r_2 < \cdots$ such that if $s \geq r_k$ then

$$\|\bar{\psi}_s(b)\bar{\psi}_s(b') - \bar{\psi}_s(bb')\| < 1/k$$

$$\|\bar{\psi}_s(b) + \beta\bar{\psi}_s(b') - \bar{\psi}_s(b + \beta b')\| < 1/k$$

$$\|\bar{\psi}_s(b)^* - \bar{\psi}_s(b^*)\| < 1/k$$

$$\|\bar{\psi}_s(b)\| < \|b\| + 1/k$$

for all $b, b' \in B_{n_k+100}$, $|\beta| \leq 1$. Any continuous, increasing function r such that $r(t_k) \geq r_k$ will suffice. □

We shall refer to any uniform almost homomorphism $\{\bar{\psi}_{r(t)} \circ \bar{\varphi}_t\}$ as in the Lemma as a *composition* of $\{\bar{\varphi}_t\}$ and $\{\bar{\psi}_t\}$.

Lemma 3. *The underlying C^*-algebra almost homomorphisms of any two compositions are homotopic.*

Proof. Let r and r' satisfy the conclusions of Lemma 2. It suffices to show that the underlying homomorphism of the compositions constructed from r and $r+r'$ are homotopic. The path of parametrizations $r+\lambda r'$ ($\lambda \in [0, 1]$) gives rise to a family of compositions which can be regarded as a u.a.h. from $\{A_k\}$ to $C[0,1]$. The underlying almost homomorphism from A to $C[0,1]$ is a homotopy as desired. □

Lemma 4. *Let $\{\varphi_t\}\colon A \to B$ and $\{\psi_t\}\colon B \to C$ be almost homomorphisms and let $\{A_k\}$ be a generating system for A. There exist composable almost homomorphisms $\{\bar{\varphi}_t\}\colon \{A_k\} \to B$ and $\{\bar{\psi}_t\}\colon \{B_n\} \to C$ representing $\{\varphi_t\}$ and $\{\psi_t\}$.*

Proof. By Lemma 1, there exists a u.a.h. $\{\bar{\varphi}_t\}\colon \{A_k\} \to B$ representing $\{\varphi_t\}$. Choose a generating system $\{B_n\}$ for B such that say $\{\bar{\varphi}_t(a) : a \in A_k \quad t \leq k\} \subset B_k$ for every k (observe that this is possible since the sets $\{\bar{\varphi}_t(a) : a \in A_k \quad t \leq k\}$ are compact). By Lemma 1 again, there exists an almost homomorphism $\{\bar{\psi}_t\}\colon \{B_n\} \to C$ which represents $\{\psi_t\}$. Then $\{\bar{\varphi}_t\}$ and $\{\bar{\psi}_t\}$ are composable. □

Lemma 5. *Let $\{\bar{\varphi}_t\}$, $\{\bar{\psi}_t\}$ and $\{\bar{\varphi}'_t\}$, $\{\bar{\psi}'_t\}$ be two composable pairs of uniform almost homomorphisms representing the same C^*-algebra almost homomorphisms, $\{\varphi_t\}\colon A \to B$ and $\{\psi_t\}\colon B \to C$. There exists a continuous, increasing function r such that for every $s \geq r$ both of the compositions $\{\bar{\psi}_{s(t)} \circ \bar{\varphi}_t\}$ and $\{\bar{\psi}'_{s(t)} \circ \bar{\varphi}'_t\}$ are defined (in the sense of Lemma 2) and both represent the same almost homomorphism of C^*-algebras.*

Proof. We may fix the pair $\{\bar{\varphi}'_t\}$, $\{\bar{\psi}'_t\}$ to be a particular choice of representing almost homomorphisms: we shall choose them, as in the proof of Lemma 1, to be defined and equicontinuous on all of A and B, respectively. Choose $1 < t_1 < t_2 \cdots$, converging to infinity, so that the first set of relations in the proof of Lemma 2 are satisfied for both $\{\bar{\varphi}_t\}$ and $\{\bar{\varphi}'_t\}$ (here we are considering the latter as being defined on $\{A_k\}$, the generating system on which $\{\bar{\varphi}_t\}$ is defined). We have that for every $b \in \bigcup B_n$,

$$\lim_{s\to\infty}(\bar{\psi}_s(b) - \bar{\psi}'_s(b)) = \lim_{s\to\infty}(\bar{\psi}_s(b) - \psi_s(b)) + \lim_{s\to\infty}(\psi_s(b) - \bar{\psi}'_s(b)) = 0$$

In fact, using a compactness argument, it is easily seen that there exist $r_1^{(1)} < r_2^{(1)} < r_3^{(1)} \cdots$ such that if $s \geq r_k^{(1)}$ then

$$\|\bar{\psi}_s(\bar{\varphi}_t(a)) - \bar{\psi}'_s(\bar{\varphi}_t(a))\| < 1/k$$

for all $a \in A_k$, $t \le t_k$. Let $\{B_n\}$ be the generating system on which $\{\bar\psi_t\}$ is defined, and choose $n_1 < n_2 < n_2 \cdots$ so that $\bar\varphi_t(a) \in B_{n_k}$ for all $a \in A_{k+100}$ and all $t \le t_{k+1}$. Choose $r_1^{(2)} < r_2^{(2)} < r_3^{(2)} < \cdots$ so that if $s \ge r_k^{(2)}$ then

$$\|\bar\psi_s(b)\bar\psi_s(b') - \bar\psi_s(bb')\| < 1/k$$

$$\|\bar\psi_s(b) + \beta\bar\psi_s(b') - \bar\psi_s(b + \beta b')\| < 1/k$$

$$\|\bar\psi_s(b)^* - \bar\psi_s(b^*)\| < 1/k$$

$$\|\bar\psi_s(b)\| < \|b\| + 1/k$$

for all $b, b' \in B_{n_k+100}$, $|\beta| \le 1$. Let $\{B'_k\}$ be a generating system for B such that B'_k contains the set of all elements of the form $\bar\varphi_t(a)$, $\bar\varphi'_t(a)$, or $\bar\varphi_t(a) - \bar\varphi'_t(a)$, where $a \in A_{k+100}$ and $t \le t_{k+1}$. Choose $r_1^{(3)} < r_2^{(3)} < r_3^{(3)} < \cdots$ so that if $s \ge r_k^{(3)}$ then

$$\|\bar\psi'_s(b)\bar\psi'_s(b') - \bar\psi'_s(bb')\| < 1/k$$

$$\|\bar\psi'_s(b) + \beta\bar\psi'_s(b') - \bar\psi'_s(b + \beta b')\| < 1/k$$

$$\|\bar\psi'_s(b)^* - \bar\psi'_s(b^*)\| < 1/k$$

$$\|\bar\psi'_s(b)\| < \|b\| + 1/k$$

for all $b, b' \in B'_k$, $|\beta| \le 1$. Writing $\bar\psi_s(\bar\varphi_t(a)) - \bar\psi'_s(\bar\varphi'_t(a))$ as

$$\begin{aligned}\bar\psi_s(\bar\varphi_t(a)) - \bar\psi'_s(\bar\varphi'_t(a)) &= \left\{\bar\psi_s(\bar\varphi_t(a)) - \bar\psi'_s(\bar\varphi_t(a))\right\} \\ &\quad + \left\{\bar\psi'_s(\bar\varphi_t(a)) - \bar\psi'_s(\bar\varphi'_t(a)) + \bar\psi'_s(\bar\varphi_t(a) - \bar\varphi'_t(a))\right\} \\ &\quad - \bar\psi'_s\left(\bar\varphi_t(a) - \bar\varphi'_t(a)\right)\end{aligned}$$

we see that it suffices to choose any continuous, increasing function r such that $r(t_k) \ge \max\{r_k^{(1)}, r_k^{(2)}, r_k^{(3)}\}$. □

Theorem 1. *Define a composition of two almost homomorphisms* $\{\varphi_t\}\colon A \to B$ *and* $\{\psi_t\}\colon B \to C$ *to be the underlying almost homomorphism of any composition of u.a.h.'s* *representing* $\{\varphi_t\}$ *and* $\{\psi_t\}$*. Then composition of almost homomorphisms passes to a well defined map* $[\![A,B]\!] \times [\![B,C]\!] \to [\![A,C]\!]$.

