Differential topology, spring 2008

May 2, 2008

1 Main definitions and examples

Definition 1 Topological and a smooth manifold.

Example 1 $S^n, T^n, \mathbf{RP}^n, \mathbf{CP}^n$ and lens space.
Classical groups: $GL(n), SL(n), O(n), SO(n)$.

Exercise 1 Compute the transition functions for S^n (2 different atlases) and $\mathbf{RP}^n$.

Example 2 Homeomorphism between $SO(3)$ and $\mathbf{RP}^3$.

Definition 2 Smooth maps between manifolds. Immersion, submersion, embedding.

Example 3 Irrational line on a torus.

Example 4 Immersions of a disc in $\mathbf{R}^2$.

Example 5 Immersion of $\mathbf{RP}^2$ in $\mathbf{R}^3$.

Exercise 2 Construct an embedding of $\mathbf{RP}^2$ to $\mathbf{R}^4$ given by a componentwise quadratic map $\mathbf{R}^3 \to \mathbf{R}^4$.

Example 6 Hopf bundle $S^3 \to S^2$ as an example of submersion.

Exercise 3 Are the fibers of the Hopf bundle pairwise linked?

Example 7 Grassman manifolds $G_{k,n}$ and $G^+_{k,n}$: local charts.

Exercise 4 1. $G_{k,n} \cong G_{n-k,n}$.
2. $G^+_{2,4} \cong S^2 \times S^2$.
3. $G^+_{2,n+1}$ is homeomorphic to the set $\{z_0^2 + \cdots + z_n^2 = 0 \subset \mathbf{CP}^n$.
4. Define the Grassmannians of affine subspaces of a given dimension and find their dimensions.

Definition 3 Smooth submanifolds.

Exercise 5 A submanifold is a manifold.

Theorem 1 *1. Let $f : M^m \to N^n$ be an immersion at point $a \in M$. Then there is a coordinate change in a neighborhood of point $f(a)$ such that in a neighborhood of a the map f is a linear embedding $\mathbf{R}^m \to \mathbf{R}^n$.*
2. Let $f : M^m \to N^n$ be a submersion at point $a \in M$. Then there is a coordinate change in a neighborhood of point a such that in a neighborhood of a the map f is a linear projection $\mathbf{R}^m \to \mathbf{R}^n$.

Corollary 2 *Let $f : M^m \to N^n$ be a smooth map, $X^k \subset N$ a submanifold. If f is a submersion at each point of $f^{-1}(X)$ then $f^{-1}(X) \subset M$ is a submanifold of codimension $n - k$.*

Exercise 6 $\{z_0^n + z_1^n + z_2^n = 0\} \subset \mathbf{CP}^2$ is a smooth compact surface. What is its genus?

Example 8 $SL(n, \mathbf{R}) = \det^{-1}(1)$ is a submanifold of $Mat(n, \mathbf{R}) = \mathbf{R}^{n^2}$. Likewise for $O(n) = f^{-1}(E)$ where $f(A) = AA^*$.

2 Two technical theorems

Definition 4 Smooth partition of unity.

Theorem 3 *Given an open covering of a manifold, there exists an associated partition of unity.*

Definition 5 Regular and singular points and values of a smooth map $f : M^m \to N^n$.

Lemma 4 *Let $f : M^n \to N^n$ be a smooth map, M compact. Then, for every regular value $y \in N$, the set $f^{-1}(y)$ is finite (maybe void). The function $\#f^{-1}(y)$ is locally constant on the set of regular values.*

Proof. The set $f^{-1}(y)$ is closed and discrete, hence compact. Let $f^{-1}(y) = \{x_1, \dots, x_n\}$ and U_i be an open neighborhood of x_i which is mapped by f diffeomorphically. Then $M - \cup U_i$ is closed, hence compact. Hence $f(M - \cup U_i)$ is closed and the function $\# f^{-1}(y)$ is constant in the neighborhood $(\cap f(U_i)) - f(M - \cup U_i)$. $\square$

Application: Fundamental Theorem of Algebra. Let $p : \mathbf{C} \to \mathbf{C}$ be a polynomial. Extend it to a map $S^2 \to S^2$ sending ∞ to ∞. Then all points, except the roots of $p'(z)$, are regular. Hence the set of singular values is finite, and any two can be connected by a chain of intersecting neighborhoods over which p has the same number of preimages by Lemma 4. This number is not zero (since the map $p(z)$ has some image), hence it is non-zero for every regular value. Thus zero is in the image (as either regular or singular value).

