WRIGHT'S THEOREM ON WARING'S PROBLEM IN SHORT INTERVALS

1. The theorem

I was drawn to the following generalization of a result of E. Maitland Wright [1937] by a paper of Kirsti Biggs [2018].

Theorem 1.1 (Wright, 1937)**.** . *Let* $k \in \mathbb{N}$ *with* $k \geq 2$ *and* $s \in \mathbb{N}$. *Then there is a positive number* $\xi(k, s)$ *such that there are infinitely many* n *for which the equation*

$$x_1^k + \cdots x_s^k = n \tag{1.1}$$

has no solutions in positive integers x_i *with*

$$\left| x_i^k - \frac{n}{s} \right| < \xi(s, k) n^{1-\frac{1}{2k}}. \tag{1.2}$$

The exponent $1 - \frac{1}{2k}$ here is quite surprising. I would have expected that $1 - \frac{1}{k}$ would have been best possible for large s. The proof shows that no large n of the form

$$sm^k + skm^{k-1}$$

can have the desired representation. It would seem that n could be varied by a small amount from this value, and it would be of some interest to know what is the maximal interval in which there is no representation.

2. The proof

The proof is quite explicit. The number $\xi \in (0, 1)$ will otherwise be at our disposal, and will eventally be chosen in terms of s and k, and we suppose m is large compared with s, k and ξ and then define

$$n = sm^k + skm^{k-1}.$$

We argue by contradiction. Suppose that there are x_i $(1 \le i \le s)$ such that (1.1) holds and

$$\left|x_i^k - \frac{n}{s}\right| < \xi n^{1-\frac{1}{2k}}.$$

Let

$$a_i = x_i - m.$$

Since we cannot have $x_i = m$ for every i we have

$$\sum_{i=1}^{s} a_i^2 \ge 1. \tag{2.1}$$

Also, as m is large we have $x_i > 0$ and

$$\begin{aligned}\left|x_i^k - \frac{n}{s}\right| &= \left|x_i - \left(\frac{n}{s}\right)^{1/k}\right| \left(x_i^{k-1} + \cdots + \left(\frac{n}{s}\right)^{(k-1)/k}\right)\\ &\ge \left|x_i - \left(\frac{n}{s}\right)^{1/k}\right| \left(\frac{n}{s}\right)^{(k-1)/k}.\end{aligned}$$

We also have

$$\begin{aligned}&m^{k-1}\left|\left(\frac{n}{s}\right)^{1/k} - m\right|\\ &\quad\le \left(\left(\frac{n}{s}\right)^{(k-1)/k} + \cdots + m^{k-1}\right)\left|\left(\frac{n}{s}\right)^{1/k} - m\right|\\ &\quad= \left|\frac{n}{s} - m^k\right|.\end{aligned}$$

Thus

$$\begin{aligned}m^{k-1}|a_i| &= m^{k-1}|x_i - m|\\ &\le m^{k-1}\left|x_i - \left(\frac{n}{s}\right)^{1/k}\right| + m^{k-1}\left|\left(\frac{n}{s}\right)^{1/k} - m\right|\\ &\le m^{k-1}\left(\frac{s}{n}\right)^{(k-1)/k}\left|x_i^k - \frac{n}{s}\right| + \left|\frac{n}{s} - m^k\right|\\ &\le m^{k-1}\left(\frac{s}{n}\right)^{(k-1)/k}\xi n^{1-\frac{1}{2k}} + km^{k-1}.\end{aligned}$$

Hence

$$|a_i| \leq s\xi n^{\frac{1}{2k}} + k \leq 2s\xi n^{\frac{1}{2k}} \leq 3s^2\xi m^{\frac{1}{2}}.$$

We are supposing that (1.1) holds. Thus

$$0 = \left| \sum_{i=1}^{s} (m+a_i)^k - sm^k - skm^{k-1} \right|$$

$$= \left| \sum_{i=1}^{s} \sum_{j=1}^{k} \binom{k}{j} m^{k-j} a_i^j - skm^{k-1} \right| \quad (2.2)$$

and so

$$0 \geq km^{k-1} \left| \sum_{i=1}^{s} a_i - s \right| - \left| \sum_{j=2}^{k} \binom{k}{j} m^{k-j} \sum_{i=1}^{s} a_i^j \right|.$$

Thus

$$\left| \sum_{i=1}^{s} a_i - s \right| \leq \left| \sum_{j=2}^{k} \binom{k}{j} k^{-1} m^{1-j} \sum_{i=1}^{s} a_i^j \right|$$

$$\leq \sum_{j=2}^{k} \binom{k}{j} k^{-1} m^{1-j} s \left(3s^2 \xi m^{\frac{1}{2}} \right)^j$$

$$\leq C(k,s)\xi^2.$$

for some positive number $C(k,s) \geq 1$ depending only on k and s. Thus if ξ is taken to be, for example,

$$\frac{1}{\sqrt{2C(k,s)}}$$

we find that the integer

$$\sum_{i=1}^{s} a_i - s$$

is 0.

Inserting this in (2.2) we have

$$0=\sum_{j=2}^{k}\sum_{i=1}^{s}\binom{k}{j}m^{k-j}\sum_{i=1}^{s}a_i^j.$$

If $k=2$ this gives

$$\sum_{i=1}^{s}a_i^2=0$$

which contradicts (2.1). Thus we may suppose that $k\geq 3$. Then

$$\binom{k}{2}m^{k-2}\sum_{i=1}^{s}a_i^2=\left|\sum_{j=3}^{k}\sum_{i=1}^{s}\binom{k}{j}m^{k-j}a_i^j\right|$$
$$\leq\sum_{j=3}^{k}\binom{k}{j}m^{k-j}\left(3\xi s^2m^{\frac{1}{2}}\right)^{j-2}\sum_{i=1}^{s}a_i^2.$$

and so, by (2.1). we have

$$\binom{k}{2}\leq C_1(k,s)m^{-1/2}$$

which is impossible when m is sufficiently large.

References

[2018] Kirsti Biggs, Almost equal summands in Waring's Problem with shifts, Monatshefte für Mathematik 188(2019), 31–35.

[1937] E. M. Wright, The representation of a number as a sum of four 'almost equal' squares, Oxford Quarterly J. of Math. 8(1937), 278–279.