Dimensional analysis as the *other* language of physics

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We review the use of dimensional analysis as a tool for the systematic study and analysis of physical concepts and phenomena at multiple levels in the physics curriculum. After reviewing the methodology of its use and citing examples from classical physics, we illustrate how it can be applied to problems in quantum mechanics, including research-level problems, noting both its power and its limitations.
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I. INTRODUCTION

Physics majors are expected to have a diverse skill set or "tool box" after completing an undergraduate degree program. Their experiences generally include gaining a broad physical intuition, being able to perform "back-of-the-envelope" numerical calculations, having experimental skills across different technical areas (optics, electronics, etc.), being adept at the use of graphical and statistical data analysis and visualization tools, and possessing substantial programming abilities. But above all, students majoring in physics must typically meet the most formal requirements in terms of mathematics coursework of any program, aside from mathematics itself.

The central role played by advanced mathematics in physics is perhaps most famously described in a quote by Eugene Wigner:[1]

> *"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."*

On the other hand, abstract, advanced, or very specialized mathematical techniques can just as often confuse, confound, and even discourage students at many stages of their physics careers. Furthermore, such methods are often well beyond the level of students early in their physics education, and may never be amenable for use in a "concepts"-type course or in service classes for non-physics majors.

It can be useful, then, to have other unifying descriptions or themes that cut across all subfields of physics, ones that still allow students to explore, in a serious and nontrivial way, both classical and more contemporary physics topics, but with a minimal amount of abstract mathematical machinery. We will argue here that dimensional analysis can be such a powerful tool.

Dimensional analysis should be familiar, in one sense, to all students in physics, even if only from the admonition to "make sure that the units work in your final answer" or similar *a posteriori* checks. We have in mind here, however, a more proactive and *a priori* methodology to confront (if not solve completely) many problems, by asking "up front" if one can extract the necessary functional dependence of some intermediate or final answer on the physical parameters relevant to the problem. Having identified such variables, it can be the case that the physical dimensions of the desired final result can, in fact, completely determine (or at least heavily constrain) the dependence on those values, either as being rational powers of combinations of the parameters and/or as functions of dimensionless ratios. This type of approach can be useful to physics students at every stage of their careers, but seems to be increasingly unfamiliar to many.

Dimensional analysis in this form is familiar in engineering, and many of the standard textbooks on the subject[2] have such an audience in mind. Even when this topic is discussed in *Physics Today*, the emphasis can still be on more applied areas.[3] One intent of this review is to encourage the use of such methods in the undergraduate physics curriculum, at all levels of instruction, and especially beyond standard problems in classical physics, in part by providing a number of useful examples.

More fundamentally (and perhaps more metaphorically), we want to stress the fact that dimensional analysis can be used, almost literally, as *another* language to describe physical phenomena. The fact that almost all physical quantities have dimensions that can be written in terms of rational powers of a handful (quite literally, five) of basic dimensions is a powerful unifying theme in physics. We will explore how a basic "alphabet" of dimensions, including length (L), time (T), mass (M), charge (Q), and temperature (Θ), can be used to "encode" the description of the vast majority of physical quantities that students will encounter. These dimensional quantities mesh well with five of the seven MKSA-Kelvin units used in the current International System of Units (SI); the other two SI units, a measure of quantity (Avogadro's number) and a unit of luminous intensity (candela), are used far less frequently in problems arising in physics and so will not be discussed here.[4]

In Sec. II, we provide a lightning review of introductory physics and the "alphabet" of dimensional constructs necessary for describing most physical quantities. Along the way, we will also collect some fundamental constants of nature, which we will think of as "basic words" that appear to be essential constructs. We focus especially on the constants c, e, $\hbar$, and k_B, as the SI will begin to make use of them as base quantities in "A more fundamental International System of Units" starting in 2018.[5]

In Sec. III, we then review the standard methods of dimensional analysis as used to predict the dependences (or constraints) on physical systems in an *a priori* way. Then, in Sec. IV, we provide a number of more modern examples of the use of dimensional analysis as applied to problems in quantum physics, even approaching recent research problems. We conclude in Sec. V with some general comments.

II. REVIEW OF DIMENSIONAL ANALYSIS: ESTABLISHING THE "ALPHABET" AND SOME UNIFYING "WORDS"

In a typical introductory physics text (calculus-based or not), one usually encounters classical mechanics first, beginning with the study of one-dimensional kinematics. In this context, we need only two dimensional quantities, length and time, which we will denote by L and T. We will use the (seemingly) standard notation that $[\chi]$ is to be interpreted (or read) as "the dimensions of χ," so that we have

$$[x(t)] = L, \quad [v(t)] = \frac{L}{T}, \quad \text{and} \quad [a(t)] = \frac{L}{T^2}, \tag{1}$$

for position, velocity, and acceleration, respectively. While we might advocate the use of specific *units* such as meters and seconds in most problem-based applications, one could, in principle, just as logically use furlongs and fortnights.

Extending classical mechanics to include Newton's laws, we need the concept of inertial mass, with a new dimension (M), where some familiar dimensionful quantities are given by

$$[F] = \frac{ML}{T^2}, \quad [E] = [KE] = [PE] = [W] = \frac{ML^2}{T^2},$$
$$[P] = \frac{M}{LT^2}, \quad \text{and} \quad [\rho] = \frac{M}{L^3}, \tag{2}$$

for force, energy (kinetic, potential, etc., including work), pressure, and density, respectively. Much of classical mechanics, fluid mechanics, many engineering applications, and even the formalism of quantum mechanics require only these three dimensions, L, T, and M.