Exercise 7 Prove that the extension of a polynomial to the Riemann sphere is a smooth map at infinity as well. Is ∞ a regular point? Why does the above argument fail for the function $expz$?

Theorem 5 (Sard Lemma). *The set of singular values has zero measure.*

Definition 6 Manifold with boundary.

Exercise 8 The boundary ∂M is a submanifold (without boundary).

Example 9 Let $f : M \to \mathbf{R}$ be a smooth function and c a regular value. Then $\{x \in M \mid f(x) \geq c\}$ is a manifold with boundary.

Exercise 9 There exist four non-diffeomorphic 1-dimensional manifolds: $S^1, \mathbf{R}, (0, 1], [0, 1]$.

Theorem 6 (modification of Corollary 2.) *Let M be a manifold with boundary, $f : M^m \to N^n$ a smooth map, $X^k \subset N$ a submanifold. If f is a submersion at each point of $f^{-1}(X)$ and $f_{|\partial M}$ is a submersion at each point of $f^{-1}(X) \cap \partial M$ then $f^{-1}(X) \subset M$ is a submanifold with boundary of codimension $n - k$ and $\partial(f^{-1}(X)) = f^{-1}(X) \cap \partial M$.*

Exercise 10 Prove Theorem 6.

Corollary 7 *Let M be a compact manifold with boundary. Then there is no smooth map $f : M \to \partial M$ such that $f_{|\partial M}$ is the identity.*

Proof. By Sard Lemma, almost every $y \in \partial M$ is a regular value of f, and obviously, of $f_{|\partial M}$. Then, by Theorem 6, $f^{-1}(y)$ is a manifold whose boundary is y. But $f^{-1}(y)$ is closed, hence compact, hence it must be either S^1 or $[0, 1]$, but neither has a single point boundary. $\Box$

Application: Brouwer's Fixed Point Theorem. A continuous map $g : D^n \to D^n$ can be uniformly approximated by a smooth map $f : D^n \to D^n$. If g has no fixed points then, by compactness of D^n, there exists $\varepsilon > 0$ s.t. $|g(x) - x| > \varepsilon$ for all x. Then the same holds for a smooth approximation f (with smaller ε). Finally, construct $h : D^n \to S^{n-1}$ as the intersection point of the ray $[f(x), x)$ with ∂D^n. However this map cannot exist by Corollary 7.

Exercise 11 Prove that h above is smooth.

3 Tangent vectors

Definition 7 Tangent vector to M as the velocity of a smooth curve on M and as the directional derivative of a smooth function on M.

Coordinate change of the components of a tangent vector.

Definition 8 The differential of a smooth map $f : M \to N$.

Definition 9 Tangent bundle TM. Its structure of a smooth manifold.

Example 10 TS^{n-1} is diffeomorphic to the space of oriented lines in $\mathbf{R}^n$.

Exercise 12 $TS^{n-1} = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid z_1^2 + \cdots + z_n^2 = 1\}$.

Definition 10 Smooth vector field on a smooth manifold.

Example 11 A non-vanishing vector field on odd-dimensional sphere.

Exercise 13 Construct a non-vanishing vector field on odd-dimensional projective space.

Exercise 14 Construct three linearly independent vector fields on S^{4n-1}.

Theorem 8 (Whitney) *Let M^n be a closed smooth manifold. Then M admits an immersion in $\mathbf{R}^{2n}$ and embedding in $\mathbf{R}^{2n+1}$.*

Proof. 1). Using partition of unity, embed M in R^N for N large enough.
2). Let $\pi : R^N \to \mathbf{R}^{N-1}$ be the orthogonal projection along unit vector v. As long as v is not tangent to v, the map π is an immersion of M. Consider the Gauss map $T_1 M \to S^{N-1}$; its image consists of such "bad" vectors v. By Sard's Lemma, a.e. vector is not in the image, as long as $N > 2n$. This is iterated until one reaches $\mathbf{R}^{2n}$.
3). Again project to a hyperplane. For π to be an embedding, v should not be collinear with the vectors xy where $x, y \in M$. Now, the role of Gauss map is played by the map $M \times M - \Delta \to S^{N-1}$. As long as $N > 2n+1$, the image is a null set and a.e. $v \in S^{N-1}$ will do. □

Definition 11 Lie group.

