Geometry of Universal Covers through Cayley Graphs

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1. Introduction

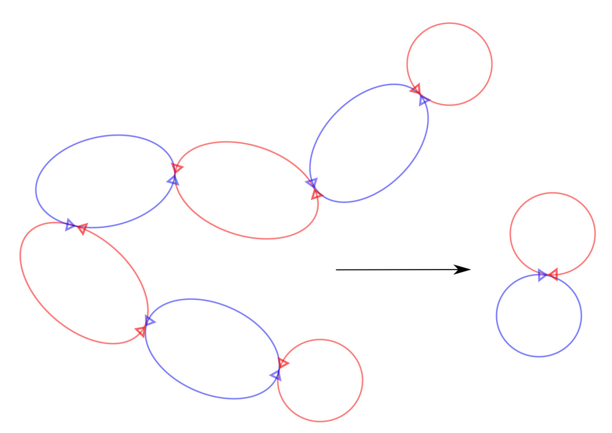
Let (X,d) be a compact length space and $\pi:\tilde{X}\to X$ a covering map. Then there is a unique way to "lift" the metric to $\tilde{X}$. We study the geometric properties of $(\tilde{X},\pi^*d)$.

2. Diameter Problem

The most elementary property of a metric space is its diameter, so it may be natural to ask whether one could bound the diameter of $\tilde{X}$ in terms of the diameter of X and the number of sheets of the cover.

Theorem (Ivanov)[3]**.** Let $\tilde{X}$ be an n-fold cover of X. If $\tilde{X}$ is equipped with the lifted metric, then

$$diam(\tilde{X})\le n\cdot diam(X).$$



Ivanov bound is attained in plenty of examples. In here $n=6$.

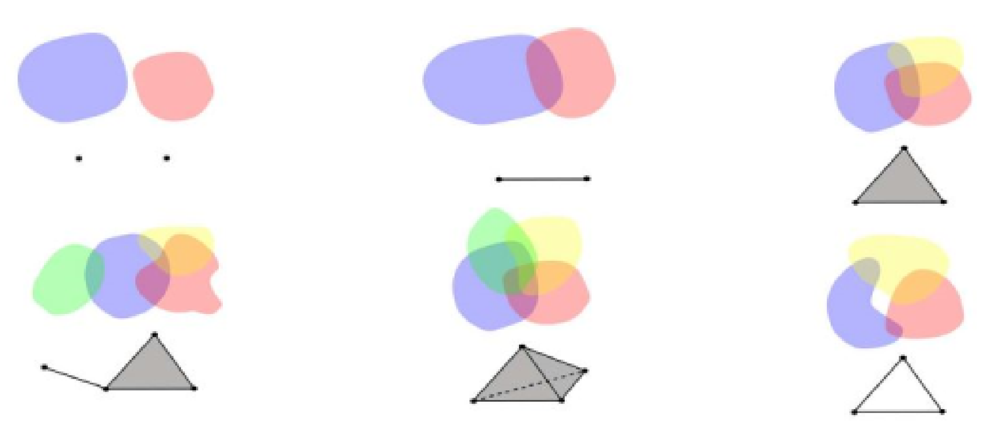
Of course, if we re-scale X by some factor, $\tilde{X}$ will suffer the same transformation, so we may assume that $diam(X)=1$.

3. Universal Covers

However, Anton Petrunin[4] noticed that in the particular case when the cover is universal, the upper bound should be asymptotically much less than the one given for general coverings by Ivanov.

To study the diameter of the universal cover $\tilde{X}$, one could study instead the Cayley graph of the Fundamental Group $\pi_1(X,x)$. The precise approach is the following:

Take a point $x\in X$ and take $\{x_i\}_{i=1}^n=\pi^{-1}(x)$. Consider the cover by the closed balls $B_i=B(x_i,1)\subset\tilde{X}$ and its nerve N.



The nerve of a cover $\{B_i\}_i$ is the simplicial complex $N\subset\Delta^{n-1}\subset\mathbb{R}^n$ with one vertex for every element of the cover and encodes when subsets of the cover intersect.

4. Nerve Lemma

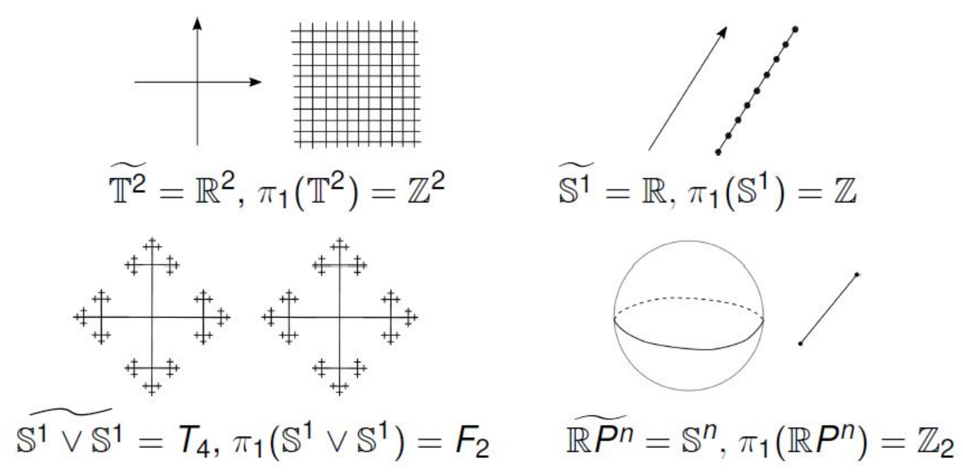
The simplicial complex N satisfies the following properties:

☞ N is simply connected.