It is often in the context of classical mechanics that students encounter one of the first true "unifications" in physics, namely, the law of universal gravitation

$$F_G = \frac{Gm_1m_2}{r^2}, \tag{3}$$

with the same force applying to apples and planets, and with Newton's constant G as a truly fundamental constant of nature. For future reference, we note that

$$[G] = \frac{L^3}{MT^2}. \tag{4}$$

We also note in passing that the principle of equivalence states that inertial and gravitational masses are one and the same, so the same basic M dimensionality is used for both.

Later in an introductory course, when one treats electricity and magnetism (EM), a useful "bumper sticker" or "fortune cookie" description might be

> *"Electric fields are caused by static charges and magnetic fields are caused by moving charges"*

and one can choose the required new dimensional quantity to be charge, denoted here by $[q] = Q$. The situation regarding units for EM problems has a much more storied history than that for mechanical dimensions, with extensive discussions[6,7] in the very early pages of this journal. We note that in a recent edition of the most widely used graduate textbook on electrodynamics,[8] the author not only includes an exhaustive discussion of the relative merits of MKS versus CGS units, but even chooses to "switch gears" between the two, halfway through the book. We also recall that for metrological reasons the basic SI dimensionful quantity currently related to charge is actually the Ampere, hence the MKSA label.

Imagining, as we do, that our discussion will be mostly used for undergraduates (majors and non-majors alike, at all levels), we choose to focus on charge (Q) as the new basic dimension underlying EM, and are happy to admit that we will always have in the back of our mind an MKSA-like system, but with the Coulomb as the explicit realization of that new dimension, not the Ampere. In that context, some of the basic quantities encountered in EM problems have dimensions

$$[I] = \frac{Q}{T}, \quad [V] = [\mathcal{E}] = \frac{ML^2}{QT^2}, \quad [C] = \frac{Q^2T^2}{ML^2},$$
$$[R] = \frac{ML^2}{Q^2T}, \quad \text{and} \quad [B] = \frac{M}{QT}, \tag{5}$$

for current, voltage/EMF, capacitance, resistance, and magnetic field, respectively.

If we continue to imagine an MKSA-type description of electric and magnetic fields, we can use the forms of Gauss' law and Ampere's law in this language (in differential form, for static fields), namely,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{6}$$

or similarly the electric and magnetic force laws of Coulomb and Biot–Savart

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \quad \text{and} \quad \mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2}\, dl', \tag{7}$$

to describe two new fundamental constants governing the laws of electricity (ϵ_0) and magnetism (μ_0). The extension of these results by Maxwell also led to another of the great early unifications of physics, bringing together electricity, magnetism, and optics, with their fundamental constants being related by $c = 1/\sqrt{\epsilon_0\mu_0}$. Finally, in contrast to the situation for mechanics, there is a fundamental quantum of charge, namely, e, which will be important for many physical processes.

Thermal physics, covering both thermodynamics and statistical mechanics, introduces a new physical dimensionality, and a new fundamental constant. The concept of absolute temperature (hereafter denoted as T_K, to avoid confusion with the T used here to represent the time dimension) as described by the ideal gas law, $PV \propto T_K$, requires a new dimension, which is conventionally written as $[T_K] = \Theta$. The fundamental constant associated with this field of study is Boltzmann's constant k_B, appearing in the ideal gas law as well as in the definition of entropy, $S = k_B \log(\Omega)$; we note for future reference that

$$[k_B] = \frac{ML^2}{\Theta T^2}. \tag{8}$$

The "unification" sometimes cited here is the connection between the microstates of a physical system and its macroscopic properties.

The final topic considered in most introductory textbooks is "modern physics," which includes both special relativity and quantum mechanics, along with their applications or realizations in nature. While these approaches are indeed radically new ways of thinking of the way nature works (especially in the realm of the very fast and/or very small) they do not require any new dimensional "letters" in the alphabet: the set L, T, M, Q, and Θ suffices. A discussion of modern physics does, however, introduce two fundamental constants of nature, namely, Planck's constant $\hbar$ and the speed of light c, which have dimensions

$$[\hbar] = \frac{ML^2}{T} \quad \text{and, of course,} \quad [c] = \frac{L}{T}. \tag{9}$$

Of the five most foundational fundamental constants included here, G, e, c, $\hbar$, and k_B, we note again that the SI system of units will begin using the last four as base quantities within the next decade. In a review describing that change, Newell[5] notes that "…the gravitational constant, G—which might seem a reasonable choice for a fundamental constant more directly linked to the traditional base mechanical units—is inherently difficult to measure," citing a recent review article.[9] The important (and dynamic) connection between dimensional analysis and metrology[10] is one that is useful to recall in this context, as are popular treatments focusing on describing these fundamental constants.[11]