Example 12 $GL(n, \mathbf{R}), SL(n, \mathbf{R}), O(n)$. Computation of tangent spaces:

$$T_A SL(n, \mathbf{R}) = \{B \mid Tr(A^{-1}B) = 0\}, \quad T_A O(n) = \{B \mid AB^* + BA^* = 0\}.$$

Example 13 The tangent bundle to a Lie group G is trivial; left- and right-invariant vector fields.

Exercise 15 Compute the tangent space to the Lie group $U(n)$.

Exercise 16 Prove that $SU(2) = S^3$.

Exercise 17 Prove that $T_P G_{k,n} = L(P, P^\perp)$..

Definition 12 Riemannian metric.

Theorem 9 *Every smooth manifold carries a Riemannian metric.*

Proof. Use a partition of unity to paste Euclidean metrics in local charts. □

Definition 13 One-parameter group of diffeomorphishs (flows) φ_t. The corresponding vector field

$$v(x) = \lim_{t \to 0} \frac{d\varphi_t(x)}{dt}.$$

Fundamental Theorem of ODE.

Lemma 10

$$\frac{\partial \Delta}{\partial s \partial t}(x) = (D_u D_v - D_v D_u)(f)(x)$$

where f is a test function, $\Delta = f(\varphi_t \psi_s(x) - \psi_s \varphi_t(x))$ and u, v are the vector fields corresponding to the flows φ, ψ.

Lemma 11 *The differential operator $[D_u, D_v]$ has first order and equals D_w where w is a new vector field.*

Definition 14 Commutator of vector fields (Lie bracket): $[u, v] := w$. Expression in local coordinates.

Example 14 The space spanned by the fields on the line $\partial_x, x\partial_x, x^2\partial_x$ is closed under commutation.

Exercise 18 Let A, B be $n \times n$ matrices, let $u(x) = Ax, v(x) = Bx$ be linear vector fields in $\mathbf{R}^n$. Compute $[u, v]$.

Theorem 12 *If $[u, v] = 0$ then the respective flows commute.*

Definition 15 Lie algebra: skew-symmetric bilinear satisfying the Jacobi identity.

Lemma 13 *The bracket of vector fields satisfies the Jacobi identity.*

Example 15 1. $\mathbf{R}^3$ with cross-product.
2. Matrix Lie algebras with the bracket $[A, B]$.

Exercise 19 Consider the following 3-dimensional Lie algebras: $(\mathbf{R}^3, \times)$, the one of Example 14, $sl(2, \mathbf{R}), o(3)$. Find all isomorphic pairs among them.

Lie algebra of a Lie group (the case of matrix groups). The 1-parameter subgroup $\varphi_t = \exp(tA)$ corresponding to a tangent vector $A \in T_E G := g$. Derivation of the structure of a Lie algebra on g with $[A, B] = AB - BA$.

Lemma 14 *If $TrA = 0$ then $\exp(A) \in SL(n, \mathbf{R})$; and if $A* = -A$ then $\exp(A) \in O(n)$.*

4 Degree of smooth map

Definition 16 Smooth homotopy of maps. Isotopy of diffeomorphisms.

Let M^n be closed, N connected and $f : M \to N$ smooth. For a regular value $y \in N$, consider Card$\{f^{-1}(y)\}$ mod 2.

Lemma 15 *Let y be regular value for f and g, and $f \sim g$. Then Card$\{f^{-1}(y)\}$=Card$\{g^{-1}(y)\}$ mod 2.*

Lemma 16 *The group of diffeomorphisms isotopic to identity acts transitively on N: any point can be sent to any other point.*

Theorem 17 *If x, y are regular values for $f : M \to N$ then Card$\{f^{-1}(x)\}$=Card$\{f^{-1}(x)\}$ mod 2.*

Definition 17 This common value is called degree of f mod 2 and denoted $\deg_2 f$.

