A THEOREM OF JARNIK

Theorem. *(Jarník (1926); cf Bombieri & Pila (1989)) Let $\mathcal{C}$ be a simple closed curve in the plane, of arc length L. The number of 'lattice points' (m,n), $m,n \in \mathbb{Z}$, lying on $\mathcal{C}$ is at most $L+1$. If C is strictly convex, then the number of lattice points on $\mathcal{C}$ is $\ll 1+L^{2/3}$, and this estimate is best-possible.*

Proof. Let k be the number of lattice points on the curve. If $k=0$ or 1, then the problem is trivial. Thus we suppose that $k \geq 2$ throughout. We label the lattice points as $P_j=(m_j,n_j)$ $1 \leq j \leq k$ in order along the curve (for example clockwise starting from some convenient point) and define $P_0=(m_0,n_0)=(m_k,n_k)$. Let $q_j=m_j-m_{j-1}$, $a_j=n_j-n_{j-1}$. Then the length of the curve from P_{j-1} to P_j is at least the length of the shortest distance between them, namely $(a_j^2+q_j^2)^{1/2}$. Thus $\sum_{j=1}^k (a_j^2+q_j^2)^{1/2} \leq L$. Moroever as the points P_j are distinct and the a_j and q_j are integers we have $(a_j^2+q_j^2)^{1/2} \geq 1$ and so $k \leq L$.

Now suppose that $\mathcal{C}$ is strictly convex. The ratio a_j/q_j (in the extended number system) represents the gradient of the straight line l_j joining P_{j-1} and P_j and these lines can be divided into four groups of of consecutive lines l_j according as $-1 \leq a_j/q_j < 1, q_j > 0$; $-1 \leq q_j/a_j < 1, a_j > 0$; $-1 \leq a_j/q_j < 1, q_j < 0$; $-1 \leq q_j/a_j < 1, a_j < 0$. The strict convexity implies that in each group the ratios are distinct (and indeed form a strictly monotonic sequence). Let k_i denote the number of members of the i–th group, so that $k_1+k_2+k_3+k_4=k$. Thus it suffices to show that $k_i \ll 1+L^{2/3}$. Moreover we may suppose that $k_i \geq 4$. Since the ratios are distinct each one has a unique representation as a/q with $(a,q)=1$, $q \geq 1$ and $-q \leq a < q$. Thus the number of members of the i–th group with denominator not exceeding Q in absolute value is bounded by $1+\sum_{q\leq Q} 2q \leq 1+2Q^2$. Let $Q=\frac{1}{3}(k_i)^{1/2}$. Then $1+2Q^2=1+\frac{2}{9}k_i \leq \frac{1}{2}k_i$. Hence for at least $\frac{1}{2}k_i$ of the ratios at least one of a_i or q_i exceeds $\frac{1}{3}(k_i)^{1/2}$ in absolute value. Hence $\frac{1}{2}k_i\frac{1}{3}(k_i)^{1/2} \leq \sum_{j=1}^k (a_j^2+q_j^2)^{1/2} \leq L$ and it follows that $k_i \ll L^{2/3}$.

To show that this is best possible we observe that the number $F(Q)$ of fractions a/q with $1 \leq a \leq q \leq Q$ and $(a,q)=1$ (the number of Farey fractions of order Q) is $\sum_{q\leq Q}\phi(q)=\frac{3}{\pi^2}Q^2+O(Q\log Q)$. Now consider the fractions a_j/q_j with $0 \leq a_j \leq Q$, $1 \leq q_j \leq Q$ and $(a_j,q_j)=1$ indexed in increasing order, so that $0=a_1/q_1 < a_2/q < \dots$. Their number is $1+F(Q)+\sum_{2\leq a_j \leq Q}\phi(a_j)=2F(Q)=\frac{6}{\pi^2}Q^2+O(Q\log Q)$. We list these fractions in order as $0<\frac{a_1}{q_1}<\frac{a_2}{q_2}<\dots$. Then we construct points P_j by taking a suitable origin, e.g. $P_0=(0,0)$ and define successively $P_j=P_{j-1}+(q_j,a_j)$. Let the last point constructed be P_J. We now add further points by taking the configuration of points just constructed, rotating it through 90° and moving P_0 to coincide with P_J. We then rotate and translate two more times to obtain a complete circuit of points. Now we join the points by line segments and consider the resulting convex polygon. The number of integer points on the curve is asymptotically $\frac{24}{\pi^2}Q^2$. The length of the curve is $4\sum_{1\leq q\leq Q}\sum_{0\leq a\leq Q,(a,q)=1}(a^2+q^2)^{1/2} \ll Q^3$.

References

E. Bombieri & J. Pila (1989). *The number of integral points on arcs and ovals*, Duke Math. J. **59**, 337–357.

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