III. REVIEWING CLASSICAL METHODS AND APPROACHES

We begin our review of some of the standard results of dimensional analysis with a classic example. Imagine a mass (m) oscillating in one dimension on the end of a spring (spring constant k), and subject to no other forces. Besides m and k, the only other dimensional parameter is the initial displacement x_0 of the spring from equilibrium. What can dimensional analysis tell us about the period (τ), amplitude of motion (A), and total energy (E) of this system, in terms of these parameters? The dimensions of the various quantities in question are

$$[m] = M, \quad [k] = \left[\frac{F}{x}\right] = \frac{M}{T^2}, \quad \text{and} \quad [x_0] = L,$$

so if we assume a power-law dependence, we can write $\tau = C_\tau m^\alpha k^\beta x_0^\gamma$, where α, β, and γ are arbitrary powers to be determined, and C_τ is a dimensionless constant. Taking the dimensions of both sides, we have

$$[\tau] = [C_\tau]\,[m]^\alpha\,[k]^\beta\,[x_0]^\gamma \quad \to \quad T = M^\alpha \left(\frac{M}{T^2}\right)^\beta L^\gamma, \tag{10}$$

since $[C_\tau]$ is dimensionless. Then, matching basic dimensions, we require that

$$M: \qquad 0 = \alpha + \beta + 0, \tag{11}$$

$$L: \qquad 0 = 0 + 0 + \gamma, \tag{12}$$

$$T: \qquad 1 = 0 - 2\beta + 0, \tag{13}$$

which is satisfied by $\alpha = 1/2$, $\beta = -1/2$, and $\gamma = 0$, so that $\tau \propto \sqrt{m/k}$. Recall that the exact answer one obtains from solving the differential equation is $\tau = 2\pi\sqrt{m/k}$.

This is the type of result that is the most straightforward, where dimensional analysis alone determines how the result depends on the physical parameters, along with an undetermined *dimensionless* constant of $\mathcal{O}(1)$, hopefully within an order of magnitude of unity, say $10^{\pm 1}$. If instead we ask how the amplitude A of the motion could depend on m, k, and x_0, we naturally find that $A \propto m^0\,k^0\,x_0^1 \propto x_0$. If we try the same approach for the total energy, we write $E = C_E\,m^\alpha\,k^\beta\,x_0^\gamma$, and matching dimensions, we find that $\alpha = 0$, $\beta = 1$, and $\gamma = 2$, so that $E \propto kx_0^2$ which is also correct, with $C_E = 1/2$.

If instead of an initial x_0, we give the mass an initial velocity v_0, the analysis proceeds as above, also giving $\tau \propto \sqrt{k/m}$, but now $A \propto v_0\sqrt{m/k}$, and $E \propto mv_0^2$, again with the number of dimensionful parameters matching exactly the number of dimensional constraints.

However, considering the more general problem with both an initial displacement and an initial speed, we find that the system is under-constrained, since assuming

$$E \propto m^\alpha\,k^\beta\,x_0^\gamma\,v_0^\delta \to \frac{ML^2}{T^2} = (M)^\alpha \left(\frac{M}{T^2}\right)^\beta (L)^\gamma \left(\frac{L}{T}\right)^\delta \tag{14}$$

gives

$$M: \qquad +1 = \alpha + \beta + 0 + 0, \tag{15}$$

$$L: \qquad +2 = 0 + 0 + \gamma + \delta, \tag{16}$$

$$T: \qquad -2 = 0 - 2\beta + 0 - \delta. \tag{17}$$

We can solve for three of the exponents in terms of the fourth, so that we might write $\alpha = 1 - \beta$, $\delta = 2 - 2\beta$, and $\gamma = 2\beta$. Thus, the energy can be written in the form

$$\begin{aligned} E &= C_\beta\,m^{1-\beta}\,k^\beta\,x_0^{2\beta}\,v_0^{2-2\beta} = C_\beta\,mv_0^2\left(\frac{kx_0^2}{mv_0^2}\right)^\beta \\ &= C_\beta\,mv_0^2\,\Pi^\beta, \end{aligned} \tag{18}$$

where the analysis itself has suggested a new dimensionless ratio, defined by

$$\Pi \equiv \frac{kx_0^2}{mv_0^2}. \tag{19}$$

This result does, of course, reproduce the two earlier special cases, since for $\beta = 1$ we have the pure-displacement case of $E \propto kx_0^2$, while for $\beta = 0$ we have $E \propto mv_0^2$.

Since the dependence in Eq. (18) is possible for any value of β, and any integral value of β is allowed, along with an arbitrary dimensionless constant C_β, we can write even more generally that

$$\begin{aligned} E &= (mv_0^2)(C_0\,\Pi^0 + C_1\,\Pi^1 + C_2\,\Pi^2 + \cdots) \\ &= (mv_0^2)\left[\sum_{\beta=0}^{\infty} C_\beta \Pi^\beta\right] = mv_0^2\,F(\Pi), \end{aligned} \tag{20}$$

so that in fact any function of the dimensionless ratio Π is allowed. In this case, of course, the appropriate function is a very simple one, since

$$E = \frac{mv_0^2}{2} + \frac{kx_0^2}{2} = \frac{mv_0^2}{2}\left(1 + \frac{kx_0^2}{mv_0^2}\right) \to F(\Pi) = \frac{1}{2}(1 + \Pi). \tag{21}$$

This example is a model for the more general results made famous by Buckingham,[12] who was one of the first to systematically describe methods of dimensional analysis, leading to the so-called Buckingham Pi theorem.[13] If the system of dimensional equations is under-constrained, having M dimensionful variables but only N constraints, there can be $I = M - N$ different dimensionless ratios, so the general result might depend on $F(\Pi_1, \Pi_2,\ldots,\Pi_I)$. Even in cases where one obtains such a general result, depending on an unknown function, considering limiting cases can still provide information on the functional form of $F(\Pi)$. We will see explicit cases of nontrivial forms for $F(\Pi)$ in examples below.