Proof. It follows from Lemma 5 that the homotopy class of a product does not depend on the choice of representing almost homomorphisms. It is then clear that it only depends on the asymptotic equivalence class of the almost homomorophisms involved (since equivalent almost homomorphisms have the same representatives). To see that it only depends on the homotopy class of the almost homomorphisms involved, form compositions of homotopies. (There is a small technical point that arises here: one must form an almost homomorphism $B[0,1] \to C[0,1]$, starting from an almost homomorphism $B \to C$. The obvious method of doing so requires that, for example, the almost homomorphism $B \to C$ be comprised of an equicontinuous family of maps. However, using the fact that composition is well defined on asymptotic equivalence classes, by Lemma 1 we may assume this extra condition.) □

Theorem 2. *The composition law* $[\![A,B]\!] \times [\![B,C]\!] \to [\![A,C]\!]$ *is associative.*

Proof. Given almost homomorphisms $\{\varphi_t\}: A \to B$, $\{\psi_t\}: B \to C$, and $\{\theta_t\}: C \to D$, by reparametrizing first $\{\varphi_t\}$, then $\{\psi_t\}$, and then $\{\theta_t\}$, and choosing generating systems appropriately, we can find representing uniform almost homomorphisms $\{\bar{\varphi}_t\}: \{A_k\} \to B$, $\{\bar{\psi}_t\}: \{B_k\} \to C$, and $\{\bar{\theta}_t\}: \{C_k\} \to D$, such that

$$\|\bar{\varphi}_t(a)\bar{\varphi}_t(a') - \bar{\varphi}_t(aa')\| < 1/k$$

$$\|\bar{\varphi}_t(a) + \alpha\bar{\varphi}_t(a') - \bar{\varphi}_t(a + \alpha a')\| < 1/k$$

$$\|\bar{\varphi}_t(a)^* - \bar{\varphi}_t(a^*)\| < 1/k$$

$$\|\bar{\varphi}_t(a)\| < \|a\| + 1/k$$

for $a, a' \in A_{k+100}$ and $t \leq k$; such that similar estimates hold for $\{\bar{\psi}_t\}$ on $\{B_k\}$, and $\{\bar{\theta}_t\}$ on $\{C_k\}$; and such that

$$\{\bar{\varphi}_t(a) : a \in A_{k+100}, \quad t \leq k\} \subset B_k$$

and

$$\{\bar{\psi}_t(b) : b \in B_{k+100}, \quad t \le k\} \subset C_k.$$

Then $\{\bar{\psi}_t \circ \bar{\varphi}_t\}$, $\{\bar{\theta}_t \circ \bar{\psi}_t\}$, and $\{\bar{\theta}_t \circ \bar{\psi}_t \circ \bar{\varphi}_t\}$ are all compositions, in the sense of Lemma 2, the latter being a composition of both of the pairs $(\{\bar{\theta}_t \circ \bar{\psi}_t\}, \{\bar{\varphi}_t\})$ and $(\{\bar{\theta}_t\}, \{\bar{\psi}_t \circ \bar{\varphi}_t\})$.$\square$

It is clear that the class of the identity almost homomorphism ($\varphi_t(a) \equiv a$) in $[\![A, A]\!]$ acts as an identity with respect to this composition law. Thus we obtain a category with objects the separable C^*-algebras and morphisms the homotopy classes of almost homomorphisms.

Definition 5. *Denote by* **A** *the category so obtained.*

We shall denote by $[\![f]\!]$ or $[\![\varphi]\!]$ the morphism in **A** corresponding to a $*$-homomorphism f or almost homomorphism $\{\varphi_t\}$.

Every $*$-homomorphism $f\colon A \to B$ determines an almost homomorphism: $f_t(a) \equiv f(a)$. This gives us a functor $\mathbf{C^*\text{-}Alg} \to \mathbf{A}$. Note that if one of the morphisms involved in a composition in **A** is the class of a $*$-homomorphism then the composition is equal to the class of the composition (in the obvious sense) of the almost homomorphism with that $*$-homomorphism.

3. Tensor Products

Let A and B be C^*-algebras. We shall denote by $A \otimes B$ the *maximal* tensor product of A and B. Recall that this is a completion of the algebraic tensor product $A \odot B$, and is characterized by the property that if $\pi\colon A \odot B \to C$ is any $*$-homomorphism of $A \odot B$ into a C^*-algebra C, such that $\|\pi(a \otimes b)\| \leq \|a\|\|b\|$ for every elementary tensor $a \otimes b$, then π extends to a $*$-homomorphism from $A \otimes B$ to C.

In all the appearances of the tensor product in these notes (apart from the generalities of this section) at least one of the two C^*-algebras involved will be nuclear, and so the maximal tensor product will agree with the spatial one.

Let $\{\varphi_t\}\colon A \to B$ and $\{\psi_t\}\colon C \to D$ be almost homomorphisms. Starting from these, define a map $\varphi \times \psi\colon A \times C \to \mathcal{C}(B \otimes D)$ by

$$\varphi \times \psi(a, c)(t) = \varphi_t(a) \otimes \psi_t(c).$$

Passing to the quotient $\mathcal{Q}B$, we obtain a $*$-homomorphism $A \odot C \to \mathcal{Q}(B \otimes D)$ and by the above characterization of the tensor product, this determines an almost homomorphism from $A \otimes C$ to $B \otimes D$ (up to asymptotic equivalence), which we shall denote by $\{\varphi_t \otimes \psi_t\}$.

Lemma 1. *The tensor product passes to a pairing* $[\![A, B]\!] \times [\![C, D]\!] \to [\![A \otimes C, B \otimes D]\!]$.

Proof. The proof is simply a matter of taking tensor products of homotopies (and making use of the "diagonal" $*$-homomorphism $C[0,1] \otimes C[0,1] \to C[0,1]$). $\square$

Given $f \in [\![A, B]\!]$ and $g \in [\![C, D]\!]$, we shall denote the "tensor product" by $f \otimes g \in [\![A \otimes C, B \otimes D]\!]$.

Lemma 2. *The tensor product construction satisifies the following identities:*

$$f \otimes g = (f \otimes 1)(1 \otimes g) = (1 \otimes g)(f \otimes 1)$$

$$(f_1 \otimes 1)(f_2 \otimes 1) = (f_1 f_2 \otimes 1)$$

$$(1 \otimes g_1)(1 \otimes g_2) = (1 \otimes g_1 g_2).$$

Thus the tensor product is a functor from $\mathbf{E} \times \mathbf{E}$ *to* $\mathbf{E}$*. It agrees with the usual tensor product on morphisms determined by* $*$*-homomorphisms.*

Proof. In order to compute the compositions, use generating systems of the form

$$\underbrace{A_k \otimes C_k + A_k \otimes C_k + \cdots A_k \otimes C_k}_{\leq 2^k \text{ times}}$$

and uniform almost homomorphisms of the form $\overline{\varphi_t \otimes \psi_t} = \bar{\varphi}_t \otimes \bar{\psi}_t$. The identities are then straightforward. □

4. Definition of E-theory

Definition 1. *Denote by* $\mathbf{E}$ *the category whose objects are the separable* C^**-algebras, and for which the the set of morphisms from* A *to* B *is*

$$E(A, B) = [\![C_0(\mathbf{R}) \otimes A \otimes \mathcal{K}, C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}]\!].$$

Composition of morphisms is given by composition of homotopy classes of almost homomorphisms.