Exercise 20 $\deg_2(f \circ g) = \deg_2 f \deg_2 g$.

Exercise 21 Find degree mod 2 of a rational function considered as a mapping $\mathbf{RP}^1 \to \mathbf{RP}^1$.

Exercise 22 Let $\partial W = M^n$. If a map $f : M \to N$ extends to W then $\deg_2 f = 0$.

Corollary 18 *Let M be a closed manifold. Then the identity map is not homotopic to the trivial map.*

Corollary 19 *The sphere is not a retract of the disk.*

Orientation.

Definition 18 Oriented atlas and oriented manifold. Orientation preserving/reversing diffeomorphisms.

Definition 19 Orientation of the boundary of an oriented manifold. Orientation of the product of oriented manifolds.

Exercise 23 Let $A \in O(n)$. When is $A : S^{n-1} \to S^{n-1}$ orientation preserving?

Exercise 24 Let M, N be oriented and $f : M \times N \to N \times M$ be $f(x, y) = (y, x)$. When is f orientation preserving?

Orientation and π_1: orienting two-fold covering. Orientability of simply-connected manifolds.

Example 16 Orientability of $\mathbf{RP}^n$.

Exercise 25 Investigate orientability of $G^+_{k,n}$ and $G_{k,n}$.

Exercise 26 The tangent bundle TM is always orientable.

Exercise 27 Are Lie groups orientable?

Degree of a smooth map of oriented manifolds.

Definition 20 Counting preimages of a regular value with signs.

Lemma 20 *Let W^{n+1} be compact and oriented, $\partial W = M^n$ with the boundary orientation, and let $f : W \to N^n$ be a smooth map where N is also oriented. Then if $y \in N$ is a regular value for $f|_M$ then* $\deg(f, y) = 0$.

Lemma 21 *If $f, g : M^n \to N^n$ are homotopic maps of oriented manifolds and y isa regular value for both then* $\deg(f, y) = \deg(g, y)$.

Example 17 If $p(z)$ is a complex polynomial of degree n then the respective map $\mathbf{CP}^1 \to \mathbf{CP}^1$ has degree n. This implies the Fundamental Theorem of Algebra.

Exercise 28 Find the degree of the map of $SO(n)$ given by $A \mapsto A^2$.

5 Applications of degree

1). Even-dimensional sphere has no non-vanishing vector fields. Otherwise the identity and minus identity would be homotopic; but they have different degrees.

2). Degree of a map $S^1 \to S^1$ is its only homotopic invariant. Application: winding number of a closed smooth plane curve.

Exercise 29 The winding number of a plane curve has the opposite parity to the number of its double points.

3). Normal Gauss map of a smooth hypersurface in $\mathbf{R}^n$. The degree of the Gauss map is the algebraic number of critical points of the height function.

Exercise 30 Find the degree of the Gauss map of the standard surface $M_g \subset \mathbf{R}^3$.

Exercise 31 Prove that the unit normal field along M_g with $g \neq 1$ does not extend as a non-vanishing vector field inside M_g.

4). Linking number of closed oriented curves in $\mathbf{R}^3$. Diagrammatic computation of the linking number.

Exercise 32 How does linking number depend on the orientations and the order of the curves?

Exercise 33 Give an example of linked curves with linking number zero.

5). Degree from algebraic topological point of view. Degree of $f : M^n \to N^n$ as $f_* : H_n(M, \mathbf{Z}) \to H_n(N, \mathbf{Z})$.

Theorem 22 *The two definitions yield the same.*

6). Hopf Theorem.

Theorem 23 *The homotopy classes of smooth maps $M^n \to S^n$ where M is closed and oriented are in 1-1 correspondence with $\mathbf{Z}$ given by the degree.*

Index of an isolated zero of a vector field

Definition 21 Index $ind_v(x)$ in a domain in $\mathbf{R}^n$ as the degree of a map from sphere to sphere.

Lemma 24 *Index is invariant under diffeomorphisms of a domain.*

Thus the definition extends to smooth manifolds.

