Reminder: The Virial Theorem for Gaseous Systems

Assume you have a gravitationally bound system made up of an ideal gas in hydrostatic equilibrium. In the case of monotonic gas particles, the internal energy of this system is

\[ E_i = \int_0^{\mathcal{M}_T} u \, d\mathcal{M} = \int_0^{\mathcal{M}_T} \frac{3}{2} \frac{N_A}{\mu} kT \, d\mathcal{M} \]

If we substitute in the ideal gas law, this becomes

\[ P = \frac{\rho N_A}{\mu} kT \quad \Longrightarrow \quad E_i = \int_0^{\mathcal{M}_T} \frac{3}{2} \frac{P}{\rho} \, d\mathcal{M} \]

Now take the equation of hydrostatic equilibrium, multiply by \( 4 \pi r^3 \), and integrate over the entire cloud

\[ \int_0^R \frac{dP}{dr} \cdot 4\pi r^3 \, dr = - \int_0^R \frac{GM(r)}{r} \rho \cdot 4\pi r^2 \, dr \]
Virial Theorem

When integrated by parts, the left side of this

\[ \int_0^R \frac{dP}{dr} \cdot 4\pi r^3 dr = P(r) 4\pi r^3 \bigg|_0^R - \int_0^R 12\pi r^2 P \, dr \]

So

\[ P(R) 4\pi R^3 - \int_0^R 12\pi r^2 P \, dr = - \int_0^R \frac{G M(r)}{r} \rho \cdot 4\pi r^2 dr \]

The first term on the left side goes away, since the pressure at the surface is zero, and the right side of the equation is just the total gravitational potential of the cloud. Thus

\[ \int_0^R 12\pi r^2 P \, dr = 2 \int_0^R \frac{3}{2} P \cdot 4\pi r^2 \, dr = -E_{\text{grav}} \]

Or, since \( dM = 4 \pi r^2 \rho \, dr \)

\[ 2 \int_0^M 3 \frac{P}{2} dM = E_{\text{grav}} \implies 2 E_i + E_{\text{grav}} = 0 \]
Virial Theorem

Note that the result of this simple derivation, \( 2 E_i + E_{\text{grav}} = 0 \) does not depend on \( M(r) \), \( \rho(r) \), or \( T(r) \), and can be generalized for other equations of state. For an ideal gas

\[
P = \frac{\rho N_A}{\mu} k T = \frac{2}{3} \rho u
\]

but more generally, we can write \( P = (\gamma - 1) \rho u \), where \( \gamma = 5/3 \) for an ideal gas. Under this law, the internal energy becomes

\[
E_i = \int_0^{\mathcal{M}_T} \frac{1}{\gamma - 1} \frac{P}{\rho} \, d\mathcal{M}
\]

And the virial theorem becomes

\[
3(\gamma - 1) E_i + E_{\text{grav}} = 0
\]
Virial Theorem

Finally, let’s consider the cloud’s total energy, \( W = E_i + E_{\text{grav}} \).

\[
W = -\frac{E_{\text{grav}}}{3(\gamma - 1)} + E_{\text{grav}} = \frac{3\gamma - 4}{3\gamma - 3} E_{\text{grav}}
\]

Note the implication of this equation. Because the system must have a non-zero temperature, it must radiate some of its energy into space. Thus, through energy conservation

\[
\mathcal{L}_T + \frac{dW}{dt} = 0 \implies \mathcal{L}_T = -\frac{3\gamma - 4}{3\gamma - 3} \frac{dE_{\text{grav}}}{dt}
\]

This means that as the cloud radiates, its gravitational energy will get more negative, but this decrease will be less than the energy radiated. The remaining energy will go into heating the cloud. In the case of an ideal gas, half the energy radiated will go into \( E_{\text{grav}} \), the other half into \( E_i \).

Note also that as long as \( \gamma > 4/3, W > 0 \). If \( \gamma < 4/3 \), the total energy will be negative, and the system will collapse.
Jeans Criterion

A cloud will collapse if the gravitational potential is stronger than its thermal (and magnetic) support. In other words $|E_{\text{grav}}| > 2E_i$. As we just saw, the internal energy is

$$E_i = \int_0^{M_T} \frac{1}{\gamma - 1} \frac{P}{\rho} \, dM = \int_0^{M_T} \frac{1}{\gamma - 1} \frac{1}{\mu m_H} kT \, dM$$

For simplicity, let’s take a uniform density, isothermal cloud. For such a cloud

$$|E_{\text{grav}}| = \int_0^R \frac{GM(r)}{r} \, dM = \frac{3}{5} \frac{G M^2}{R} = 2E_i = \frac{2}{\gamma - 1} \frac{1}{\mu m_H} kT M$$

or

$$M_J = \frac{5}{3} \frac{2}{\gamma - 1} \frac{1}{G \mu m_H} kT R$$
Jeans Criterion

If we substitute density for radius via \( \rho = \frac{M}{\frac{4}{3} \pi R^3} \), this becomes

\[
M_J > \left\{ \frac{5}{3} \frac{2}{\gamma - 1} \frac{1}{G \mu m_H} kT \right\}^{\frac{3}{2}} \left\{ \frac{3}{4\pi \rho} \right\}^{\frac{1}{2}} \sim 10^5 M_\odot \left( \frac{T}{100 \text{ K}} \right)^{\frac{3}{2}} \left( \frac{n}{\text{cm}^{-3}} \right)^{-\frac{1}{2}}
\]

This is the Jeans mass. If we re-write the criterion in terms of the sound speed, \( c_s = (\gamma P/\rho)^{1/2} \), and assume an ideal, monotonic gas (\( \gamma = 5/3 \)), gravity will overcome the thermal energy when

\[
M_J > \frac{9}{2} \frac{c_s^3}{G^{3/2} \sqrt{\pi \rho}}
\]

Similarly, we can replace mass with density to derive the Jean’s length

\[
R_J > \left( \frac{15kT}{4\pi G \mu m_H \rho} \right)^{1/2} = \frac{3}{2} c_s \left( \frac{1}{\pi G \rho} \right)^{1/2}
\]

All scales larger than this are unstable to collapse.
Another way of looking at the collapse problem is to compare the timescale for free fall collapse to the timescale for a pressure wave to propagate across the cloud and restore equilibrium.