☞ The 1-skeleton Γ of N is the Cayley Graph of the fundamental group $\pi_1(X,x)$ with generators the classes containing loops of length ≤ 2.

☞ Γ is $(2,1)$-quasi-isometric to $\tilde{X}$.

☞ $diam(\tilde{X})\le 2(diam(\Gamma)+1)$.



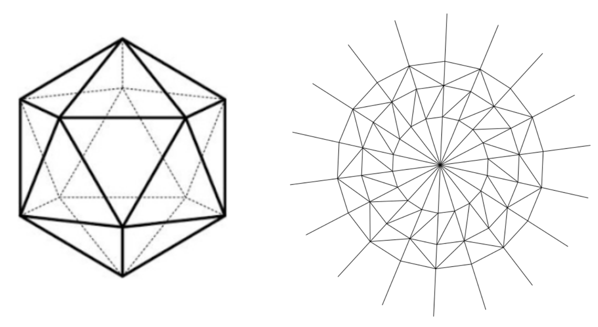
Universal covers are quasi-isometric to Cayley Graphs of Fundamental Groups

5. Spiderweb Graphs

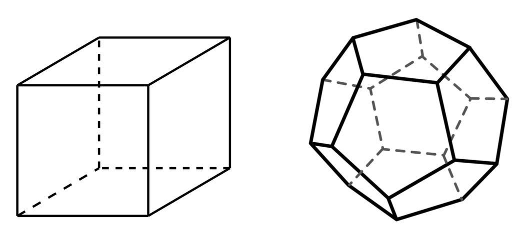
Lemma. Let G be a group, S a finite set of generators, and Γ the corresponding Cayley graph. Then the following are equivalent:

- Γ is the 1-skeleton of a simply connected 2-simplex.
- The 2-simplex obtained from Γ by attaching a triangle on every cycle of length 3 is simply connected.
- G has a presentation $\langle S|R\rangle$, where R consists of words of no more than 3 letters.

Definition. A graph satisfying the first point of the previous lemma will be called a Spiderweb Graph.



Spiderweb Graphs



Non-Spiderweb Graphs

6. Effective Bound

Proposition [5]**.** Let Γ be a spiderweb Cayley graph of size n, then

$$diam(\Gamma)\le\sqrt{4n+1}-2$$

Corollary. For an n-sheeted universal cover $\tilde{X}$ of a compact length space X with $diam(X)=1$, then

$$diam(\tilde{X})\le\sqrt{4n+1}$$

7. Asymptotic Bound

Theorem (Benjamini, Finucane, Tessera)[1]**.** Let G_n be a sequence of fnite graphs such that the isomorphism group of each one acts transitively on the vertices and $diam(G_n)\to\infty$. Assume that for some $p>0$,

$$\limsup_{n\to\infty}\frac{diam(G_n)}{|G_n|^p}>0.$$

Then a subsequence of $G_n/diam(G_n)$ converges in the Gromov-Hausdorff sense to a finite dimensional torus.

Corollary. Let Γ_n be a sequence of spiderweb Cayley graphs such that $diam(\Gamma_n)\to\infty$, then for any $p>0$,

$$\lim_{n\to\infty}\frac{diam(\Gamma_n)}{|\Gamma_n|^p}\to 0.$$

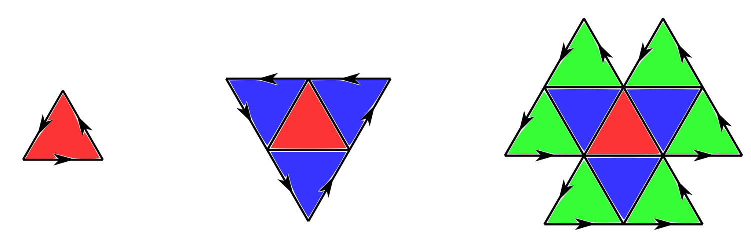
Corollary. For the universal cover $\tilde{X}$ of a compact length space X with $diam(X)=1$, we have for any $p>0$,

$$diam(\tilde{X})=o\left(|\pi_1(X,x)|^p\right).$$

8. Examples

Abelian case[4]**.** Petrunin constructed a sequence of compact length spaces X_n such that $|\pi_1(X_k)|=3\cdot 2^k$, but $diam(\tilde{X}_k)\sim k$ as follows:

- Start with an equilateral Reuleaux triangle.
- Attach an equilateral Reuleaux triangle on each side of the figure you have.
- Repeat the previous step $k-1$ times.
- Identify all sides as one.



Nonabelian case[4]**.** Greg Kuperberg used the Coxeter presentation of the symmetric group S_k to build a sequence of spaces X_k such that $|\pi_1(X_k)|=k!$, but $diam(\tilde{X}_k)\sim k^2$.

9. Why?

We conjecture that this phenomenom is related to a concentration behaviour tied to the Jordan-Turing-Gelander Theorem[2], which states that manifolds cannot be approximated by vertex transitive graphs unless they are tori (therefore, have holes).

Heuristically, a sequence of vertex transitive spiderweb graphs cannot concentrate around a finite dimensional space, so it should behave like a Levi-family and the "volume" will increase much more than the diameter.

10. References

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3. Petrunin, A. (2009). Diameter of m-fold cover. https://mathoverflow.net/questions/7732/diameter-of-m-fold-cover.
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5. Zamora, S. (2017). On the diameter of universal cover. arXiv:1807.08827

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