Using the tensor product constructed in the previous section, we see that there is a natural functor, $f \mapsto 1 \otimes f$ from $\mathbf{A}$ to $\mathbf{E}$, and hence a natural functor from $\mathbf{C^*\text{-}Alg}$ to $\mathbf{E}$. We shall denote by $[f]$ or $[\varphi]$ the morphism determined by an $*$-homomorphism f or an almost homomorphism φ.

The most obvious feature of the functor $\mathbf{C^*\text{-}Alg} \to \mathbf{E}$ is that it is *stable*: the morphisms $A \to A \otimes \mathcal{K}$ in $\mathbf{C^*\text{-}Alg}$, given by $a \mapsto a \otimes e$, where e is a rank one projection, become isomorphisms in $\mathbf{E}$.

This stability is part of a slightly richer structure: denote by $M \colon \mathbf{A} \to \mathbf{A}$ the functor of "tensoring with $\mathcal{K}$":

$$\{A \xrightarrow{f} B\} \mapsto \{A \otimes \mathcal{K} \xrightarrow{f \otimes 1} B \otimes \mathcal{K}\}.$$

Fix a rank one projection $e \in \mathcal{K}$ and a $*$-isomorphism $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, and use these to define natural transformations $1_{\mathbf{A}} \to M$ and $M^2 \to M$ in the obvious way. The functor M, together with these natural transformations, forms what is known as a *monad* in the category $\mathbf{A}$ (see Mac Lane's book, *Categories for the Working Mathematician*, to which the reader is referred for a discussion of other category theory ideas mentioned below). In fact, this monad is *idempotent*: the natural transformation $M^2 \to M$ is an isomorphism. It follows from the abstract nonesense of monads that $\mathbf{E}$ can be defined in the following way: the objects are the separable C^*-algebras, as before, but the set of morphisms from A to B is now

$$\mathbf{E}(A, B) = [\![C_0(\mathbf{R}) \otimes A, C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}]\!].$$

Composition of say f and g is given by the the following composition of morphisms in $\mathbf{A}$:

$$C_0(\mathbf{R}) \otimes A \xrightarrow{f} C_0(\mathbf{R}) \otimes B \otimes \mathcal{K} \stackrel{g\otimes 1}{\longrightarrow} C_0(\mathbf{R}) \otimes C \otimes \mathcal{K} \otimes \mathcal{K} \cong C_0(\mathbf{R}) \otimes C \otimes \mathcal{K}.$$

(This is known as the *Kleisli construction.*) The maps $C_0(\mathbf{R}) \otimes A \to C_0(\mathbf{R}) \otimes A \otimes \mathcal{K}$ give an isomorphism from the old to the new description of $\mathbf{E}$.

We wish to exhibit an additive structure on the category $\mathbf{E}$. In order to reduce the number of details, we recall a few ideas from category theory.

Let $\mathbf{X}$ be a category which has finite products and a zero object. Let $A \times B$, together with the morphisms $\pi_A\colon A \times B \to A$ and $\pi_B\colon A \times B \to B$, be a product in $\mathbf{X}$. The pair of morphisms $1_A\colon A \to A$ and $0\colon A \to B$ determine (by the universal property of the product) a map $\sigma_A\colon A \to A \times B$. Similarly, we have a morphism $\sigma_B\colon B \to A \times B$. The product $(A \times B, \pi_A, \pi_B)$ is called a *biproduct* if the data $(A \times B, \sigma_A, \sigma_B)$ is a coproduct in $\mathbf{X}$. We say that $\mathbf{X}$ *has biproducts* if every product is in fact a biproduct.

Suppose that $\mathbf{X}$ has biproducts. Then we may add morphisms in $\mathbf{X}$ as follows: given $f_1, f_2\colon A \to B$, let $f_1 + f_2$ be the composition

$$A \stackrel{\Delta}{\longrightarrow} A \times A \stackrel{(f_1,f_2)}{\longrightarrow} B$$

(here Δ is the diagonal map, coming from the structure of $A \times A$ as a product; and (f_1, f_2) is induced from the maps $f_1, f_2 \colon A \to B$, and the structure of $A \times A$ as a coproduct). This makes each morphism set $\mathbf{X}(A, B)$ into an abelian monoid (with the zero morphism being the zero element) in such a way that composition is bilinear.

If, for each object A in $\mathbf{X}$ there is a morphism $i \colon A \to A$ such that $i + 1 = 0$ then the monoids $\mathbf{X}(A, B)$ are in fact abelian groups, and $\mathbf{X}$ is an additive category.

Lemma 1. *The category* $\mathbf{E}$ *has products and a zero object. In fact the functor* $\mathbf{C^*\text{-}Alg} \to \mathbf{E}$ *preserves these structures.*

Proof. Straightforward. □

Recall that the product in $\mathbf{C^*\text{-}Alg}$ is denoted (somewhat perversely) $A \oplus B$.

Lemma 2. *The category* $\mathbf{E}$ *has biproducts.*

Proof. It is easily verified that the morphisms σ_A and σ_B are given by the natural inclusions of A and B into $A \oplus B$. We can use the fact that $\mathcal{K} \cong M_2(\mathcal{K})$ to show that $A \oplus B$ is a coproduct. Given a pair of almost homomorphisms

$$\{\varphi_t\} \colon C_0(\mathbf{R}) \otimes A \otimes \mathcal{K} \to C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}$$

and

$$\{\psi_t\} \colon C_0(\mathbf{R}) \otimes B \otimes \mathcal{K} \to C_0(\mathbf{R}) \otimes B \otimes \mathcal{K},$$

define an almost homomorphism $\{\theta_t\} \colon A \oplus B \colon M_2(C_0(\mathbf{R}) \otimes B \otimes \mathcal{K})$ by

$$\theta_t(a \oplus b) = \begin{pmatrix} \varphi_t(a) & 0 \\ 0 & \psi_t(b) \end{pmatrix}$$

It is easily verified that $[\theta] \colon A \oplus B \to D \otimes M_2 \cong D$ is the unique morphism in $\mathbf{E}$ such that $[\theta]\sigma_A = [\varphi]$ and $[\theta]\sigma_B = \psi$. □

The addition in $E(A, B)$ determined by the biproduct structure on $\mathbf{E}$ is the usual notion of "direct sum".

Lemma 2. *Denote by* $i = i \otimes 1 \otimes 1: C_0(\mathbf{R}) \otimes A \otimes \mathcal{K} \to C_0(\mathbf{R}) \otimes A \otimes \mathcal{K}$ *the* $*$*-automorphism given by reversing* $\mathbf{R}$. *Then* $1 + [i] = 0$ *in* $E(A, A)$.

Proof. As is well-known, the $*$-homomorphism $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}: C_0(\mathbf{R}) \to C_0(\mathbf{R}) \otimes M_2$ may be connected to zero through $*$-homomorphisms. □

5. The Difference Construction

Let

$$0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$$

be a short exact sequence of C^*-algebras which is split by some $*$-homomorphism $s: A/I \to A$. We shall construct a morphism $\rho: A \to I$ in $\mathbf{E}$ such that:

$$\rho[i] = 1_I, \quad [i]\rho + [s][p] = 1_A \quad \sigma\pi = 1_{A/I}. \tag{1}$$

Let $\{u_t\}_{t \in [1,\infty)}$ be a quasicentral approximate unit for $I \triangleleft A$. To be precise, we mean that:

(i) $u_t \in I$ and $0 \leq u_t \leq 1$;

(ii) $t \mapsto u_t$ is norm continuous and increasing:

(iii) $u_t a - a \to 0$ for every $a \in I$; and

(iv) $u_t a - a u_t \to 0$ for every $a \in A$.