Dimensional analysis methods have been applied to other classical mechanics systems[14] and to dispersion in waves,[15] but is perhaps more familiarly used in fluid dynamics. Jensen[16] notes that the drag force (F_D) on a spherical object (radius R), moving at constant speed v through a medium of density ρ and viscosity η (where $[\eta] = M/LT$), can be written in terms of these parameters as

$$F_D \propto \rho^\alpha \eta^\beta R^\gamma v^\delta = \rho^\alpha \eta^{1-\alpha} R^{1+\alpha} v^{1+\alpha} \propto \eta R v \left(\frac{\rho R v}{\eta}\right)^\alpha. \tag{22}$$

As expected, this form correctly reproduces both the Stokes' law limit for small, slow particles in a highly viscous material ($F = 6\pi\eta R v$) for $\alpha = 0$, and the limit appropriate for large, fast-moving objects, where $F \propto \rho R^2 v^2$, for $\alpha = 1$. Noting Buckingham's theorem, we expect more generally that the force law would be given by $F = \eta R v G(\Pi)$, where in this case the dimensionless ratio is

$$\Pi = \frac{\rho R v}{\eta} \equiv \mathrm{Re}, \tag{23}$$

where Re is the famous Reynolds number and $G(\Pi)$ is the undetermined function. In contrast to the straightforward case of the harmonic oscillator, where the general form of the energy is known, the complex nonlinear problem of fluid flow admits no such simple, closed-form solution for $G(\Pi)$. But this approach does single-out the correct combination of variables as an important dimensionless ratio of direct relevance to categorizing different kinds of fluid flow. There are literally dozens of dimensionless numbers (ratios of physical quantities) used in fluid dynamics alone,[17] so this situation is exemplary of many other physical and/or engineering systems.

Perhaps even more importantly, such dimensionless ratios can be used in scaling arguments, so that systems with different parameters, but identical dimensionless ratios, can exhibit similar behavior—an idea used in applications such as wind tunnels. The outcome of identifying scaling behavior can be one of the most important aspects of determining the dimensional "content" of a physical result through such methods.

Going beyond mechanics, consider next the problem of finding the time-dependent current in a capacitor (C) being charged through a resistance (R) by an external battery ($\mathcal{E}$). We can proceed by identifying $I(t) \propto \mathcal{E}^\alpha R^\beta C^\gamma t^\delta$, and matching dimensions (using Eq. (5)) to find that

$$I(t) \propto \mathcal{E}^{+1} R^{-1-\delta} C^{-\delta} t^\delta = \frac{\mathcal{E}}{R}\left(\frac{t}{RC}\right)^\delta \to \frac{\mathcal{E}}{R} F(\Pi), \tag{24}$$

where $\Pi \equiv t/RC$, implicitly identifying the time constant for the problem. The standard analysis of the time-dependent differential equation of course gives $F(\Pi) = e^{-\Pi}$, so we see that non-power-law functional dependences of dimensionless ratios are consistent with Buckingham's Pi theorem. An interesting, and far more open-ended, application of dimensional analysis to a Lenz's law problem has been presented elsewhere.[18]

As a more advanced example in electrodynamics, consider the power (P) radiated by an accelerating point charge. The charge (q) and acceleration (a) have obvious places (and dimensions), and since this problem involves both electric and magnetic phenomena, we might expect both ϵ_0 and μ_0 to play a role. Thus, we write

$$\begin{aligned} P &= C_P\, \epsilon_0^\alpha\, \mu_0^\beta\, q^\gamma\, a^\delta \to \frac{ML^2}{T^3} \\ &= \left(\frac{Q^2T^2}{ML^3}\right)^\alpha \left(\frac{ML}{Q^2}\right)^\beta (Q)^\gamma \left(\frac{L}{T^2}\right)^\delta. \end{aligned} \tag{25}$$

This is easily solved to find $\alpha = 1/2$, $\beta = 3/2$, $\gamma = 2$, and $\delta = 2$, giving

$$P = C_P\, \epsilon_0^{1/2}\, \mu_0^{3/2}\, q^2\, a^2 = C_P \frac{\mu_0 q^2 a^2}{c}, \tag{26}$$

if we make use of the connection that $c = 1/\sqrt{\epsilon_0\mu_0}$; this expression is consistent with the standard result,[19] where $C_P = 1/6\pi$. The derivation of the power radiated by an accelerating charge is a challenging one in advanced electrodynamics, but can be studied in this way using much simpler techniques.

If we ask instead whether a particle moving at constant speed v_0 can radiate, replacing $a \to v_0$ in this analysis, we find that *no* solution of the resulting system of equations is possible. This is a very useful result, reinforcing the fact that charges in uniform motion do not radiate, simply on dimensional grounds. Dimensional analysis has also been applied to the problem of the gravitational power radiated by a body in a circular orbit,[20] but because there are fewer constraint equations (just three since only M, L, and T are involved in mechanics problems, and not Q as in EM) and more dimensional variables, more assumptions are required to connect with the answer as derived in general relativity.