Exercise 34 Construct a function $f(x,y)$ such that ∇f has an isolated zero at thew origin with the index $k = 1, 0, -1, -2, ...$

Exercise 35 Find the index of a linear vector field Ax where $A \in GL(n, \mathbf{R})$.

Exercise 36 Find the indices of the field ∇h where h is the height function on the round sphere in $\mathbf{R}^n$.

Exercise 37 Find the index of $\nabla f(x,y)$ at a critical point of f using the second derivative test.

Theorem 25 (Poincaré-Hopf) *Let M be a closed manifold with a vector field v. Then $\sum ind_v(x) = \chi(M)$.*

Plan of Proof. 1. Let $N \subset \mathbf{R}^n$ be a domain, v a vector field in N with the outward direction on ∂N, and $\Gamma : \partial N \to S^{n-1}$ the Gauss map. Then $\sum ind_v(x) = \deg \Gamma$.
2. Definition of non-degenerate zero. Computation of the index as the sign of the Jacobian $\partial v(x)/\partial x$.
3. Embed M in $\mathbf{R}^N$ and consider the tubular neighborhood N_ε, the locus of points at distance $\leq \varepsilon$ from M.

Lemma 26 *For ε small enough, N_ε is a manifold with boundary, the field v extends to N_ε as a field w such that the zeros of v and w coincide and have the same indices, and w has the outward direction along the boundary ∂N_ε.*

Exercise 38 The normals to M are orthogonal to ∂N_ε.

4. A degenerate isolated zero of a vector field can be perturbed in a neighborhood of the zero so that the degenerate zero is replaced by finitely maany non-degenerate ones with the same total index.
5. Given a triangulation of M, construct a vector field whose zeros are the centers of the simplices with the indices $(-1)^d$ where d is the dimension of the respective simplex. □

Corollary 27 *If M is odd-dimensional then $\chi(M) = 0$.*

Exercise 39 Let M be a manifold with boundary and v a vector field having the outward (inward) direction on ∂M. What can you say about $\sum ind_v(x)$?

Exercise 40 Let a closed manifold M carry a smooth field of directions. Prove that $\chi(M) = 0$.

6 Differential forms

Recall linear algebra: tensor product of vector spaces, symmetric and exterior powers of a space, dual space, adjacent linear map, etc..

Example 18 Symplectic space is a space V with non-degenearate skew-symmetric bilinear form ω.

Exercise 41 1. A symplectic space is even-dimensional.
2. Let V be a vector space. Define a natural symplectic form in $V \oplus V^*$.
3. Let (V^{2n}, ω) be a symplectic space. A subspace U is called isotropic if $\omega|_U = 0$. Prove that $\dim U \leq n$.
4. An isotropic subspace $U \subset V^{2n}$ is called Lagrangian if $\dim U = n$. Prove that the set Λ_n of all Lagrangian subspaces is a smooth manifold and find its dimension.

Definition of a differential k-form on a smooth manifold. The wedge product $\Omega^p(M) \otimes \Omega^q(M) \to \Omega^{p+q}(M)$. Cotangent bundle T^*M. Differrential of a function as a 1-form.

Exercise 42 Wedge product is associative and skew-commutative: $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

Exercise 43 Let α be a 1-form. Prove: if $\alpha \wedge \beta = 0$ then there exists a form γ such that $\beta = \alpha \wedge \gamma$.

Example 19 If α is a 1-form then $\alpha \wedge \alpha = 0$. If (V^{2n}, ω) is a symplectic space then $\omega^{\wedge n} \neq 0$.

Exercise 44 let $v \in \mathbf{R}^3$. Define a 1-form α_v by $\alpha_v(u) = u \cdot v$, and a 2-form ω_v by $\omega_v(u, w) = \det(v, u, w)$. Prove that $\alpha_v \times \alpha_u = \omega_{v \times u}$ and $\alpha_v \wedge \omega_u = (u \cdot v) \det$.

Exercise 45 Rewrite the forms in polar coordinates: $xdy - ydx, dx \wedge dy$.

Example 20 The Liouville 1-form on T^*M and its local expression as pdq.

Integration of differential forms.

Definition 22 k-chain σ is $f : P^k \to M$ where P is an oriented convex polyhedron. Vector space of chains. For a k-form, definition of $\int_\sigma \omega$.

