1. **Rouché's Theorem**

Theorem 1.1 (Estermann)**.** *Suppose that f and g are holomorphic on a domain $\mathcal{D}$, that $\mathcal{C}$ is a simple closed contour in $\mathcal{D}$ and that*

$$|f(z) - g(z)| < |f(z)| + |g(z)| \tag{1.1}$$

for all $z \in \mathcal{C}$. Then f and g have the same number of zeros inside $\mathcal{C}$.

Proof. Clearly $f(z)g(z) \neq 0$ for $z \in \mathcal{C}$. Let $F(z) = g(z)/f(z)$ whenever $f(z) \neq 0$. Then for $z \in \mathcal{C}$ we have $|1 - F(z)| < 1 + |F(z)|$. Le $\mathcal{L} = \mathbb{C} \backslash \{z \in \mathbb{R} : z \leq 0\}$. Then and for $z \in \mathcal{C}$ we have $F(z) \in \mathcal{L}$. By considering the family of open disks $D(z, \delta_z)$ where $z \in \mathcal{C}$ and $\delta_z > 0$ it follows that there is an open subset $\mathcal{A}$ of $\mathcal{D}$ such that $\mathcal{C} \subset \mathcal{A}$, $F(\mathcal{A}) \in \mathcal{L}$, and F is holomorphic on $\mathcal{A}$. Now we can define $G(z) = \log F(z)$ for $z \in \mathcal{A}$ so that G is holomorphic on $\mathcal{A}$ and $G'(z) = \frac{F'}{F}(z)$. For example, take $G(z) = \log |F(z)| + i \arg F(z)$, the principal value of the logarithm. Now for $z \in \mathcal{C}$, $G'(z) = \frac{g'}{g}(z) - \frac{f'}{f}(z)$ and so

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \left(\frac{g'}{g}(z) - \frac{f'}{f}(z) \right) dz = 0.$$

□

Corollary 1.1 (Rouché)**.** *Suppose that f and h are holomorphic on a domain $\mathcal{D}$, that $\mathcal{C}$ is a simple closed contour in $\mathcal{D}$ and that*

$$|h(z)| < |f(z)| \tag{1.2}$$

for all $z \in \mathcal{C}$. Then f and $f + h$ have the same number of zeros inside $\mathcal{C}$.

Proof. Let $g(z) = f(z) + h(z)$ so that the hypothesis of the theorem is satisfied. Then f and $g(z) = f(z) + h(z)$ have the same number of zeros inside $\mathcal{C}$ □