(We shall not necessarily use all these properties all the time.) For $t \in [1, \infty)$ let

$$D_t = \begin{pmatrix} \sin(x) & \cos(x)(1-u_t) \\ \cos(x)(1-u_t) & -\sin(x) \end{pmatrix} \in C[-\pi/2, \pi/2] \otimes \tilde{I} \otimes M_2.$$

Note that $t \mapsto D_t$ is a norm continuous family of self-adjoint elements, and that $\|D_t\| \leq 1$ for all $t \in [1, \infty)$.

Lemma 1. *If* $f \in C_0(-1, 1)$ *then* $f(D_t) \in C_0(-\pi/2, \pi/2) \otimes I \otimes M_2$.

Proof. By the Stone-Weierstrass Theorem it suffices to prove the lemma for the functions $f(x) = (1 - x^2)$ and $f(x) = x(1 - x^2)$, and since $C_0(-\pi/2, \pi/2) \otimes I \otimes M_2$ is an ideal in

$C[-\pi/2,\pi/2]\otimes\tilde{I}\otimes M_2$, it in fact suffices to consider the first of these two functions. An explicit computation completes the proof. □

Lemma 2. *For every $a\in A$ and every $f\in C[-1,1]$,*

$$\lim_{t\to\infty}[f(D_t),\begin{pmatrix} a & 0\\ 0 & s(p(a))\end{pmatrix}]=0.$$

Proof. By the Stone-Weierstrass Theorem again, it suffices to prove the lemma for $f(D_t)=D_t$. Here the commutator is

$$[f(D_t),\begin{pmatrix} a & 0\\ 0 & s(p(a))\end{pmatrix}]=$$

$$\cos(x)(1-u_t)\begin{pmatrix} 0 & a-s(p(a))\\ s(p(a))-a & 0\end{pmatrix}-\cos(x)\begin{pmatrix} 0 & [u_t,s(p(a))]\\ [u_t,a] & 0\end{pmatrix}$$

which goes to zero as $t\to\infty$ by properties (iii) and (iv) of $\{u_t\}$. □

For $t\in[1,\infty)$ define maps $\varphi_t: C_0(-1,1)\odot A\to C_0(-\pi/2,\pi/2)\otimes I\otimes M_2$ by

$$\varphi_t(f\otimes a)=f(D_t)\begin{pmatrix} a & 0\\ 0 & s(p(a))\end{pmatrix}.$$

By the above two lemmas,together with the universal property of the (maximal) tensor product, that $\{\varphi_t\}$ passes to a $*$-homomorphism from $C_0(\mathbf{R})\otimes A$ into $\mathcal{Q}(C_0(-\pi/2,\pi/2)\otimes I\otimes M_2)$. Thus we obtain an almost homomorphism from $C_0(\mathbf{R})\otimes A$ to $C_0(-\pi/2,\pi/2)\otimes I\otimes M_2$, and so a morphism $\rho: A\to I$ in $\mathbf{E}$.

Proposition 1. *The relations* (1) *are satisfied.*

Proof. For $\lambda\in[0,1]$, let

$$D_t^\lambda=\begin{pmatrix} \sin(x) & \lambda\cos(x)(1-u_t)\\ \lambda\cos(x)(1-u_t) & -\sin(x)\end{pmatrix}\in C[-\pi/2,\pi/2]\otimes\tilde{I}\otimes M_2.$$

The argument of Lemma 1 shows that $f(D_t^\lambda)\in C_0(-\pi/2,\pi/2)\otimes\tilde{I}\otimes M_2$ for every $f\in C(-1,1)$, and the argument of Lemma 2 shows that the correspondence

$$f\otimes a\mapsto f(D_t^\lambda)\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}$$

determines an almost homomorphism from $C_0(-1,1)\otimes I$ to $C_0(-\pi/2,\pi/2)\otimes I\otimes M_2[0,1]$. This is a homotopy between the almost homomorphism $\{\varphi_t\circ i\}$ and the $*$-homomorphism $f\otimes a\mapsto f(\sin(x))\otimes\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}$ from $C_0(-1,1)\otimes A$ to $C_0(-\pi/2,\pi/2)\otimes A\otimes M_2$ (to be precise, it is a homotopy between representatives of the asymptotic equivalence classes of such). The former determines the morphism $\rho[i]$ in $\mathbf{E}$; the latter determines $1_{\mathbf{E}}$. A similar correspondence,

$$f\otimes a\mapsto f(D_t^\lambda)\begin{pmatrix} a & 0\\ 0 & s(p(a))\end{pmatrix}$$

gives rise to a homotopy connecting $\{i\circ\varphi_t\}$ to the $*$-homomorphism

$$f\otimes a\mapsto\begin{pmatrix} f(\sin(x))\otimes a & 0\\ 0 & f(-\sin(x))\otimes s(p(a))\end{pmatrix}.$$

The former determines the morphism $[i]\rho$ in $\mathbf{E}$ whilst, according to the description of the additive structure of E-theory given in Section 4, the latter determines the morphism $1_A-[sp]=1_A-[s][p]$ in $\mathbf{E}$. Finally, $[p][s]=[ps]=[1_{A/I}]=1_{A/I}$. □

We shall use this to derive a *difference construction* for almost homomorphisms. Let B be an ideal in E, and suppose that we are given two almost homomorphisms $\{\varphi_t\},\{\psi_t\}\colon A\to E$ such that $\varphi_t(a)-\psi_t(a)\in B$ for all $a\in A$ and all $t\in[1,\infty)$. Consider the C^*-algebra

$$D=E\oplus_B E=\{e_1\oplus e_2\in E\oplus E : e_1-e_2\in B\}.$$

The almost homomorphisms $\{\varphi_t\}$ and $\{\psi_t\}$ combine to form an almost homomorphism $\{\theta_t\}\colon A\to D$: $\theta_t(a)=\varphi_t(a)\oplus\psi_t(a)$. Now, there is a short exact sequence of C^*-algebras

$$0\to B\xrightarrow{j_1}D\xrightarrow{p_2}E\to 0.$$

Here j_1 maps B to $B\oplus 0\subset E$ and p_2 is the projection onto the second summand of $D\subset E\oplus E$, and we define maps j_2 and p_1 similarly. This sequence is split by the $*$-homomorphism $s\colon e\mapsto e\oplus e$, and corresponding to this splitting we have the element $\rho\in E(D,B)$.

Definition 1. *Denote by $\Delta(\varphi, \psi)$ the element $\rho[\theta]$ of $E(A, B)$.*

Proposition 2. *Denote by $j \colon B \to E$ the inclusion map. Then:*

$$[j]\Delta = [\varphi] - [\psi] \in E(A, E) \tag{2}$$

and furthermore, if $f \colon A' \to A$ is a $*$-homomorphism such that $\{\varphi_t \circ f\}$ and $\{\psi_t \circ f\}$ both map A' into $B \subset E$ then

$$\Delta[f] = [\varphi \circ f] - [\psi \circ f] \in E(A', B). \tag{3}$$

Proof. By the previous proposition, $[j_1]\rho + [s][p_2] = 1_D$, so that

$$[j_1]\rho[\theta] = [\theta] - [s][p_2][\theta].$$

Since $j = p_1 j_1$, we see that

$$[j]\Delta = [p_1][j_1]\rho[\theta] = [p_1][\theta] - [p_1][s][p_2][\theta].$$

But $p_1 s p_2 = p_2$, and so

$$[i]\Delta = [p_1][\theta] - [p_2][\theta][p_1 \circ \theta] - [p_2 \circ \theta] = [\varphi] - [\psi].$$

For (3), it suffices to show that $\rho[j_1] = 1_B$ and $\rho[j_2] = -1_B$. The first of these relations is a basic property of ρ; for the second, we have

$$[j_1]\rho[j_2] = (1_D - [s][p_2])[j_2] = [j_2] - ([j_1] + [j_2]) = -[j_1],$$

and therefore $\rho[j_2] = -1$ as claimed, since $[j_1]$ is a monomorphism. □

6. Excision

Throughout this section, let J be any ideal in a C^*-algebra A.