As an example where dimensional analysis can highly constrain, but not necessarily pinpoint, the dependence on physical quantities, consider the problem of the time dependence of the thickness $x(t)$ of an ice sheet floating on water, maintained at a constant (externally imposed) temperature difference ΔT_{ext} during the winter. One presumes that the physics of the problem must depend on the thermal properties of the ice, especially the thermal conductivity (κ) and latent heat of fusion (L_F, via the heat released through freezing one layer of ice as it flows through the existing ice sheet), and the density ρ relates the volume of ice frozen to its mass. Thus, the simplest likely power-law dependence on these quantities would be $x(t) \propto \kappa^\alpha \Delta T_{\mathrm{ext}}^\beta t^\gamma L_F^\delta \rho^\epsilon$, or, in terms of their dimensions

$$L = \left(\frac{ML}{T^3\Theta}\right)^\alpha (\Theta)^\beta (T)^\gamma \left(\frac{L^2}{T^2}\right)^\delta \left(\frac{M}{L^3}\right)^\epsilon. \tag{27}$$

Since the system is under-constrained, we can solve for four powers in terms of the fifth, and we choose to write $\epsilon = -\alpha$, $\delta = (1/2) - 2\alpha$, $\gamma = 1 - \alpha$, and $\beta = \alpha$, so that

$$x(t) \propto \kappa^{\alpha}\Delta T_{\rm ext}^{\alpha} t^{1-\alpha} L_F^{1/2-2\alpha}\rho^{-\alpha} = \left(\frac{\kappa\Delta T_{\rm ext}}{tL_F^2\rho}\right)^{\alpha} t\sqrt{L_F}. \quad (28)$$

In this case, the fact that there is a dimensionless quantity $\Pi \equiv \kappa\Delta T_{\rm ext}/tL_F^2\rho$, is not as important as the specific power-law dependences of the individual physical parameters.

For example, since the ice presumably grows thicker with time, we expect that $1 - \alpha > 0$ (hence $\alpha < 1$) on physical grounds. We would also expect, all other things being equal, that the ice thickness will be smaller if either $\Delta T_{\rm ext}$ or κ is smaller (less heat flow) or if L_F is larger (harder to freeze the ice), which suggest that $\alpha > 0$ and $1/2 - 2\alpha < 0$ (or $\alpha > 1/4$), respectively. Combining these conditions gives $1/4 < \alpha < 1$, so that the choices $\alpha = 0, 1$ are immediately disallowed on physical grounds; the dependence $L_F^{1/2-2\alpha}$ itself might then suggest (but certainly not prove) that $\alpha = 1/2$ is a natural choice, giving

$$x(t) = C\left(\frac{\kappa\Delta T_{\rm ext}}{\rho L_F} t\right)^{1/2}, \quad (29)$$

where C is a dimensionless constant. A classic "first principles" calculation[21] (from the late 19th century) equating the heat flow through a slab of ice (already of thickness x), namely,

$$\frac{dQ}{dt} = \kappa A \frac{\Delta T_{\rm ext}}{x}, \quad (30)$$

with the heat given off in freezing the next infinitesimal layer of ice, via

$$dQ = L_F\, dm = L_f[dV\,\rho] = L_F[(A\, dx)\rho], \quad (31)$$

gives the relation

$$x\, dx = \left(\frac{\kappa\Delta T_{\rm ext}}{L_F\rho}\right)dt, \quad (32)$$

which can be integrated to obtain Eq. (29), with $C = \sqrt{2}$.

IV. EXPLORING THE QUANTUM FRONTIER

If we associate the birth of quantum mechanics with the extraction of Planck's constant (h or $\hbar$) from studies of the nature of blackbody radiation, then the quantum era is well over a century old. Some of the earliest (now historical) results are well known, but can still provide an illustration of the use of dimensional analysis in a more "modern physics" setting.

For example, in discussions of blackbody radiation one often starts with the notion of the frequency-dependent intensity, namely, the power (energy per unit time) per unit area per unit frequency interval, sometimes denoted as $dR_T(f)/df$. The total intensity, integrated over all frequencies, is

$$R_T \equiv \int_0^{\infty} \frac{dR_T(f)}{df}\, df, \quad (33)$$

and both quantities can be directly related to experimental measurements. Theoretical derivations related to blackbody radiation are most easily done, however, using the notion of the frequency-dependent energy density (energy per unit volume, per unit frequency interval), often written as $\rho(f)$, and we will apply the methods described above to see how $\rho(f)$ and R_T depend on the fundamental physical constants of the problem. We first note that the dimensions of these two quantities are

$$[R_T] = \frac{\text{energy}}{\text{area}\cdot\text{time}} = \frac{M}{T^3} \quad \text{and} \quad [\rho(f)] = \frac{\text{energy}}{\text{volume}\cdot\text{frequency}} = \frac{M}{TL}. \quad (34)$$

Since the problem clearly involves quantum mechanics, photons (hence relativity), and thermal physics, we can expect $\hbar$, c, T_K, and k_B all to appear, and for frequency-dependent results, also f. In the spirit of the original derivation, we will actually use $h = 2\pi\hbar$, which of course has the same dimensions.

Starting with $\rho(f)$, for example, we try $\rho(f) \propto h^{\alpha} f^{\beta} c^{\gamma} T_K^{\delta} k_B^{\epsilon}$, or in terms of dimensions

$$\frac{M}{TL} = \left(\frac{ML^2}{T}\right)^{\alpha}\left(\frac{1}{T}\right)^{\beta}\left(\frac{L}{T}\right)^{\gamma}(\Theta)^{\delta}\left(\frac{ML^2}{\Theta T^2}\right)^{\epsilon}. \quad (35)$$

