COUNTEREXAMPLES TO THE COARSE BAUM-CONNES CONJECTURE

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Unpublished Note, 1999

1. Introduction

Let X be a discrete, bounded geometry metric space.[1] Associated to X is a C^*-algebra $C^*(X)$ which has proved very useful in C^*-algebraic approaches to the Novikov conjecture in manifold theory. The *coarse Baum-Connes conjecture* proposes a formula for the operator K-theory of $C^*(X)$ in terms of the K-homology groups of certain 'thickenings' of X. Its interest derives from the fact that if X is the metric space underlying a finitely generated discrete group G then the validity of the coarse Baum-Connes conjecture for X implies the validity of the Novikov higher signature conjecture for G. The coarse conjecture has been proved in a number of cases. Most notably, Yu has shown that the Coarse Baum-Connes conjecture holds for any X which embeds uniformly in Hilbert space.

The purpose of this note is to sketch the construction of some counterexamples to the Baum-Connes conjecture. These are inspired by Gromov's recent observation that expanding sequences of graphs do not embed uniformly in Hilbert space, although our examples are more properly situated within the theory of Kazhdan's property T than in expander graph theory. As they stand, our examples are far from the finitely generated group examples of interest to the Novikov conjecture.

2. Property T

2.1. Notation. Throughout the entire note we shall denote by Γ an infinite, torsion-free discrete group with the following properties

- Γ has property T.
- Γ is residually finite.
- Γ has *linear type.*

We shall review the salient consequence of linear type in Section 5. For now we note that uniform lattices in semisimple groups have this property, so there is a reasonably broad class of groups Γ with the properties described in 2.1.

[1] *Discrete* means that all sets are open; *bounded geometry* means that for every $R > 0$ the quantity $\sup_{x\in X} \#\{y \in X : d(x, y) \leq R\}$ is finite.

Since Γ is a property T group it is finitely generated. Fix a finite, symmetric[2] generating set S.

Denote by $\Gamma_1, \Gamma_2, \ldots$ a decreasing, nested sequence of finite-index normal subgroups of Γ such that $\cap_n \Gamma_n = \{e\}$.

2.2. Definition. Denote by X_n the metric space whose underlying set is Γ/Γ_n and whose metric is the right-translation invariant distance function associated to the generating set S (or to be precise, the image of S in Γ/Γ_n). Let X_Γ be any metric space which is the disjoint union of the X_n, with the given metric on X_n and the separation between X_n and X_m (the minimum distance between a point of one and a point of the other) tending to infinity as n and m tend to infinity.

Although it is not relevant to the main argument, let us note the following fact (due to Gromov).

2.3. Lemma. *The metric space X admits no uniform embedding into a Hilbert space.*

Proof. Suppose that $f\colon X \to H$ is a uniform embedding, and denote by f_n the restriction of f to $X_n \subseteq X$. Adjusting f by an overall scale factor if necessary, and adjusting each f_n by a translation in H, we obtain at a sequence of functions $f_n\colon X_n \to H$ such that

(i) there is some proper function $\phi\colon [0,\infty) \to [0,\infty)$, which is independent of n, such that
$$\phi(d(x,x')) \le \|f_n(x) - f_n(x')\| \le d(x,x'),$$
for all n and all $x, x' \in X_n$.

(ii) $\sum_{x\in X_n} f_n(x) = 0$.

Denote by $\ell^2_0(X_n, H)$ the Hilbert space of H-valued functions on X_n which have total sum zero. By property T, there is some $\varepsilon > 0$, independent of n, such that if $h \in \ell^2_0(X_n, H)$ then
$$\sum_{s\in S}\sum_{x\in X_n} \|h(x) - h(sx)\|^2 \ge \varepsilon \sum_{x\in X_n} \|h(x)\|^2.$$

Applying this to f_n we get that
$$\sum_{s\in S}\sum_{x\in X_n} 1 \ge \varepsilon \sum_{x\in X_n} \|f_n(x)\|^2,$$
or
$$|S| \cdot |X_n| \ge \varepsilon \sum_{x\in X_n} \|f_n(x)\|^2.$$

The last inequality implies that within X_n there is a set Y_n comprised of half or more of the elements of X_n such that
$$y \in Y_n \quad \Rightarrow \quad \|f_n(y)\|^2 \le 2\varepsilon^{-1}|S|.$$

[2] *Symmetric* means that if $s \in S$ then $s^{-1} \in S$.