Lemma 1. *Let $\{u_t\}_{t\in[1,\infty)}$ be a quasicentral approximate unit for $J \triangleleft A$ (as in the previous section). If $f \in C_0[-1,1)$ then $f(2u_t-1)a \to 0$ for every $a \in J$, and $[f(2u_t-1),a] \to 0$ for every $a \in A$.*

Proof. By the Stone-Weierstrass Theorem it suffices to consider the function $f(x) = 1-x$. For this function the result is clear. □

Denote by C_p the mapping cone of the projection $p\colon A \to A/J$,

$$C_p = \{a \oplus f \in A \oplus C_0[-1,1) : p(a) = f(-1)\},$$

and denote by $j\colon J \to C_p$ the inclusion $j\colon a \mapsto j \oplus 0$. The main technical result of this section is the following:

Theorem 1. *The morphism $[j]\colon J \to C_p$ in $\mathbf{E}$ is invertible.*

Proof. We shall construct two almost homomorphisms $\{\varphi_t\}, \{\psi_t\}\colon C_p \to C_p$ and apply the difference construction of the previous section to define an inverse morphism $C_p \to J$ in $\mathbf{E}$. Let $\{\varphi_t\}$ be the identity. To construct $\{\psi_t\}$, fix a linear section $s\colon A/J \to J$ (not necessarily multiplicative, or even continuous), and a quasicentral approximate unit $\{u_t\}$, and define a map $\Theta\colon C_0[-1,1) \odot A/J \to \mathcal{C}A$ by

$$\Theta(f \otimes x)(t) = f(2u_t - 1)s(x).$$

It follows from Lemma 1 that Θ passes to a $*$-homomorphism from $C_0[-1,1) \odot A/J$ to $\mathcal{C}A/\mathcal{C}_0J$ such that $\|\Theta(f \otimes x)\| \leq \|f\|\|x\|$ and by the universal property of the maximal tensor product, we obtain a $*$-homomorphism $\Theta\colon C_0[-1,1) \otimes A/J \to \mathcal{C}A/\mathcal{C}_0J$. Composing with any section $\mathcal{C}A/\mathcal{C}_0J \to \mathcal{C}A$, and then with the evaluation maps $e_t\colon \mathcal{C}A \to A$, we obtain an almost homomorphism $\{\theta_t\}\colon C_0[-1,1) \otimes A/J \to A$. Note that $\{\theta_t\}$ has the property that for all $F \in C_0[-1,1) \otimes A/J = A/J[-1,1)$, and all $t \in [1,\infty)$, $p(\theta_t(F)) = F(-1)$. Because of this, we can define $\{\psi_t\}: C_p \to C_p$ by

$$\psi_t(a \oplus F) = \theta_t(F) \oplus F.$$

The difference element $\Delta(\varphi,\psi) \in E(C_p, J)$ is inverse to $[j] \in E(J, C_p)$. Indeed, the composition $[j]\delta$ is equal to $[\varphi] - [\psi]$ by relation (2) of the previous section, and $[\varphi] = 1_{C_p}$ by definition, whilst $[\psi]$ factors through the contractible C^*-algebra $C_0[-1,1) \otimes A/J$, and is hence zero. Furthermore, by relation (3), $\delta[j] = [\varphi \circ j] - [\psi \circ j]$, and $\varphi \circ j = 1_J$, whilst $\psi \circ j = 0$. □

Lemma 2. *For every B the sequence of abelian groups*

$$E(B, C_p) \to E(B, A) \to E(B, A/J)$$

is exact in the middle.

Proof. Given an almost homomorphism $\{\varphi_t\}: C_0(\mathbf{R}) \otimes B \otimes \mathcal{K} \to C_0(\mathbf{R}) \otimes A \otimes \mathcal{K}$ and a homotopy

$$\{\theta_t\}: C_0(\mathbf{R}) \otimes B \otimes \mathcal{K} \to C_0(\mathbf{R}) \otimes A \otimes \mathcal{K}[-1,1) \cong C_0(\mathbf{R}) \otimes A[-1,1) \otimes \mathcal{K}$$

connecting the composition $\{p \circ \varphi_t\}$ to zero, define an almost homomorphism $\{\psi_t\}: B \to C_0(\mathbf{R}) \otimes C_p \otimes \mathcal{K}$ by

$$\psi_t(x) = \varphi_t(x) \oplus \theta_t(x).$$ □

Theorem 2. *For every C^*-algebra B the functor $E(B, \ )$ is half-exact.*

Proof. This follows immediately from Theorem 1 and the above proposition. □

By following the usual construction, we obtain from the short exact sequence of C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

a long exact sequence in the groups $E(B,\ )$. Recall from this construction that the connecting morphism

$$E(B, C_0(\mathbf{R}) \otimes A/J) \to E(B, J)$$

is given by composing in **E** with the morphism

$$C_0(-1,1) \otimes A/J \quad \xrightarrow{[i]} \quad C_p \quad \xrightarrow{[j]^{-1}} \quad /J$$

(here $i\colon C_0(-1,1) \otimes A/J \to C_p$ denotes the natural inclusion $F \mapsto 0 \oplus F$; the morphism $[j]$ is invertible by virtue of Theorem 1).

It follows from the proof of Theorem 1 that this morphism is induced from the almost homomorphism $C_0(-1,1) \otimes A/J \to J$ determined by the formula $f \otimes x \mapsto f(2u_t - 1) \otimes s(x)$.

In order to treat the problem of excision for the groups $E(\ , B)$, it is necessary to consider some aspects connected to Bott Periodicity. Denote by $S\colon \mathbf{E} \to \mathbf{E}$ the suspension functor

$$\{A \quad \xrightarrow{f} \quad B\} \mapsto \{A \otimes C_0(\mathbf{R}) \quad \xrightarrow{f \otimes 1} \quad B \otimes C_0(\mathbf{R})\}.$$

(We obtain this from the tensor product structure in **A**.)

Theorem 3. *Let*

$$0 \to \mathcal{K} \to T_0 \to C_0(\mathbf{R}) \to 0$$

be the (reduced) Toeplitz extension. Denote by $\delta_A\colon S^2 A \to A \otimes \mathcal{K} \cong A$ the connecting morphism in **E** *associated with the short exact sequence*

$$0 \to A \otimes \mathcal{K} \to A \otimes T_0 \to A \otimes C_0(\mathbf{R}) \to 0.$$

Then δ_A is an isomorphism, for every A, and for every morphism f in **E** *the diagram*

$$\begin{array}{ccc} S^2 A & \xrightarrow{S^2 f} & S^2 B \\ \delta_A \downarrow & & \downarrow \delta_B \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes.