This can be solved (say in terms of α) to give $\epsilon = 1 - \alpha$, $\delta = 1 - \alpha$, $\gamma = -3$, and $\beta = 2 + \alpha$, or

$$\begin{aligned}\rho(f) &= C\left[(k_BT_K)f^2c^{-3}\right]\left(\frac{hf}{k_BT_K}\right)^{\alpha}\\ &= C\left[\frac{f^2}{c^3}(hf)\right]\left(\frac{hf}{k_BT_K}\right)^{\alpha-1}\\ &\to \frac{f^2}{c^3}(hf)\,F\left(\Pi \equiv \frac{hf}{k_BT_K}\right),\end{aligned} \quad (36)$$

where the analysis has correctly identified the appropriate dimensionless ratio Π, and $F(\Pi)$ is an arbitrary function thereof. This result does, of course, agree with the standard form

$$\rho(f) = \frac{f^2}{c^3}(hf)\left[\frac{8\pi}{e^{hf/k_BT_K} - 1}\right]. \quad (37)$$

For the total intensity, since we've integrated over frequency, we have only four dimensional parameters and might therefore expect a more determined result (four equations in four unknowns). If we write $R_T \propto h^{\alpha}c^{\beta}T_K^{\gamma}k_B^{\delta}$, we find by matching dimensions that

$$R_T = Ch^{-3}c^{-2}k_B^4T_K^4 = \left(C\frac{k_B^4}{h^3c^2}\right)T_K^4, \quad (38)$$

which is indeed consistent with the Stefan–Boltzmann equation $R_T = \sigma T_K^4$, where $\sigma \equiv (2\pi^5/15)(k_B^4/h^3c^2)$.

One of the benchmark quantum mechanical systems is the hydrogen atom, for which one can easily obtain order-of-magnitude estimates of most relevant quantities, including energies and spatial extension, using dimensional analysis. With the understanding that this is a problem involving the electrostatic attraction of two fundamental charges (e), that the electron sets the relevant mass scale (m_e, just as the lighter mass sets the scale in two-body classical orbit problems), and that it involves quantum mechanics, we can propose for the energy scale that

$$E_H = C_H e^{\alpha} \epsilon_0^{\beta} m_e^{\gamma} \hbar^{\delta} \rightarrow \left(\frac{ML^2}{T^2}\right)$$
$$= (Q)^{\alpha} \left(\frac{Q^2 T^2}{ML^3}\right)^{\beta} (M)^{\gamma} \left(\frac{ML^2}{T}\right)^{\delta}.$$

Matching dimensions, one finds that $\alpha = 4$, $\beta = -2$, $\gamma = 1$, and $\delta = -2$, so that

$$E_H = C_H \frac{e^4}{\epsilon_0^2} \frac{m}{\hbar^2}. \tag{39}$$

In this case, comparing to the standard expression for the H atom energy levels, one finds that $C_H = -1/32\pi^2$ for the ground state; if one assumes that the appropriate fundamental constant for electrostatics that should be used for dimensional analysis problems is actually $4\pi\epsilon_0$, then the estimate above would be much closer to the ground-state energy scale, with $C_H = -1/2$. Using the same dimensional arguments, one can find that the associated length scale has dimensions $(\hbar^2/m_e)(\epsilon_0/e^2)$, which is indeed proportional to the Bohr radius. In any dimensional-analysis approach to the physical parameters for a given quantum state ψ_n, we can often find the dimensionful dependences correctly, but we should keep in mind that each one may depend on an undetermined function of the quantum number(s) and in the case of the H-atom energies this is $F(n, l, m) = F(n) = 1/n^2$, providing a different example of an undetermined dimensionless pre-factor.

Students can certainly hone their dimensional analysis skills against such problems with historical and well-known answers, but one of the reasons for pursuing this method is to be able to approach new physics in new contexts. As a more recent example, consider the study of the interactions of neutrons with Earth's gravitational field. It was shown long ago[22] that one could measure the local value of g by watching neutrons "drop" in the Earth's gravitational field, a purely classical effect. The effect of terrestrial gravity on the phase of the wave function of an otherwise free neutron—a much more foundational topic—was also measured more than 35 years ago.[23] Far more recently, however, evidence for quantized states of a neutron[24] in the gravitational potential of the earth has been published.

We can think of this last system as a textbook problem of a particle in a potential described by

$$V(z) = \begin{cases} +\infty & z < 0, \\ m_n g z & z \geq 0, \end{cases} \tag{40}$$

which can be treated in position space,[25] in momentum space,[26] or by WKB methods.[27] One can repeat our now-standard dimensional analysis for the energy scales associated with the quantum states of the neutron in this potential, starting with $E \propto m_n^{\alpha} g^{\beta} \hbar^{\gamma}$, to find that $E \propto (m_n g^2 \hbar^2)^{1/3}$. If we substitute numbers, we find that these energies are of order 10^{-36} J $\sim 10^{-12}$ eV; the wavelength of radiation associated with transitions between such quantum states would be of order $\lambda_{\gamma} = 2\pi\hbar c/E \sim 10^6$ m, which is not remotely experimentally accessible. But if we instead ask for the typical spatial extension l of the quantum states, which gives an estimate of how high above the flat surface the neutron will hover, we find that $l \sim (\hbar^2/m_n^2 g)^{1/3} \approx 10\,\mu$m, which is small but still measurable. The experimental evidence for the quantum structure of the neutron's bound states was made by actually probing the spatial extent of the neutron wave function on such table-top length scales.[24]

In approaching the "full-blown" versions of many quantum problems (such as the H atom or the neutron in a gravitational field) by actually solving the Schrödinger equation, one of the first steps is often to rewrite the differential equation in dimensionless form, which often automatically identifies the appropriate length, time, and energy scales. But for students who never get that far in the physics curriculum, dimensional analysis can provide a way to extract some of the truly interesting science in the problem, in a quantitative manner, without focusing on the details of the mathematical derivations.