Properties: the integral is linear in σ and ω; it is invariant under orientation preserving diffeomorphisms.

Exercise 46 Let $f : M \to N$ be a k-fold covering of compact n-dimensional manifolds, and ω an n-form on N. Prove that $\int_M f^*(\omega) = k \int_N \omega$.

Integral and degree.

Theorem 28 *Let $f : M \to N$ be a map of closed manifolds of dimension n and ω an n-form on N. Then $\int_M f^*(\omega) = (\deg f) \int_N \omega$.*

Sketch of Proof. For regular values this is the preceding exercise. Let Δ be the set of singular values. Then $\int_\Delta \omega = 0$ by Sard. On the other hand, the Jacobian of f is degenerate at points of $f^{-1}(\Delta)$, hence $\int_{f^{-1}(\Delta)} f^*(\omega) = 0$. $\square$

Example 21 *Gauss integral.* Let $\gamma_1(t_1)$ and $\gamma_2(t_2)$ be two closed curves in $\mathbf{R}^3$. The Gauss map $T^2 \to S^2$ is given by

$$(t_1, t_2) \mapsto \frac{\gamma_2(t_2) - \gamma_1(t_1)}{|\gamma_2(t_2) - \gamma_1(t_1)|},$$

and its degree is the linking number of the curves. A computation using Theorem 28 yields:

$$\mathrm{lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int \int \frac{(\gamma_1' \times \gamma_2') \cdot (\gamma_2 - \gamma_1)}{|\gamma_2 - \gamma_1|^3} \, dt_1 dt_2.$$

Exterior differential of a differential form

Definition of $d\omega$ and computation in local coordinates.

Exercise 47 1. $d(\omega + \eta) = d\omega + d\eta$.
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$.
3. $d^2 = 0$.
4. For $f : M \to N$ and $\omega \in \Omega^*(N)$, one has: $d(f^*\omega) = f^*(d\omega)$.

Theorem 29 (Stokes) *For a $k+1$-chain σ and k-form ω, one has:*

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega.$$

Example 22 1. The classical *Green theorem*: σ is a plane domain and $\omega = fdx + gdy$ is a 1-form. Then

$$\int_{\partial\sigma} fdx + gdy = \int_\sigma (g_x - f_y) dxdy.$$

2. The *divergence theorem*: σ is a domain in 3-space and

$$\omega = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy.$$

Then $d\omega = divF dV$ and

$$\int_\sigma divF dV = \int_{\partial\sigma} (F \cdot n) dA.$$

3. The classical *Stokes theorem*: σ is an oriented surface in 3-space with boundary, F a vector field along σ and $\omega = F_1 dx + F_2 dy + F_3 dz$. Then

$$\int_\sigma (curlF \cdot n) dA = \int_{\partial\sigma} (F \cdot T) ds.$$

Example 23 Stokes Theorem implies that there does not exist a retraction $B^n \to S^{n-1}$.

Definition 23 De Rham cohomology.

Example 24 If M is connected then $H^0_{DR}(M) = \mathbf{R}$. Computations: $H^*_{DR}(\mathbf{R})$, $H^*_{DR}(S^1)$. If M^n is a closed oriented manifold then $H^n_{DR}(M) = \mathbf{R}$.

Exercise 48 *Compute*
1. $H^*_{DR}(T^2)$.
2. $H^*_{DR}(S^2)$.
3. $H^*_{DR}(\mathbf{RP}^2)$.

Theorem 30 (De Rham) *For a closed manifold* M*, one has:* $H^*_{DR}(M) = H^*(M, \mathbf{R})$.

A particular case:

Theorem 31 (Poncaré Lemma) *If* ω *is a closed differential form in* $\mathbf{R}^n$ *then* ω *is exact.*

Proof. Sketch. Let $h : \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$ be the map $h(x,t) = tx$. For a k-form ω, one has: $h^*\omega = \omega_1 + dt \wedge \omega_0$ where $\omega_{0,1}$ does not contain dt. Define

$$A(\omega) = \int_0^1 \omega_0 dt.$$

One has: $Ad + dA = \text{id}$; hence if $d\omega = 0$ then $\omega = dA(\omega)$. $\Box$

Example 25 S^{n-1} is not a retract of B^n: an argument using Stokes Theorem.