Proof. By Cuntz's version of the Bott Periodicity Theorem, $\delta_{A*}: E(B, S^2A) \to E(B, A)$ is an isomorphism for every B, and therefore δ_A itself is an isomorphism. To prove the commutativity of the diagram, since $\delta = [j]^{-1}[i]$, it suffices to show that the diagrams

$$\begin{array}{ccc} S^2A & \xrightarrow{S^2 f} & S^2B \\ 1_A \otimes [i] \downarrow & & \downarrow 1_B \otimes [i] \\ A \otimes C_p & \xrightarrow[f\otimes 1]{} & B \otimes C_p \end{array} \quad \text{and} \quad \begin{array}{ccc} A \otimes \mathcal{K} & \xrightarrow{f\otimes 1_{\mathcal{K}}} & B \otimes \mathcal{K} \\ 1_A \otimes [j] \downarrow & & \downarrow 1_B \otimes [j] \\ A \otimes C_p & \xrightarrow[f\otimes 1_{C_p}]{} & B \otimes C_p \end{array}$$

commute. The commutativity of these follows from Lemma 2 of Section 3. □

Lemma 3. *For all A and B the suspension operation $E(A, B) \to E(SA, SB)$ is an isomorphism.*

Proof. Theorem 3 shows that the iterated functor $S^2: \mathbf{E} \to \mathbf{E}$ is naturally isomorphic to the identity, and hence that $S^2: E(A, B) \to E(S^2A, S^2B)$ is an isomorphism. It follows from a simple computation that $S: E(A, B) \to E(SA, SB)$ must then be an isomorphism as well. □

Lemma 4. *For every C^*-algebra B the sequence of abelian groups*

$$E(A/J, B) \to E(A, B) \to E(C_p, B)$$

is exact in the middle.

Proof. Denote by $q: C_p \to A$ the natural projection $a \oplus f \mapsto a$. We shall show that if $[\varphi] \in E(A, B)$ and if $[\varphi \circ q] = 0$ then there is some $[\theta] \in E(S(A/J), SB)$ such that $[\theta \circ p] = S[\varphi] \in E(SA, SB)$. In view of the previous lemma, this will suffice.

For simplicity, denote by A', B', etc, the C^*-algebras $C_0(\mathbf{R}) \otimes A \otimes \mathcal{K}$, $C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}$, etc. Note that $C_{p'} \cong (C_p)'$. It will simplify matters a little further if we replace the interval $[-1, 1)$ appearing in the definition of C_p with $[0, 1)$, and also identify $C_0(\mathbf{R})$ with $C_0(0, 1)$.

Let $\{\varphi_t\}: A' \to B'$ be an almost homomorphism and let $\{\eta_t\}: C_{p'} \to B'[0, 1)$ be a homotopy connecting $\{\varphi_t \circ q'\}$ to the zero almost homomorphism $C_{\prime} \to B'$. We have the inclusion $i' \colon S(A'/J') \cong A'/J'(0,1) \to C_{p'}$, and since $q' \circ i' = 0 \colon S(A'/J') \to A'$, it follows that the composition of $\{\eta_t\}$ with i' in fact maps $S(A'/J')$ into $SB' \cong B'(0,1) \subset B'(0,1)$. Thus we set $\{\theta_t\} = \{\eta_t \circ i'\} \colon S(A'/J') \to SB'$. It remains to construct a homotopy $\{\psi_t\} \colon S(A'/J') \to SB'[0,1]$ connecting $\{\theta_t\}$ (at say $\lambda = 1 \in [0,1]$) to $\{\varphi_t \circ q' \otimes 1\}$ (at $\lambda = 0 \in [0,1]$). Regard $SB'[0,1]$ as the C^*-algebra of continuous, B'-valued functions of $x \in [0,1]$ and $\lambda \in [0,1]$, vanishing at $x = 0,1$ (and regard SA' as A'-valued functions on $[0,1]$, vanishing at 0 and 1). For $t \in [1,\infty)$ define ψ_t by

$$\psi_t(f)(x,\lambda) = \theta_t \left(f((1-\lambda)x) \oplus p' f_{(1-\lambda)x} \right) (\lambda x).$$

Here, f_a denotes the function f translated by λ: $f_a(x) = f(x+a)$ (and $f_a(x) = 0$ if $x + a > 1$). It is easily verified that this is a homotopy as required. □

Putting Lemma 4 and Theorem 1 together we obtain:

Theorem 4. *For every C^*-algebra B, the functor $E(\ ,B)$ is half-exact.* □

7. Characterization of E-Theory

We shall prove:

Theorem 1. *Let $F \colon \mathbf{C^*\text{-}Alg} \to \mathbf{Ab}$ be any stable, half-exact, homotopy invariant functor. Then F factors uniquely throught the category $\mathbf{E}$:*

$$\begin{array}{ccc} \mathbf{C^*\text{-}Alg} & \rightarrow & \mathbf{E} \\ & \searrow & \downarrow \\ & & \mathbf{Ab} \end{array}$$

This sharpens Cuntz's results on pairing such functors with KK-theory. Furthermore, by considering the functors $\mathbf{X}(F(A), F(\ ))$ we easily obtain from this the following result:

Theorem 2. *Let $F \colon \mathbf{C^*\text{-}Alg} \to \mathbf{X}$ be any functor into an additive categroy which is stable, homotopy invariant, and half-exact on morphisms groups (in both variables). Then F factors uniquely through $\mathbf{E}$.* □

Note that the functor $\mathbf{C^*\text{-}Alg} \to \mathbf{E}$ itself satisfies the hypotheses of this theorem, by the results of the previous section, and so Theorem 2 actually characterizes $\mathbf{E}$.

Lemma 1. *Let J be a contractible ideal in a C^*-algebra A and denote by $p\colon A \to A/J$ the quotient map. Then the morphism $[p] \in E(A, A/J)$ is invertible, and $F(p)\colon F(A) \to F(A/J)$ is an isomorphism for every F as in Theorem 1.*

Proof. Since $F(J) = 0$, the long exact sequence of F-groups shows that $F(p)$ is an isomorphism. From the long exact sequences in the groups $E(B,\ )$, it follows that the map $[p]\colon E(B, A) \to E(B, A/J)$ is an isomorphism for every B. As noted before, this implies that $[p]$ is an isomorphism. □

Let $\{\varphi_t\}\colon A \to B$ be an almost homomorphism, and let $\Phi\colon A \to \mathcal{Q}B$ be the corresponding $*$-homomorphism. Let $A_\Phi = \Phi[A] \subset \mathcal{Q}B$ and let $B_\Phi \subset \mathcal{C}B$ be the inverse image of A_Φ under the projection $\mathcal{C}B \to \mathcal{Q}B$. Note that we have a short exact sequence

$$0 \to \mathcal{C}_0 B \to B_\Phi \xrightarrow{p} A_\Phi \to 0,$$

and that $\mathcal{C}_0 B$ is contractible.

Lemma 2. *Denote by $e\colon B_\Phi \to B$ the $*$-homomorphism given by evaluation at $1 \in [1, \infty)$. There exists an almost homomorphism $\{\psi_t\}\colon A \to B_\Phi$ such that the two triangles in the diagram*

$$\begin{array}{ccc} A & \xrightarrow{[\![\varphi]\!]} & B \\ [\![\Phi]\!]\big\downarrow & [\![\psi]\!]\searrow & \big\uparrow[\![e]\!] \\ A_\Phi & \xleftarrow[{[\![p]\!]}]{} & B_\Phi \end{array}$$

in **A** *commute.*

Proof. Define an almost homomorphism $\{\psi_t\}\colon A \to \mathcal{C}B$ by $\psi_t(a)(s) = \varphi_{r(s,t)}(a)$, where

$$r(s,t) = \begin{cases} s & \text{if } s \geq t, \\ t & \text{if } s < t. \end{cases}$$

We note that each ψ_t differs from the map $A \to \mathcal{C}B$ determined by $\{\varphi_t\}$ only by elements in $\mathcal{C}_0 B$. Therefore each ψ_t in fact maps A into B_Φ. By composing with $e\colon B_\Phi \to B$ we recover