But within Y_n there must be points y_n and y_n' with $\lim_{n\to\infty} d(y_n, y_n') = \infty$, whereas of course $\|f_n(y_n) - f_n(y_n')\|^2 \leq 2\sqrt{2}\varepsilon^{-1}|S|$. This contradicts (i) above. □

The same argument shows that no expanding sequence of graphs embeds uniformly into Hilbert space (which was Gromov's observation).

The lemma shows that the space X is at least a candidate for a counterexample to coarse Baum Connes, since Yu's theorem does not apply to it. The counterexample itself will be based on the *Kazhdan projection* in the full group C^*-algebra of Γ. This is the projection characterized by the following well known result.

2.4. Theorem. *Let Γ be a property T group. There is a unique projection $p \in C^*_{\max}(\Gamma)$ with the property that the image of p in any unitary representation of Γ is the orthogonal projection onto the Γ-fixed vectors.* □

3. C*-Algebras Associated to a Metric Space

To begin this section X may be any discrete, bounded geometry metric space.

3.1. Definition. Denote by $B(X)$ the algebra of $X \times X$ complex matrices $[a_{xy}]$ such that

(a) $\sup_{x,y} |a_{xy}| < \infty$; and
(b) $\sup\{\, d(x,y) \mid a_{xy} \neq 0 \,\} < \infty$.

The algebra $B(X)$ is represented in an obvious way on $\ell^2(X)$ and we denote by $B^*(X)$ the C^*-algebra completion in this representation. This is the *uniform C^*-algebra* associated to the metric space X.

3.2. Definition. Denote by $\mathcal{K}(H)$ the C^*-algebra of compact operators on a separable Hilbert space H. Denote by $C(X)$ the algebra of $X \times X$ matrices $[a_{xy}]$ satisfying conditions (a) and (b) of Definition 3.1 above, but where now the entries a_{xy} are elements of $\mathcal{K}(H)$. The algebra $C(X)$ is represented in an obvious way on $\ell^2(X, H)$ and we denote by $C^*(X)$ the C^*-algebra completion in this representation. This is the *coarse C^*-algebra* associated to the metric space X.

The coarse Baum-Connes conjecture is an assertion concerning the K-theory groups of the C^*-algebra $C^*(X)$. But before turning to K-theory and $C^*(X)$, let us take a quick look at some aspects of the algebra $B^*(X)$.

Recall that a C^*-algebra A is *exact* if and only if for every short exact sequence of C^*-algebras

$$0 \to J \to B \to C \to 0$$

the tensor product sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes C \to 0,$$

constructed using the minimal (a.k.a. spatial) tensor product, is exact.

An optimistic conjecture in C^*-algebra theory asserts that every reduced group C^*-algebra $C^*_r(G)$ is exact. The evidence for this is that for a wide variety of groups one can construct an *amenable* action of G on a compact topological space X. The hypothesis of amenability implies that the reduced crossed product $C_r^*(G, C(X))$ is a nuclear C^*-algebra. Now, nuclear C^*-algebras are exact, C^*-subalgebras of exact C^*-algebras are exact, and of course $C_r^*(G)$ is a C^*-subalgebra of $C_r^*(G, C(X))$, and so the existence of an amenable action implies the exactness of $C_r^*(G)$. This line of reasoning applies to discrete subgroups of Lie groups, Coxeter groups, hyperbolic groups, groups of finite asymptotic dimension, etc.

