EXPANDERS, UNIFORM EMBEDDINGS AND EXACT C^*-ALGEBRAS

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Introduction. Gromov has recently announced the existence of groups (even groups of finite type) which do not embed uniformly in Hilbert space. His ideas raise the serious possibility that counterexamples to various analytic variations on the Novikov conjecture may not be far off. I'll sketch the very first steps that can be taken in this direction.

Expanders. In what follows it will be convenient to view (connected) graphs as metric spaces (X, d) for which

(a) the distance function is integer-valued; and
(b) there exists between any two points $x_0, x_n \in X$ with $d(x_0, x_n) = n$ a chain of points $x_0, x_1, \dots, x_{n-1}, x_n$ such that $d(x_j, x_{j+1}) = 1$.

(Graphs which are not connected correspond to 'metric' spaces for which the distance function is allowed to assume the value ∞. We will have occasion to deal with these later on.)

The *valence* $\operatorname{val}(x)$ of a point x in a graph X is the number of $y \in X$ such that $d(x, y) = 1$. The valence of X is the supremum of the valences of its vertices. Throughout this note we shall deal with finite-valence graphs.

The *Laplace operator* on a finite-valence graph X is the positive Hilbert space operator

$$\Delta \colon \ell^2(X) \to \ell^2(X)$$

defined by

$$\Delta f(x) = \operatorname{val}(x) f(x) - \sum_{d(x,y)=1} f(y).$$

The Laplace operator is positive (in the sense of Hilbert space operator theory) since

$$\langle \Delta f, f \rangle = \sum_{d(y,z)=1} |f(y) - f(z)|^2 \geq 0.$$

If X is a finite and connected graph then the kernel of Δ is comprised precisely of the constant functions. We are interested in the lowest non-zero eigenvalue of Δ, which we shall denote by $\lambda_1(X)$. For the purposes of this note an *expander* is a sequence $\{X_n\}_{n=1}^{\infty}$ of finite, connected graphs such that

(a) $\lim_{n\to\infty} |X_n| = \infty$;
(b) $\sup_n \operatorname{val}(X_n) < \infty$; and
(c) $\inf_n \lambda_1(X_n) > 0$.

It is not so easy to build an expander! It is an instructive exercise to show that $\inf_n \lambda_1(X_n) = 0$ for sequences of trees, portions of tessellations in the euclidean or hyperbolic plane, etc.

Expanders and Property T. If G is a group with a given symmetric[1] set of generators S then the Cayley graph of G with respect to S is the metric space comprised of G with the left-invariant word-length metric associated to S. The Cayley graph is connected and its valence is $|S|$.

Suppose now that Γ is an infinite property T group with a symmetric finite generating set S. Suppose that $\Gamma_1, \Gamma_2, \dots$ is a sequence of finite quotient groups of Γ such that $|\Gamma_n| \to \infty$, and denote by $S_1, S_2, \dots$ the sequence of generating sets for these groups obtained by projecting S into each group Γ_n. Alon and Milman made the following elegant and simple observation:

Lemma. *If X_n is the Cayley graph of Γ_n with respect to S_n then the sequence $\{X_n\}_{n=1}^\infty$ is an expander.*

(The point is that small positive eigenvalues for Δ on $\ell^2(X_n)$ correspond to almost fixed vectors for Γ in the natural representation on $\ell^2_0(\Gamma)$, where the subscript 0 denotes functions which are orthogonal to the constant functions.)

Expanders and Uniform Hilbert Space Embeddings. Let $\{X_n\}_{n=1}^\infty$ be a sequence of connected graphs. A *uniform embedding* of $\{X_n\}_{n=1}^\infty$ into a Hilbert space H is a sequence of functions $f_n\colon X_n \to H$ such that

(a) $d(f_n(x), f_n(y)) \le d(x,y)$, for all n and all $x, y, \in X_n$; and
(b) for every $R > 0$ there is some $k > 0$ (independent of n) such that if $x, y \in X_n$ and $d(x,y) \ge k$ then $d(f_n(x), f_n(y)) \ge R$.