In the absence of gas or magnetic pressure, the free-fall collapse time is $\tau_{ff} \sim (1 / G \rho)^{1/2}$, while that for a pressure wave is simple $\tau_s \sim R / c_s$. Equating these expression (while replacing density with radius) yields

$$\mathcal{M}_J \sim \frac{c_s^3}{\sqrt{\rho G^3}}$$

Note that the Jeans Criterion is somewhat of a swindle, since it assumes that the gas immediately outside the cloud is static. (In fact, it will be collapsing as well.) Note also that for a better estimate, one needs to include external and magnetic pressure in the virial theorem, i.e.,

$$2 E_i + E_{grav} + B - 3 P_{ext} V = 0$$
As gas collapses in a cloud core, the density increases, causing the Jeans mass to decrease. This leads to fragmentation, as smaller regions of the cloud are pushed over the Jeans mass.

- The increase in density causes the cloud to become opaque to IR photons. The trapped energy heats the cloud, increases the gas pressure, and brings the system into hydrostatic equilibrium.

- Due to the cloud’s angular momentum, an accretion disk is formed. From the virial theorem, half the energy of accretion heats the star, half is radiated, so

\[ \mathcal{L}_{\text{acc}} = \frac{G M \dot{M}}{2R} \]
Collapse and Fragmentation

• In general, the timescale for accretion, \( \tau_{\text{acc}} \sim \frac{\dot{M}}{M} \), is shorter than the protostar’s thermal timescale. The protostar cannot thermally adjust to the added energy, and therefore heats up adiabatically.

• Eventually, the protostar becomes hot enough to disassociate molecular hydrogen. When this happens, the energy input does not go into heating the gas, but disassociating the hydrogen. This means that the specific heat of the protostar is \( \gamma < \frac{4}{3} \), and, through the virial theorem, the star must collapse on a dynamical timescale. This collapse releases energy, which goes into further \( \text{H}_2 \) disassociation (rather than heating the gas). Very quickly, all the molecular hydrogen is destroyed, and the protostar settles into a new hydrostatic equilibrium. Similar episodes occur when the temperature becomes hot enough to ionize hydrogen and helium, \( T \sim 10^4 \text{ K} \).
Herbig-Haro Objects

- Bipolar outflows, powered by the accretion, break through where the disk has the lowest density (out the poles). These patches of nebulosity are called Herbig-Haro objects.
Young Stellar Objects (YSOs)

- Finally, accretion slows down, and the star creates energy via gravitational collapse. The pre-main sequence phase begins where, from the virial theorem, the interior temperature increases as

\[
\frac{GM}{R} \sim kT \implies T \sim M^{2/3} \rho^{1/3}
\]
Pre-Main Sequence Evolution

- Prior to hydrogen ignition, the star simply evolves via contraction, on a thermal timescale

\[ t_{KH} = \frac{GM^2}{RL} \]

Evolution proceeds nearly vertically, down the Hayashi track, which, by definition, is the location of fully convective stars. (In other words, \( \mathcal{L} \) changes, but \( T \) is nearly constant.)
Pre-Main Sequence Evolution

- Prior to hydrogen ignition, the star simply evolves via contraction, on a thermal timescale

\[ t_{KH} = \frac{GM^2}{RL} \]

Eventually, the protostar heats up until a radiative zone is formed in the core. This new mechanism of energy transport changes the protostar’s structure and evolution; instead of moving vertically, it moves horizontally on the HR diagram, i.e., at constant \( \mathcal{L} \) but changing \( T \).
Pre-Main Sequence Evolution

- Prior to hydrogen ignition, the star simply evolves via contraction, on a thermal timescale

\[ t_{KH} = \frac{GM^2}{RL} \]

Note that due to the thermal timescale (and the strong dependence of luminosity with mass), high mass stars will evolve much more quickly than low mass stars.
**T Tauri Stars**

Low mass, variable, gravitationally contracting objects are called T Tauri stars. Based on the strength of Hα, (which presumably comes from the disk) they are sub-classed as “Classical”, “Weak-Lined”, and “Naked”.

Because T Tauri stars are young, they are rotating relatively fast, and because they have convective envelopes, their magnetic fields can become wrapped up (like the Sun), causing flares. Young stars are often x-ray sources (and the x-ray emission can last longer than the disk).
Lithium and Young Stars

T Tauri stars have convective envelopes. Brittle species like lithium will be destroyed by the high temperatures at the bottom of the convective layer. The presence of lithium is thus an age indicator.
Types of Young Stars

**T Tauri Stars:** F, G, K, and M stars with Hα and excess IR emission (presumably from their disk)

**FU Orionis Stars:** T Tauri stars with extreme variability (both in brightness and spectral type) presumably from flares

**Herbig Ae Be Stars:** A- and B-type counterparts to T Tauri stars, with Hα emission and IR excess

**UX Orionis Stars:** Herbig Ae Be stars with extreme variability (in color and magnitude) due to changes in the extinction caused by dust in their disks.