$\{\varphi_t\}$ from $\{\psi_t\}$. On the other hand, each ψ_t, composed with the projection $B_\Phi \to A_\Phi$, becomes simply the $*$-homomorphism $\Phi\colon A \to C$ determined by $\{\varphi_t\}$. $\square$

Lemma 3. *Every morphism in* $\mathbf{E}$ *is a composition of morphisms of the form* $[f]\colon A \to B$, *or* $[f]^{-1}\colon B \to A$ *(if* $[f]$ *is invertible), where* $f\colon A \to B$ *is a* $*$*-homomorphism.*

Proof. The isomorphism $A \to S^2A \otimes \mathcal{K}$ given by stability and Bott Periodicity is of the form described in the statement of the lemma. By naturality of this isomorphism, it suffices to prove the lemma for morphisms from $S^2A \otimes \mathcal{K}$ to $S^2B \otimes \mathcal{K}$ given by almost homomorphisms of the form

$$\varphi_t \otimes 1_{C_0(\mathbf{R}^2)\otimes\mathcal{K}}\colon C_0(\mathbf{R}) \otimes A \otimes \mathcal{K} \otimes C_0(\mathbf{R}^2) \otimes \mathcal{K} \to C_0(\mathbf{R}) \otimes B \otimes \mathcal{K} \otimes C_0(\mathbf{R}^2) \otimes \mathcal{K}$$

Interchanging the copies of $\mathcal{K}$ (a homotopy) and exchanging by rotation the first copy of $C_0(\mathbf{R})$ with one of the others (another homotopy) we obtain a morphism $S^2A \otimes \mathcal{K} \to S^2B \otimes \mathcal{K}$ which is in the image of the functor $\mathbf{A} \to \mathbf{E}$. It follows from the above lemma that any morphism $[\varphi]$ in this image is of the required form, since we have

$$[\varphi] = [e][p]^{-1}[\Phi]. \qquad \square$$

Proposition 1. *Let* F *be a functor as in Theorem 1. Fix* A *and let* $x \in F(A)$*. There is a unique natural transformationm* $\alpha\colon E(A,\ ) \to F$ *(of functors on* $\mathbf{C^*}$**-Alg**, *necessarily additive) such that* $\alpha_A(1_A) = x$.

Proof. Define new functors F_n by $F_n(A) = F(C_0(\mathbf{R}^n) \otimes A)$. All the F_n satisfy the hypotheses of Theorem 1, and by Cuntz's version of the Bott Periodicity Theorem, F_2 is naturally isomorphic to F, so it suffices to prove the theorem for F_2. Let $x \in F_2(A)$. Given an almost homomorphism $\{\varphi_t\}\colon C_0(\mathbf{R}) \otimes A \to C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}$, we may form the composition of homomorphisms

$$F_1(C_0(\mathbf{R}) \otimes A) \xrightarrow{F_1(\Phi)} F_1(C) \xrightarrow{F_1(p)^{-1}} F_1(D)$$

$$\xrightarrow{F_1(e)} \quad F_1(C_0(\mathbf{R}) \otimes B \otimes \mathcal{K}) \quad \xrightarrow{\cong} \quad F_1(C_0(\mathbf{R}) \otimes B),$$

giving us a homomorphism

$$F_2(A) \to F_2(B).$$

Define $\alpha_B(\{\varphi_t\})$ to be the image of $x \in F_2(A)$ under this map. By performing the same construction for a homotopy, it is easily seen that $\alpha_B(\{\varphi_t\})$ depends only on the homotopy class of $\{\varphi_t\}$, and so passes to a map $\alpha_B\colon E(A,B) \to F_2(B)$. The naturality of α is clear, as is the fact that $\alpha_A(1_A) = x$. The uniqueness assertion follows immediately from the above lemma. □

Proof of Theorem 1. The construction of α in the last proposition determines a function $E(A,B) \to \mathrm{Hom}(F(A), F(B))$. (The fact that each morphism in $E(A,B)$ determines a *linear* map from $F(A)$ to $F(B)$ follows from the uniqueness part of the proposition. In fact, it is easy to show that the natural transformations α of the previous proposition are necessarily additive, and so this function is a group homomorphism.) The fact that this actually determines a functor $\mathbf{E} \to \mathbf{Ab}$ follows from another appeal to the uniqueness part of the proposition, as of course does the uniqueness of this functor. For details, see the proof of the corresponding result for KK-theory. □

Theorem 3. *The functor* $\mathbf{C^*\text{-}Alg} \to \mathbf{E}$ *factors uniquely through the category* $\mathbf{KK}$ *constructed out of* KK*-theory. The induced maps* $KK(A,B) \to E(A,B)$ *are isomorphisms for all* K*-nuclear* C^**-algebras* A *and arbitrary* B.

Proof. The first statement follows from the fact that $\mathbf{C^*\text{-}Alg} \to \mathbf{KK}$ is characterized as the universal functor from $\mathbf{C^*\text{-}Alg}$ to an additive category which is homotopy invariant, stable and split exact (see my article on characterizing KK-theory). The second assertion follows upon applying Theorem 1 to the functors $KK(A,\ )$, for K-nuclear C^*-algebras A. By Skandalis's work, these functors are half-exact, and so we obtain natural transformations $E(A,\ ) \to KK(A,\ )$ which provide inverses to the maps $KK(A,B) \to E(A,B)$. □

8. Examples of Almost Homomorphisms

The purpose of this section is to describe some constructions involving almost homomorphisms that should be of interest in applications to differential topology. I have not checked the details in most of what follows, and in the last subsection I shall go no further than to sketch out a few ideas.

1. *Unbounded Kasparov Elements*

Let A and B be C^*-algebras, and let $(\mathcal{E}, D)$ be an unbounded Kasparov (A, B)-bimodule. Thus:

(i) $\mathcal{E}$ is a $\mathbf{Z}/2$-graded Hilbert B-module, equipped with a zero-graded action of A (we shall suppress any symbol for this representation, and regard elements of A as operators on $\mathcal{E}$);

(ii) D is a grading degree one, self-adjoint, unbounded operator on $\mathcal{E}$ (thus the resolvents $(D \pm i)^{-1}$ are everywhere defined, bounded operators on $\mathcal{E}$, with $(D+i)^{-1\,*} = (D-i)^{-1}$, and $f(D)$ is of degree one for every bounded odd function f); such that

(iii) for a dense set in A, the commutator $[a, D]$ is densely defined and extends to a bounded operator on $\mathcal{E}$; and

(iv) the resolvent $(D \pm i)^{-1}$ is in the ideal $\mathcal{K}(\mathcal{E})$ of compact operators.

From this data we can construct an almost homomorphism $\{\varphi_t\}\colon C_0(\mathbf{R}) \otimes A \to C_0(\mathbf{R}) \otimes \mathcal{K}(\mathcal{E})$ as follows. Denote by $\mathcal{S}\mathcal{E}$ the Hilbert $C_0(\mathbf{R}) \otimes B$-module $C_0(\mathbf{R}) \boxtimes_{\mathbf{C}} \mathcal{E}$, and observe that $\mathcal{K}(\mathcal{S}\mathcal{E}) \cong C_0(\mathbf{R}) \otimes \mathcal{K}(\mathcal{E})$. Denote by $M\colon \mathcal{S}\mathcal{E} \to \mathcal{S}\mathcal{E}$ the operator of pointwise multiplication by $M(x) = x$. Let D act on $\mathcal{S}\mathcal{E}$ as $1 \boxtimes D$, and denote by γ the grading operator. Then let

$$F_t = \gamma M + t^{-1} D, \qquad (t \in [1, \infty)).$$

This is a family of self adjoint, degree one operators on $\mathcal{S}\mathcal{E}$, and the family of resolvents $t \mapsto (F_t \pm i)^{-1}$ is norm continuous. Define

$$\varphi_t(f \otimes a) = f(F_t) a \qquad (t \in [1, \infty)).$$

It is a simple matter to check that this does indeed determine an almost homomorphism (compare sections 5 and 6).