If G is a finitely generated group, and if X is its underlying metric space (with right-translation invariant word-length metric), then the left regular representation of G on $\ell^2(X)$ includes $C_r^*(G)$ as a C^*-subalgebra of the C^*-algebra $B^*(X)$. In fact it is not hard to show that:

3.3. Lemma. *If X is the metric space underlying a finitely generated group G then*

$$B^*(X) \cong C_r^*(G, C(\beta X)),$$

where βX is the Stone-Cech compactification of X, and the inclusion of $C_r^*(G)$ into $B^*(X)$ is the natural inclusion into the reduced crossed product. □

If a group acts amenably on any compact space at all then it acts amenably on its Stone-Cech compactification. So if G is exact by virtue of an amenable action on a compact space then the uniform C^*-algebra of its underlying metric space is nuclear. In fact the exactness of $B^*(X)$ follows from the exactness of $C_r^*(G)$:

3.4. Lemma. *If G is a finitely generated group and if X is its underlying metric space then $C_r^*(G)$ is exact if and only if $B^*(X)$ is exact.*

Proof. The exactness of $B^*(X)$ certainly implies the exactness of $C_r^*(G)$ since $C_r^*(G)$ is a C^*-subalgebra of $B^*(X)$. Suppose, conversely, that $C_r^*(G)$ is exact and that we are presented with an exact sequence of C^*-algbras

$$0 \to J \to B \to C \to 0.$$

Since tensoring with a commutative C^*-algebra preserves exactness the sequence

$$0 \to C(\beta G) \otimes J \to C(\beta G) \otimes B \to C(\beta G) \otimes C \to 0$$

is exact. It is a result of Kirchberg and Wassermann that if G is exact then the operation of reduced crossed product by G preserves exact sequences. But in the present case upon forming reduced crossed products we obtain the exact sequence

$$0 \to B^*(X) \otimes J \to B^*(X) \otimes B \to B^*(X) \otimes C \to 0,$$

as required. □

The above lines of thought might suggest the optimistic conjecture that the rough C^*-algebra of any bounded geometry, discrete metric space is exact[3]. We shall see that this is wrong.

[3] The extreme optimist might be led to conjecture that the rough C^*-algebra is always nuclear.

3.5. Proposition. *If X is the metric space constructed in the previous section then the C^*-algebra $B^*(X)$ is not exact.*

Proof. We will prove that $B^*(X)$ is not exact by constructing within it a C^*-subalgebra C which is not exact. There is an obvious choice for C: take the image of the left regular representation of $C^*(G)$ on the direct sum of the Hilbert spaces $\ell^2(G_n)$. It is known that associated to each irreducible, finite-dimensional unitary representation π of a property T group G there is a projection in $C^*(G)$ which acts in any unitary representation as the orthogonal projection onto the π-isotypical vectors. Denote by J the ideal in C generated by the images of all such projections. It is shown by Wassermann[4] that the sequence

$$0 \to C \otimes J \to C \otimes C \to C \otimes C/J \to 0$$

is *not* exact, and therefore C is not an exact C^*-algebra. □

3.6. Remark. The above argument is far from optimal, since a slightly more sophisticated analysis shows that we can construct an inexact $B^*(X)$ starting from any nonamenable, residually finite group and forming a metric space from a sequence of finite quotients as we did in the last section. Constructions of this type are well known in the field of exact C^*-algebra theory and elsewhere, although their relevance to the C^*-algebra theory of metric spaces seems not to have been observed previously.

4. The Assembly Map

We refer to the paper of Higson and Roe on the subject for a precise formulation of the coarse assembly map and the coarse Baum-Connes conjecture. We note that although they state the conjecture for a 'complete path metric space of bounded coarse geometry' their assertion covers the space X_Γ introduced in Section 2. This is because X_Γ is coarsely equivalent to a complete path metric space of bounded coarse geometry (replace each quotient $X_n = \Gamma/\Gamma_n$ by the corresponding Cayley graph).