At an even more fundamental level, the problem of how to combine quantum mechanics ($\hbar$) and special relativity (c) with gravitation (G) is still an open question, with approaches such as string theory and loop quantum gravity being widely debated and studied. But presumably in any such theory, these three fundamental constants of nature will be combined to set the time, length, and energy scales for which such a description would be necessary.[28] Thus, if we take combinations of the form $\hbar^{\alpha} c^{\beta} G^{\gamma}$ and match dimensions, we find

$$L_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\,\text{m}, \quad T_P = \sqrt{\frac{\hbar G}{c^5}} \sim 3 \times 10^{-44}\,\text{s},$$
$$\text{and} \quad E_P = \sqrt{\frac{\hbar c^5}{G}} \sim 10^{19}\,\text{GeV} \tag{41}$$

for the Planck length, time, and energy, respectively. The possibility of using such quantities as the basis for a system of "absolute units" has often been discussed.[29]

A problem related to the quantum Hall effect (discussed further below) is that of the discrete energy levels of a charged particle in a uniform magnetic field, giving rise to so-called Landau levels. The quantized energies E_L could depend on $\hbar$, e, m_e, and the value of the applied magnetic field B_0. Imposing the dimensional matching conditions, we find that

$$E_L \propto \hbar^1 e^1 B_0^1 m^{-1} = \hbar \left(\frac{eB_0}{m}\right) = \hbar\omega_C, \tag{42}$$

where ω_C is the classical cyclotron frequency. Even if students are not expecting to see the classical quantity ω_C, it automatically comes out of the analysis.

One of the first experiments that gave information on the mechanism of electrical conduction was the (classical) Hall effect, which can probe the sign and density of the actual charge carriers. In the standard rectangular-bar geometry (long bar of width d and thickness t), there is a current I along the long direction, with a magnetic field B in the t or thickness direction, perpendicular to the current. The ratio of the voltage induced across the bar to the current along the bar is often called the Hall resistance and can be shown via an elementary analysis to be $R_H = B/(etn_e)$, so that measurements of R_H plotted versus applied field (B) give a straight line with a slope determined by the material (electronic density n_e) and geometric (t) properties of the sample.

For some samples (MOSFETs, for example), at sufficiently low temperature and high magnetic fields, it was shown by von Klitzing *et al.*[30] that there were well-defined plateaus in the R_H versus B plots, corresponding to quantized values of resistance, given by $R_H = R_K/i$ (i an integer), where the value of R_K was found to be independent of the material properties (n_e) and the geometry (t, d, etc.). Because this seems to be a quantum phenomenon, we might guess that this new fundamental unit of resistance R_K (now known as the Klitzing constant) might depend on $\hbar$, e, perhaps the carrier mass m_e, and let's even imagine a possible dependence on ϵ_0. If we then write $R_K \propto \hbar^{\alpha} e^{\beta} m_e^{\gamma} \epsilon_o^{\delta}$ and match dimensions, we find that $\alpha = 1$, $\beta = -2$, and $\gamma = \delta = 0$, so that $R_K \propto \hbar/e^2$. The conventional definition is actually that $R_K = h/e^2 \approx 25.8\,\mathrm{k\Omega}$, and this value is now used extensively in the field of metrology.[31] To students, at any level, who have only been introduced to quantization in the context of discrete energy levels or angular momenta, the fact that there are other important manifestations of quantum mechanics that give rise to quantized physical properties, such as resistance, can be enlightening.

As two more such examples, we recall that in superconducting materials, both the circulation[32] of the superfluid velocity $\mathbf{v}_s$ and the magnetic flux[33] are quantized. If we write these quantities in terms of their definitions, namely,

$$C \equiv \oint \mathbf{v}_s \cdot d\mathbf{l} \quad \text{and} \quad \Phi_B \equiv \int_A \mathbf{B} \cdot d\mathbf{A}, \tag{43}$$

then dimensional analysis alone can be used to derive their dependence on fundamental parameters, using as input the dimensions of the factors in the integrands. For circulation, we might imagine the relevant parameters being $\hbar$ and m (only mechanics seems to be involved, classical or quantum), while for Φ_B we might guess $\hbar$, m, e, and perhaps μ_0, since it is a magnetic effect. Assuming power-law dependences in both cases, we find that

$$C \propto \frac{\hbar}{m} \rightarrow \frac{h}{2m} \quad \text{and} \quad \Phi_b \propto \frac{\hbar}{e} \rightarrow \frac{h}{2e}, \tag{44}$$

where the last expression in each case is the actual "textbook" result.

As our final example, we consider a neutron star—a quantum system that shows just how important dimensionless parameters can be. The stability of a neutron star arises from the interplay between the gravitational self-energy $V_G(r)$ of the star's mass M_* and the quantum mechanical zero-point energy $E_0(r)$ of the confined neutrons, both of which depend on the star's radius r. If we write each of these energies as a function of its expected physical parameters, we have

$$V_G(r) \propto G^{\alpha_1} M_*^{\beta_1} r^{\gamma_1} \quad \text{and} \quad E_0(r) \propto \hbar^{\alpha_2} m_n^{\beta_2} r^{\gamma_2}. \tag{45}$$

Matching dimensions in each, we then find that

$$V_G(r) \propto -\frac{GM_*^2}{r} \quad \text{and} \quad E_0(r) \propto \frac{\hbar^2}{m_n r^2}. \tag{46}$$

The first result is consistent with standard gravitational theory (though we have inserted the physically relevant minus sign by hand!), while the second is consistent with "particle in a box" quantum mechanics. Of course there are many neutrons ($N_n \sim M_*/m_n \sim 10^{57} \gg 1$) so $E_0(r)$ will no doubt also depend on a yet-to-be-determined function $F(N_n)$ of that dimensionless parameter.