Gromov has made the following observation:

Lemma. *If $\{X_n\}_{n=1}^\infty$ is an expander then there is no uniform embedding of $\{X_n\}_{n=1}^\infty$ into a Hilbert space.*

Here is a proof. Suppose our sequence of graphs satisfies

- $\lim_{n\to\infty} |X_n| = \infty$;
- $\sup_n \operatorname{val}(X_n) = v < \infty$; and
- $\inf_n \lambda_1(X_n) = \varepsilon > 0$.

Suppose given a family of functions $f_n\colon X_n \to H$ satisfiying conditions (a) and (b) above. By adjusting each f_n by a translation in H (that is, by adding a suitable constant vector-valued function to f_n) we can arrange that each f_n is orthogonal to every constant function in the Hilbert space of functions from X_n to H (we need just arrange that $\sum_{x\in X_n} f(x) = 0$). Let us assume that such an adjustment has been made — and let us note that conditions (a) and (b) still hold for the adjusted maps f_n. Now the Laplace operator can be defined on H-valued functions just as it was on scalar functions, and the expander property carries over to the

[1] *Symmetric* means that if $s \in S$ then $s^{-1} \in S$.

vector-Laplacian: $\inf_n \lambda_1(X_n) = \varepsilon > 0$. But notice that

$$\begin{aligned}\langle \Delta f_n, f_n \rangle &= \sum_{d(y,z)=1} |f(y) - f(z)|^2 \\ &\leq \sum_{d(y,z)=1} 1 \\ &\leq \mathrm{val}(X_n)|X_n| \\ &\leq v|X_n|.\end{aligned}$$

It follows from the expander property that

$$\|f_n\|^2 = \sum_{x \in X_n} \|f_n(x)\|^2 \leq v|X_n|/\varepsilon.$$

Thus at for at least half of the points $x \in X_n$ we have $\|f_n(x)\|^2 \leq 2v/\varepsilon$. But this contradicts part (b) of the definition of uniform embedding and the fact that $\lim_{n\to\infty} |X_n| = \infty$ while $\sup_n \mathrm{val}(X_n) < \infty$ since among the half of X_n we are concentrating on there must be points x_n and y_n with $\lim_{n\to\infty} d(x_n, y_n) = \infty$.

The C*-Algebra of a Graph. Let X be a finite-valence graph (it need not be connected, in which case we shall think of it as a metric space in which the value $d(x, y) = \infty$ is allowed, as discussed earlier). Denote by $B(X)$ the algebra of $X \times X$ complex matrices $[a_{xy}]$ such that

(a) $\sup_{x,y} |a_{xy}| < \infty$; and
(b) $\sup\{\, d(x, y) \mid a_{xy} \neq 0 \,\} < \infty$.

The algebra $B(X)$ is represented in an obvious way on $\ell^2(X)$ and we denote by $B^*(X)$ the C^*-algebra completion in this representation. It is sometimes called the *rough* C^*-algebra associated to the graph X (viewed as a metric space).

If G is a finitely generated group, and if X is its Cayley graph, then the right regular representation of G on $\ell^2(X)$ includes $C_r^*(G)$ as a C^*-subalgebra of the C^*-algebra $B^*(X)$. In fact it is not hard to show that

$$B^*(X) \cong C_r^*(G, C(\beta X)),$$

where βX is the Stone-Cech compactification of X, and the inclusion of $C_r^*(G)$ into $B^*(X)$ is the natural inclusion into the reduced crossed product.

Exact C*-Algebras. A C^*-algebra A is exact if and only if for every short exact sequence of C^*-algebras

$$0 \to J \to B \to C \to 0$$

the tensor product sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes C \to 0,$$

constructed using the minimal (a.k.a. spatial) tensor product, is exact.