The formulation of the conjecture for a space X requires the construction of a sequence of 'coarsenings'

$$X \to |\mathcal{U}_1| \to |\mathcal{U}_2| \to \cdots .$$

In the present context, where X is a bounded geometry discrete metric space, we may take $|\mathcal{U}_R|$ to be the 'Rips complex at scale R,' which is the simplicial complex with vertex set X, in which a $(p+1)$-tuple $(x_0, \ldots, x_p)$ is a p-simplex if and only if $d(x_i, x_j) \leq R$, for all i and j.

There is then a directed sequence of assembly maps

$$\begin{array}{ccccccc} K_*(X) & \longrightarrow & K_*(|\mathcal{U}_1|) & \longrightarrow & K_*(|\mathcal{U}_2|) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ K_*(C^*(X)) & \longrightarrow & K_*(C^*(|\mathcal{U}_1|)) & \longrightarrow & K_*(C^*(|\mathcal{U}_2|)) & \longrightarrow & \cdots \end{array}$$

[4] He shows something very slightly different, but his argument carries over with basically no change.

(they are the vertical maps in the above diagram) and the coarse Baum-Connes conjecture asserts that the induced map on direct limits is an isomorphism. Along the bottom, the maps are in fact isomorphisms, so the conjecture provides a formula for $K_*(C^*(X))$.

What does the coarsening $|\mathcal{U}_R|$ look like for our space X_Γ from Section 2? To answer the question, denote by X_Γ^K the subspace of X_Γ comprised only of the spaces $X_{K+1}, X_{K+2}, \ldots$, but not the first K quotients of Γ. Then as long as K is large enough (for a given R), the component spaces in X_Γ^K will be separated by more than R. Furthermore, for K large enough the normal subgroups Γ_n corresponding to the components $X_n = \Gamma/\Gamma_n$ of X_Γ^K will contain no non-identity element of length $3R$ or less. It follows that for K large enough the Rips complex $\mathrm{Rips}(X_\Gamma^K, R)$ at scale R of X_Γ^K is the disjoint union of the quotients $\mathrm{Rips}(\Gamma, R)/\Gamma_n$ for $n = K+1, K+2, \ldots$. The following is an easy consequence:

4.1. Proposition. *The range of the coarse Baum-Connes assembly map for X_Γ is the union over all R of the ranges of the assembly maps*

$$\prod_{n=1}^{\infty} K_*(\mathrm{Rips}(\Gamma, R)/\Gamma_n) \cong K_*(\cup_{n=1}^{\infty} \mathrm{Rips}(\Gamma, R)/\Gamma_n) \to K_*(C^*(X_\Gamma)).$$

5. Counterexamples to the Coarse Baum-Connes Conjecture

In this section we continue to focus on the metric space X_Γ constructed in Section 2.

Denote by $p_0 \in C^*(\Gamma)$ the Kazhdan projection. We noted earlier that $C^*(\Gamma)$ maps to $B^*(X_\Gamma)$, and we shall also denote by p_0 the image of the Kazhdan projection in $B^*(X_\Gamma)$. Finally there is an obvious inclusion of $B^*(X_\Gamma) \otimes \mathcal{K}(H)$ into $C^*(X_\Gamma)$, and we shall denote again by p_0 the tensor product

$$p_0 \otimes \text{rank one projection}$$

in $C^*(X_\Gamma)$. Our aim is to show that the K-theory class of $p_0 \in C^*(X_\Gamma)$ does not lie in the image of the Baum-Connes assembly map.