By a result of Baaj and Julg, every element of $KK(A,B)$ can be represented as an unbounded Kasparov element. The construction we have outlined passes to a map from $KK(A,B)$ to $E(A,B)$: it is the same as the natural transformation considered in the last section. Its obvious advantage is that we get concrete formulas (for example, for the "dual Dirac" elements of Kasparov/Mischenko).

2. *Index Theory*

Define an almost homomorphism from $C_0(\mathbf{R}^n \times \mathbf{R}^n)$ to $\mathcal{K}(L^2(\mathbf{R}^n))$ as follows. Given say a smooth, compactly supported function f on $\mathbf{R}^n \times \mathbf{R}^n$, let $\mathcal{F}_2(f)\colon \mathbf{R}^n \times \hat{\mathbf{R}}^n \to \mathbf{C}$ be the Fourier transform in the second variable. For $t \in [1,\infty)$ define $\varphi_t(f)\colon L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ by

$$\varphi_t(f)(g)(x) = \int_{\mathbf{R}^n} \mathcal{F}_2(f)(x, t^{-1}\xi) e^{2\pi i x\xi} \hat{g}(\xi)\, d\xi.$$

This determines an almost homomorphism $\{\varphi_t\}\colon C_0(\mathbf{R}^n \times \mathbf{R}^n) \to \mathcal{K}(L^2(\mathbf{R}^n))$ (compare Widom's study of asymptotic families of pseudodifferential operators.)

Let us consider for a moment the case $n = 1$. Then the above almost homomorphism may also be written as

$$\varphi_t(f_1 \otimes f_2) = f_1(M) f_2(t^{-1}D),$$

where M is multiplication by $M(x) = x$ and $D = -i\, d/dx$.

Using a partition of unity, we can define in a like manner an almost homomorphism $C_0(T^*M) \to \mathcal{K}(L^2(M))$. The induced map $K(T^*M) \to \mathbf{Z}$ on K-theory is the analytic index.

This last construction can be "lifted" to obtain a construction of the index map $K(T^*M) \to K(C_r^*(\pi_1 M))$: one simply lifts $f \in C_0(T^*M)$ to a periodic function $\tilde{f} \in C_0(T^*\tilde{M})$ and constructs from $\tilde{f}$ an equivariant smoothing operator on $L^2(\tilde{M})$. (This was pointed out to me by R. Nest.) The C^*-algebra generated by such smoothing operators is $*$-isomorphic to $C^*_r(\pi_1 M) \otimes \mathcal{K}$.

Note that if M has additional geometric structure, such as negative sectional curvatures, then using the exponential map $TM \xrightarrow{\cong} \tilde{M} \times_{\pi_1 M} \tilde{M}$, we can dispense with the partitions of unity and obtain explicit formulas for the above almost homomorphisms.

3. *Almost Flat Bundles*

I will sketch how almost homomorphisms might be applied to the theory of almost flat bundles.

By a *nearly flat* bundle, on say a compact manifold X, I mean something of the following sort: a pair (E^+, E^-) of infinite dimensional hermitian bundles over X, together with a Fredholm map $F\colon E^+ \to E^-$, and families of connections (compatible with the hermitian structure) $\nabla_t^{\pm}$ on $E^{\pm}$ $(t \in [1, \infty))$, which are asymptotically flat, and for which the corresponding parallel transport maps are intertwined by F, modulo compacts.

Impose the equivalence relation of homotopy on the set of all nearly flat bundles over X, and define $KE(X)$ to be the Grothendieck group constructed from the equivalence classes.

By rephrasing the definition in combinatorial terms, it should be possible to define $KE(X)$ for say an arbitrary CW-complex X.

Picking a basepoint $x \in X$, from parallel translation we obtain unitary almost representations $\pi_1 X \to \mathrm{Aut}(E_x^{\pm})$. Using the operator F, and the usual sort of construction, we obtain two unitary almost representations on a single Hilbert space $\mathcal{H}$ which differ by compact operators. These pass to almost homomorphisms $C^*_{max}(\pi_1 M) \to \mathcal{B}(\mathcal{H})$ which are equal, modulo compacts, and by the difference construction of Section 6, we obtain from these an element $\alpha(E^{\pm}, \nabla_t^{\pm}, F) \in E(C^*_{max}(\pi_1 X), \mathbf{C})$. In this way we define an "almost monodromy map"

$$\alpha\colon KE(X) \to E(C^*_{max}(\pi_1 X), \mathbf{C}).$$

By forgetting the connections, there is of course also a map

$$\phi\colon KE(X) \to \varprojlim_{\Delta\subset X} K(\Delta),$$

where the limit is over compact subsets of X.

Suppose now that Γ is a discrete group and that we are given a class $a \in KE(B\Gamma)$. Let us see how E-theory might be used to prove the homotopy invariance of the higher signature (for manifolds with fundamental group Γ) associated with the cohomology class $\mathrm{ch}(\phi(a))$. Let M be a closed, oriented manifold with $\pi_1 M = \Gamma$, and denote by $f\colon M \to B\Gamma$ the classifying map. We have the following classes in E-theory:

$d \in E(C(M), \mathbf{C})$	the class of the signature operator
$m \in E(\mathbf{C}, C(M) \otimes C^*_{max}\Gamma)$	the class of the Mischenko line bundle
$f^*(\phi(a)) \in E(\mathbf{C}, C(M))$	the pull-back of $(E^{\pm}, F)$ to M
$\alpha(a) \in E(C^*_{max}\Gamma, \mathbf{C})$	as above

It must be shown that the class $f^*(\phi(a)) \otimes_{C(M)} d \in E(\mathbf{C}, \mathbf{C})$ is a homotopy invariant (here we are using Kasparov-style notation for the product in E-theory). It should be possible to accomplish this in two stages:

(i) Show that $f^*(\phi(a)) = m \otimes_{C^*_{max}\Gamma} \alpha(a)$. (Thus the K-theory classes of nearly flat bundles can be reconstructed from their "almost monodromies".)

(ii) Given (i), we can use the associativity of the E-theory product to obtain:

$$\begin{aligned} f^*(\phi(a)) \otimes_{C(M)} d &= \big(m \otimes_{C^*_{max}\Gamma} \alpha(a)\big) \otimes_{C(M)} d \\ &= \alpha(a) \otimes_{C^*_{max}\Gamma} \big(m \otimes_{C(M)} d\big) \end{aligned}$$

But it is well known that the class $m \otimes_{C(M)} d \in E(\mathbf{C}, C^*_{max}\Gamma) = K(C^*_{max}\Gamma)$ is an oriented homotopy invariant, whilst of course $\alpha(a)$ does not depend on M at all. Therefore we have the desired result.

Remarks. (i) I do not know to what extent the notions of "almost flat bundle" and "nearly flat bundle" are the same.

(ii) When is the map $\alpha: KE(B\Gamma) \to E(C^*_{max}\Gamma, \mathbf{C})$ an isomorphism? (Since I have only a vague idea of what the correct definition of $KE(X)$ should be, it would not be appropriate to state a conjecture here.) In connection with this question, it is worth pointing out that $E(A, \mathbf{C})$ may also be described in terms of pairs of almost homomorphisms $A \to \mathcal{B}(\mathcal{H})$ which differ only by compact operators. (Thus, by considering such "quasi-almost homomorphisms", à la Cuntz, it is not necessary to introduce suspensions.) If $A = C^*_{max}\Gamma$ then we obtain a pair of almost representations of Γ differing by compacts (by dilation, we can assume these almost representations are unitary). It seems reasonable that one can construct a nearly flat bundle from this data.