Heuristics for convergence part of Khinchin

Suppose that the k–dimensional surface $\mathcal{S}$ in n–dimensional space is parameterised by the

$$(x_1, \dots, x_k, f_{k+1}(\mathbf{x}), \dots, f_n(\mathbf{x}))$$

with $\alpha_j \le x_j \le \beta_j$. For a given q consider those $\mathbf{x}$ for which there are $\mathbf{a}$ with

$$|x_j - a_j/q| \le \psi(q)/q \quad (1 \le j \le k) \tag{1}$$

and

$$|f_j(\mathbf{x}) - a_j/q| \le \psi(q)/q \quad (k < j \le n). \tag{2}$$

Presuming the f_j are sufficiently smooth, the a_j, by substituion for the x_j, must satisfy

$$\|qf_j(\mathbf{a}/q)\| \ll \psi(q) \quad (k < j \le n). \tag{3}$$

Moreover the k–dimensional measure of that part of $\mathcal{S}$ which satisfies the inequalities (1) and (2) is $\ll (\psi(q)/q)^k$. Hence the total measure of the subset of $\mathcal{S}$ which has an approximation with denominator q with $Q < q \le 2q$ is

$$\ll \sum_{Q<q\le 2Q} (\psi(q)/q)^k N(q)$$

where $N(q)$ is the number of $\mathbf{a}$ satisfying (3). Assuming that $\psi(q)$ is decreasing, a bound of the kind

$$\text{card}\{q, \mathbf{a} : Q < q \le 2Q, q \ll a_j \ll q, \|qf_j(\mathbf{a}/q)\| \ll \Psi \quad (k < j \le n)\} \ll \Psi^{n-k} Q^{k+1}$$

combined with partial summation should show that the total measure of points having infinitely many approximations is arbitrarily small, provided that

$$\sum_{q=1}^{\infty} \psi(q)^n$$

converges.