5.1. Definition. Denote by $\mathbb{R}_{[\infty]}$ the quotient of the direct product $\prod_{n=1}^{\infty} \mathbb{R}$ by the direct sum $\sum_{n=1}^{\infty} \mathbb{R}$:

$$\mathbb{R}_{[\infty]} = \prod_{n=1}^{\infty} \mathbb{R} \Big/ \sum_{n=1}^{\infty} \mathbb{Z}.$$

To prove that the K-theory class of $p_0 \in C^*(X_\Gamma)$ does not lie in the image of the Baum-Connes assembly map we shall construct functionals

$$\tau_0 \colon K_0(C^*(X_\Gamma)) \to \mathbb{R}_{[\infty]}$$

and

$$\tau_1 \colon K_0(C^*(X_\Gamma)) \to \mathbb{R}_{[\infty]}$$

which agree on the image of the assembly map but disagree on the K-theory class $[p]$.

5.2. Definition. If q is a self-adjoint element in a C^*-algebra A such that $\|q^2-q\| < 1/4$ then denote by $\operatorname{proj}(q) \in A$ the spectral projection in A corresponding to that part of the spectrum of q to the right of $x = 1/2$.

5.3. Definition. Denote by P_n the orthogonal projection of $\ell^2(X_\Gamma, H)$ onto the subspace $\ell^2(X_n, H)$. Observe that if $x \in C^*(X_\Gamma)$ then $\lim_{n\to\infty} \|P_n x - x P_n\| = 0$ (indeed if x has finite propagation then the commutator is eventually zero). We define $\tau_0\colon K_0(C^*(X_\Gamma)) \to \mathbb{R}_{[\infty]}$ by associating to any projection $p \in C^*(X_\Gamma)$ the sequence of dimensions of the ranges of the projections $\operatorname{proj}(P_n p P_n)$ (as an element of $\mathbb{R}_{[\infty]}$ this depends only on the K-theory class of p).

Observe that for the Kazhdan projection we have $\tau_0([p_0]) = \mathbf{1}$ (the constant sequence $(1, 1, \dots)$).

The definition of τ_1 is a little more involved, and it is here that we shall need linear type (which was mentioned in Section 2). For our purposes, the effect of linear type is to give us the following lifting principle:

5.4. Lemma. *There is a constant $C > 1$ such that if $n \geq 1$ and $R \geq 1$, and if no non-identity element of Γ_n has length CR or less, then every operator T in $C^*(|\Gamma/\Gamma_n|)$ of propogation R or less lifts to an operator $\tilde{T}$ in $C^*(|\Gamma|)$ with propagation R or less and $\|\tilde{T}\| \leq C\|T\|$.*

5.5. Definition. Define $\tau_1\colon K_0(C^*(X_\Gamma)) \to \mathbb{R}_{[\infty]}$ by associating to a projection $p \in C^*(X_\Gamma)$ the sequence of von Neumann Γ_n-dimensions of $\operatorname{proj}(\widetilde{P_n p' P_n})$, where $p' \in C^*(X)$ is a finite-propagation approximant to p with $\|p - p'\| < 1/(1000C)$ and $\widetilde{P_n p' P_n} \in \mathcal{B}(\ell^2(\Gamma, H))$ is a lifting from $C^*(|\Gamma/\Gamma_n|)$ to $C^*(|\Gamma|)$.

The lemma guarantees that τ_1 is a well-defined functional on K-theory (with values in $\mathbb{R}_{[\infty]}$).

5.6. Proposition. *The trace functionals τ_0 and τ_1 agree on the range of the coarse assembly map.*

Proof. This is, in effect, the Atiyah Γ-index theorem. □

5.7. Proposition. *If p is the Kazhdan projection, viewed as an element of $C^*(X)$, then $\tau_1([p]) = \mathbf{0}$.*

Proof. We can form liftings p' by approximating the Kazhdan projection within the group algebra $\mathbb{C}[\Gamma]$ (in the $C^*_{\max}(\Gamma)$ norm) then mapping these to operators on $\ell^2(\Gamma, H)$ by the regular representation. The resulting sequence of lifts $\widetilde{P_n p' P_n}$ converge in norm to the orthogonal projection onto the Γ-fixed subspace of $\ell^2(\Gamma, H)$; in other words the approximants converge in norm to zero. □