If we combine the expressions in Eq. (46) and then minimize the total energy as a function of r, we find for the neutron star radius

$$r_{NS} = \frac{\hbar^2}{GM_*^2 m_n} F(N_n) \sim \left(10^{-92}\,\mathrm{m}\right) F(N_n). \tag{47}$$

Note that if we had attempted to use dimensional analysis to predict $r_{NS} \propto \hbar^{\alpha} G^{\beta} M_*^{\gamma} m_n^{\delta}$ directly, we would *not* have found this unambiguous prediction—only a dependence of the form $(\hbar^2/G) M_*^{\gamma} m_n^{-3-\gamma}$—so focusing first on the two separate energy expressions in this case was a more powerful approach.

If $F(N_n)$ were only linear in the number of neutrons (each neutron having the same zero-point energy), then the prediction for the physical size of this system would be roughly 10^{-35} m, which is (accidentally!) of the same magnitude as the Planck length in Eq. (41) and would be physically irrelevant. However, this approach based only on dimensional quantities misses an extremely important aspect of quantum mechanics, namely, the Pauli exclusion principle. Because neutrons are spin-1/2 fermions, only two of them can be in the same spatial wave function, and therefore the average energy of each increases with N_n. (The estimate based on $F(N_n) \propto N_n$ would be appropriate for bosons.) A simplified calculation[34] based on filling energy levels of a three-dimensional well shows that $F(N_n) \sim N_n^{5/3}$, leading to a more realistic estimate of $r_{NS} \sim 10\,\mathrm{km}$, much closer to the actual answer.

It is hard to think of any other example where dimensional analysis does, in fact, find the correct dependences on the appropriate physical constants, but then proceeds to get the actual numerical answer more wrong! One should never forget that there are important physical constraints (such as the spin-statistics theorem) that can be completely missed by dimensional analysis; quantum physics is far more than just $\hbar$!

V. COMMENTS AND CONCLUSIONS

In this note, we have reviewed the basic assumptions and notation underlying dimensional analysis, provided examples of its use (and limitations) in the realm of classical physics, and illustrated how it can be successfully applied to more modern scenarios in the quantum regime, including some that come close to current research problems. This discussion has been in the spirit of a remark made by Lord Rayleigh[35] exactly one century ago, regarding this method (which he describes as the *principle of similitude*):

> *"I have often been impressed by the scanty attention paid even by original workers in physics to the great principle of similitude…I have thought that a few examples, chosen almost at random from various fields, may help to direct the attention of workers and teachers to the great importance of the principle."*

The examples considered here have been taken from a variety of resources, from elementary discussions in introductory-level texts, to more advanced texts for physics majors, and from recent (and not-so-recent) research papers. Large numbers of further examples are readily obtainable with a minimum of effort. The author maintains a collection of examples on a public web site,[36] and encourages readers to submit additional examples.

The examples in this article have been chosen to be as representative as possible of the different types of "outcomes" that can be obtained using dimensional analysis, including cases where:

- The dependence on all relevant parameters is completely determined;
- No solution using the relevant parameters is found to be possible;
- New dimensionless combinations are found, which can be useful for scaling arguments, either phenomenological or fundamental in nature;
- Constraints are placed on the power-law dependences, which can be used to find "likely" solutions; and
- Dimensional analysis predicts the dimensional dependences correctly, but the dimensionless factors are numerically important, sometimes indicating that a significant physical effect has been ignored.

Dimensional analysis is not the only way to approach a new problem, but it can often be used to gain significant ground along the path towards understanding the physics behind the mathematics. There are many powerful analytic techniques in physics, such as the use of symmetry, which can be brought to bear on some, but not all, problems. One measure of a professional is not only the ability to use a variety of tools, but also the knack for knowing which tool(s) to use, and when.

In deciding whether to make use of this particular tool in one's courses or curricula, we might turn to discussions arising from physics education research related to the question of the nature of effective instruction. For example, Wieman and Perkins[37] ask (and answer) an important question:

> *"But what specifically do we mean by effective physics instruction? It is instruction that changes the way students think about physics and physics problem solving and causes them to think more like experts—practicing physicists. Experts see the content of physics as a coherent structure of general concepts that describe nature and are established by experiments, and they use systematic concept-based problem-solving approaches that are applicable to a wide variety of situations."*

Whether or not one considers the fact that the dimensions of almost all important physical quantities can be "spelled" with a minimal five-letter alphabet to be *miraculous* (in the Wigner sense[1]), dimensional analysis provides a methodology that allows students to work towards mastery of all of the skills mentioned above. We advocate here that dimensional analysis be considered as another important pedagogical component, at many levels, complementing higher-level mathematics, and that it be used in the study of physics as *another* language.

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Fort Wayne Electric Company Ammeter

The Fort Wayne Jenney Electric Light Company was organized in 1884 to manufacture and sell apparatus for the nascent electric power industry. Jenney arc lamps were used illuminate the Statue of Liberty for the first time in 1885. By 1899 the name had changed to the Fort Wayne Electric Company when the company was purchased by the General Electric Company of Schenectady, New York. This heavy duty ammeter was used in industrial applications. It is in the Greenslade Collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)