It is not hard to see that C^*-subalgebras of exact C^*-algebras are exact. An optimistic conjecture in C^*-algebra theory asserts that every reduced group C^*-algebra $C_r^*(G)$ is exact. The evidence for this is that for a wide variety of groups one can construct an *amenable* action of G on a compact topological space X. The hypothesis of amenability implies that the reduced crossed product $C_r^*(G, C(X))$ is a nuclear C^*-algebra. Now, nuclear C^*-algebras are exact, and of course $C_r^*(G)$ is a C^*-subalgebra of $C_r^*(G, C(X))$, and so the existence of an amenable action implies the exactness of $C_r^*(G)$. This line of reasoning applies to discrete subgroups of Lie groups, Coxeter groups, hyperbolic groups, etc.

If a group acts amenably on any compact space at all then it acts amenably on its Stone-Cech compactification. So if G is exact by virtue of an amenable action then the rough C^*-algebra of its Cayley graph is nuclear. In fact the exactness of $B^*(X)$ follows from the exactness of $C_r^*(G)$:

Lemma. *If G is a finitely generated group and if X is its Cayley graph then $C_r^*(G)$ is exact if and only if $B^*(X)$ is exact.*

Proof. The exactness of $B^*(X)$ certainly implies the exactness of $C_r^*(G)$ since $C_r^*(G)$ is a C^*-subalgebra of $B^*(X)$. Suppose, conversely, that $C_r^*(G)$ is exact and that we are presented with an exact sequence of C^*-algbras

$$0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Since tensoring with a commutative C^*-algebra preserves exactness the sequence

$$0 \longrightarrow C(\beta G) \otimes J \longrightarrow C(\beta G) \otimes B \longrightarrow C(\beta G) \otimes C \longrightarrow 0$$

is exact. It is a result of Kirchberg and Wassermann that if G is exact then the operation of reduced crossed product by G preserves exact sequences. But in the present case upon forming reduced crossed products we obtain the exact sequence

$$0 \longrightarrow B^*(X) \otimes J \longrightarrow B^*(X) \otimes B \longrightarrow B^*(X) \otimes C \longrightarrow 0,$$

as required.

The above lines of thought might suggest the optimistic conjecture that the rough C^*-algebra of any finite valence graph is exact[2]. We shall see that this is wrong. In fact we shall see that the exactness conjecture for group C^*-algebras is almost certainly wrong too.

Exactness and Expanders. In this paragraph we will work with expanders of the specific sort considered earlier, constructed from infinite property T groups. Let us call these objects *T-expanders*.

If $\{X_n\}_{n=1}^{\infty}$ is an expander then denote by $X = \cup X_n$ the disjoint union of the X_n, which is a finite-valence but disconnected graph. We shall call both X and $\{X_n\}_{n=1}^{\infty}$ an expander (or T-expander, as the case may be).

[2]The extreme optimists might be led to conjecture that the rough C^*-algebra is always nuclear.

Proposition. *If X is a T-expander then the C^*-algebra $B^*(X)$ is not exact.*

Proof. We'll prove that $B^*(X)$ is not exact by constructing within it a C^*-subalgebra C which is not exact. There is an obvious choice for C: take the image of the right regular representation of $C^*(\Gamma)$ on the direct sum of the Hilbert spaces $\ell^2(\Gamma_n)$. It is known that associated to each irreducible, finite-dimensional unitary representation π of a property T group Γ there is a projection in $C^*(\Gamma)$ which acts in any unitary representation as the orthogonal projection onto the π-isotypical vectors. Denote by J the ideal in C generated by the images of all such projections. It is shown by Wassermann[3] that the sequence

$$0 \to C \otimes J \to C \otimes C \to C \otimes C/J \to 0$$

is *not* exact, and therefore C is not an exact C^*-algebra.

Putting various things together ...

Corollary. *If a finitely generated group G contains a T-expander as a subgraph of its Cayley graph then $C_r^*(G)$ is not exact.*

It seems very likely that Gromov has constructed examples of groups with just the above property. So the exactness conjecture for group C^*-algebras very likely to be false.

[3]He shows something very slightly different, but his argument carries over with basically no change.