Lie derivative

Definition 24 Definition of the substitution $i_v\omega$ and the Lie derivatives $L_v\omega$ and $L_v u$.

Lemma 32 $L_v u = [v, u]$

Exercise 49 $L_v(\alpha \wedge \beta) = L_v(\alpha) \wedge \beta + \alpha \wedge L_v(\beta)$.

Proposition 33 (Homotopy formula) $L_v = i_v d + d i_v$.

Exercise 50 1). $[L_u, i_v] = i_{[u,v]}$.
2). $[L_u, L_v] = L_{[u,v]}$.
3). (Cartan formula) Let α be a 1-form and u, v two vector fields. Then $(d\alpha)(u,v) = L_u(\alpha(v)) - L_v(\alpha(u)) - \alpha([u,v])$.

Frobenius theorem

Definition 25 Distribution E^k on a smooth manifold, space of tangent vector fields $V(E)$ and vanishing 1-forms $\Omega(E)$. Oriented and cooriented distributions. Integrable distribution or foliation F^k.

Example 26 1-dimensional distribution is always integrable.

Example 27 The distribution given by $dz - ydx$ is completely non-integrable: it's a contact structure in $\mathbf{R}^3$.

Example 28 Irrational linear foliation on a torus. Reeb foliation on S^3.

Theorem 34 (Frobenius) *A distrribution E is integrable iff $V(E)$ is a Lie algebra iff $\Omega(E)$ is a differential ideal, i.e., $d\Omega(E) = \Omega(E) \wedge \Omega^1(M)$.*

Exercise 51 Prove that a 1-form α defines a foliation iff $d\alpha = \alpha \wedge \beta$ for some 1-form β iff $\alpha \wedge \alpha = 0$.

Exercise 52 1). The distribution of complex tangent lines to $S^3 \subset \mathbf{C}^2$ is a contact structure.
2). The distribution on the space of plane contact elements given by the "skating condition" is a contact structure.

Exercise 53 In $\mathbf{R}^4$, consider the fields

$$u = t\partial x + z\partial y - y\partial z - x\partial t, v = y\partial x - x\partial y + t\partial z - z\partial t, w = -z\partial x + t\partial y + x\partial z - y\partial t.$$

1). Where is span u, v a 2-distribution?
2). Is this distribution integrable?
3). Same question for span u, v, w.
4). Find a 1-form defining this distribution.

Exercise 54 For a cooriented codimension one foliation, $d\alpha = \alpha \wedge \beta$. Consider the 3-form $\beta \wedge d\beta$. Prove that this form is closed and that its cohomology class is well defined by the foliation (Godbillon-Vey class).

7 Morse theory

Definition 26 Non-degenerate critical function, Hessian, Morse index, Morse function.

Exercise 55 Let $df(x) = 0$. Give a coordinate-free definition of the Hessian quadratic form on T_xM.

Exercise 56 Let A be self-adjojnt matrix with simple spectrum. Describe the critical points of $Ax \cdot x$ on the unit sphere and find their Morse indices.

Theorem 35 (Morse Lemma) *Near a non-degenerate critical point, there are coordinates in which the function is* $c + \sum x_i^2 - \sum y_j^2$.

Definition 27 Poincaré polynomials $P_t(f)$ of a Morse function and $P_t(M)$ of the manifold.

Theorem 36 (Morse inequalities) *For a Morse function, one has:* $P_t(f) = P_t(M) + (1+t)Q_t$ *where* Q_t *has non-negative coefficients.*

Proof. Let $M_c = \{x \in M | f(x) \leq c\}$. Then, if c is regular value, M_c is a manifold with boundary. If a and b are regular values and there are no singular values on the segment $[a, b]$ then M_a is diffeomorphic to M_b. If a and b are regular values and there exists a unique non-degenerate critical point with Morse index μ on $M_b - M_a$ then M_b is homotopically equivalent to M_a with a disc D^μ attached. Consider the two Poincaré polynomials of M_c. When passing a critical point, $P_t(f)$ gains t^μ, and $P_t(M)$ either gains t^μ or loses $t^{\mu-1}$. □

Corollary 37 *The number of critical points is not less than the sum of Betti numbers. One has:* $\sum (-1)^{\mu(x)} = \chi(M)$.

