A THEOREM OF JARNIK

Theorem. (Jarník (1926); cf Bombieri & Pila (1989)) Let \( C \) be a simple closed curve in the plane, of arc length \( L \). The number of ‘lattice points’ \((m, n), m, n \in \mathbb{Z}\), lying on \( C \) is at most \( L + 1 \). If \( C \) is strictly convex, then the number of lattice points on \( C \) is \( \ll 1 + L^{2/3} \), and this estimate is best-possible.

Proof. Let \( k \) be the number of lattice points on the curve. If \( k = 0 \) or 1, then the problem is trivial. Thus we suppose that \( k \geq 2 \) throughout. We label the lattice points as \( P_j = (m_j, n_j) \) \( 1 \leq j \leq k \) in order along the curve (for example clockwise starting from some convenient point) and define \( P_0 = (m_0, n_0) = (m_k, n_k) \). Let \( q_j = m_j - m_{j-1} \), \( a_j = n_j - n_{j-1} \). Then the length of the curve from \( P_{j-1} \) to \( P_j \) is at least the length of the shortest distance between them, namely \((a_j^2 + q_j^2)^{1/2}\).

Thus \( \sum_{j=1}^{k} (a_j^2 + q_j^2)^{1/2} \leq L \). Moreover as the points \( P_j \) are distinct and the \( a_j \) and \( q_j \) are integers we have \((a_j^2 + q_j^2)^{1/2} \geq 1 \) and so \( k \leq L \).

Now suppose that \( C \) is strictly convex. The ratio \( a_j/q_j \) (in the extended number system) represents the gradient of the straight line \( l_j \) joining \( P_{j-1} \) and \( P_j \) and these lines can be divided into four groups of of consecutive lines \( l_j \) according as \(-1 \leq a_j/q_j < 1, q_j > 0; -1 \leq q_j/a_j < 1, a_j > 0; -1 \leq a_j/q_j < 1, q_j < 0; -1 \leq q_j/a_j < 1, a_j < 0 \). The strict convexity implies that in each group the ratios are distinct (and indeed form a strictly monotonic sequence). Let \( k_i \) denote the number of members of the \( i \)-th group, so that \( k_1 + k_2 + k_3 + k_4 = k \). Thus it suffices to show that \( k_i \ll 1 + L^{2/3} \). Moreover we may suppose that \( k_i \geq 4 \). Since the ratios are distinct each one has a unique representation as \( a/q \) with \((a, q) = 1, q \geq 1 \) and \(-q \leq a < q \). Thus the number of members of the \( i \)-th group with denominator not exceeding \( Q \) in absolute value is bounded by \( 1 + \sum_{q \leq Q} 2q \leq 1 + 2Q^2 \). Let \( Q = \frac{1}{3}(k_i)^{1/2} \). Then \( 1 + 2Q^2 = 1 + \frac{2}{3}k_i \leq \frac{1}{2}k_i \). Hence for at least \( \frac{1}{2}k_i \) of the ratios at least one of \( a_i \) or \( q_i \) exceeds \( \frac{1}{2}(k_i)^{1/2} \) in absolute value. Hence \( \frac{1}{2}k_i \ll \frac{1}{2}(k_i)^{1/2} \leq k_i \ll L^{2/3} \).

To show that this is best possible we observe that the number \( F(Q) \) of fractions \( a/q \) with \( 1 \leq a \leq q \leq Q \) and \((a, q) = 1 \) (the number of Farey fractions of order \( Q \)) is \( \sum_{q \leq Q} \phi(q) = \frac{3}{\pi^2}Q^2 + O(Q \log Q) \). Now consider the fractions \( a_j/q_j \) with \( 0 \leq a_j \leq Q, 1 \leq q_j \leq Q \) and \((a_j, q_j) = 1 \) indexed in increasing order, so that \( 0 = a_1/q_1 < a_2/q_2 < \ldots \). Their number is \( 1 + F(Q) + \sum_{2 \leq a_j \leq Q} \phi(a_j) = 2F(Q) + \frac{6}{\pi^2}Q^2 + O(Q \log Q) \). We list these fractions in order as \( 0 < \frac{a_1}{q_1} < \frac{a_2}{q_2} < \ldots \).

Then we construct points \( P_j \) by taking a suitable origin, e.g. \( P_0 = (0, 0) \) and define successively \( P_j = P_{j-1} + (q_j, a_j) \). Let the last point constructed be \( P_j \). We now add further points by taking the configuration of points just constructed, rotating it through \( 90^\circ \) and moving \( P_0 \) to coincide with \( P_j \). We then rotate and translate two more times to obtain a complete circuit of points. Now we join the points by line segments and consider the resulting convex polygon. The number of integer points on the curve is asymptotically \( \frac{24}{\pi^2}Q^2 \). The length of the curve is \( 4 \sum_{1 \leq q \leq Q} \sum_{0 \leq a \leq Q, (a, q) = 1} (a^2 + q^2)^{1/2} \ll Q^3 \).
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REFERENCES