Definition 28 f is a perfect Morse function if $Q_t = 0$.

Exercise 57 Let $f = \sum a_i |z_i|^2$ with distinct a_i on the unit sphere in $\mathbf{C}^n$. Then f descends to a function on $\mathbf{CP}^{n-1}$. Find its critical points and their Morse indices.

Example 29 A closed convex smooth hypersurface in $\mathbf{R}^n$ has at least n diameters. They are critical points of the width function on $\mathbf{RP}^{n-1}$.

Exercise 58 1). Let $f : \mathbf{R}^n \to \mathbf{R}$ be a harmonic function. Prove that its Morse indices are $\neq 0, n$.
2). Let $f : \mathbf{C}^n \to \mathbf{C}$ be an analytic function and $g = \text{Re} f$. Prove that, for every critical point, $\mu(g) = n$

Exercise 59 Let $f(x_1, \ldots, x_n) = \sum a_j \sin x_j$ be a function on T^n with $a_j \neq 0$ for all j. Find $P_t(f)$.

Distance from a point

Let $M \subset \mathbf{R}^n$ be a submanifold, $P \in \mathbf{R}^n$ a point not on M. Let $f : M \to \mathbf{R}$ be the distance to P.

Definition 29 Focal points of M.

Lemma 38 *The critical points of f are $Q \in M$ such that $PQ \perp T_Q M$. The Morse index $\mu(Q)$ equals the number of focal points on $[PQ]$.*

Corollary 39 *For almost every point P, the function f is Morse.*

Proof. Let $N = \{(x, n0 | x \in M, n \perp T_x M\}$. Consider the map $F : N \to \mathbf{R}^n$ taking (x, n) to $x + n$. Then the focal points of M are the critical values of F. The result follows from Sard's Lemma. $\square$

Theorem 40 *Let $M^{2k} \subset \mathbf{C}^n$ be a smooth affine algebraic variety. Then $H_i(M; \mathbf{Z}) = 0$ for $i > k$.*

Proof. The quadratic form whose spectrum corresponds to the focal points is $Q'(x_j, y_j) := \text{Re } Q(z_j)$ where Q is a complex quadratic form of $z_1, \ldots, z_k$. Let J be the operator of multiplication by $\sqrt{-1}$. Then $Q(Jz) = -Q(z)$, hence $JQ' = -Q'J$. It follows that the eigenvalues of Q' come in pairs $\pm\lambda$, and at most k lie on a segment PQ. $\square$

Exercise 60 Describe the homotopy type of the variety $\sum z_j^2 = 1$ in $\mathbf{C}^n$.

Versions of Morse theory

1). *Morse theory on a manifold with boundary.* One considers $f|_{\partial M}$ and takes only those points into account where ∇f is inward. Let $R_t(f)$ be the respective Poincaré polynomial for $f|_{\partial M}$.

Theorem 41 *$P_t(f) + R_t(f) = P_t(M) + (1+t)Q_t$ where Q_t has non-negative coefficients.*

2). *Morse-Bott inequalities.* One allows f to have critical manifolds N but f has non-degenerate Hessian in the normal space to N. One defines $P_t(f) = \sum P_t(N) t^{\mu(N)}$.

Theorem 42 *$P_t(f) = P_t(M) + (1+t)Q_t$ where Q_t has non-negative coefficients.*

3). *Morse-Smale-Witten complex.* Let M have a generic Riemannian metric. Construct a complex: C_k is spanned, over $\mathbf{Z}_2$, by the critical points of f with Morse index k; define

$$d(x) = \sum_{\mu(y)=\mu(x)-1} (\text{number of } -\nabla f \text{ flow lines from } x \text{ to } y)\ y.$$

Then $d^2 = 0$, the homology is the same as that of M. The Morse inequalities are those relating the Poincaré polynomials of a graded complex and its homology.

4). *Morse-Novikov theory.* This is an extension of Morse theory in which df is replaced by a closed 1-form.