First Betti number and collapse
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Abstract
We show that when a sequence of Riemannian manifolds collapses under a lower Ricci curvature bound, the first Betti number cannot drop more than the dimension.

1 Introduction
For \( n \in \mathbb{N} \), \( c \in \mathbb{R} \), \( D > 0 \), let \( \mathcal{M}_{\text{Ric}}(n, c, D) \) (resp. \( \mathcal{M}_{\text{sec}}(n, c, D) \)) denote the class of closed \( n \)-dimensional Riemannian manifolds of Ricci curvature \( \geq c \) (resp. sectional curvature \( \geq c \)) and diameter \( \leq D \). A significant proportion of the subject consists of understanding the relationship between sequences \( X_i \in \mathcal{M}_{\text{Ric}}(n, c, D) \) and their Gromov–Hausdorff limits. Our main result concerns the first Betti number of such limit space.

Theorem 1. Let \( X_i \in \mathcal{M}_{\text{Ric}}(n, c, D) \) be a sequence with \( \beta_1(X_i) \geq r \) for each \( i \). If \( X_i \) converges in the Gromov–Hausdorff sense to a space \( X \) containing a \( k \)-regular point, then
\[
\beta_1(X) \geq r + k - n.
\]

It has been known that for a Riemannian manifold \( M \) of almost non-negative Ricci curvature, if its first Betti number equals its dimension then \( M \) is homeomorphic to a torus. This result has been recently extended to singular spaces by Mondello, Mondino, and Perales [10]. A consequence of their work and Theorem 1 is the following.

Corollary 2. For each \( n \in \mathbb{N} \), there is \( \epsilon > 0 \) such that if \( X_i \in \mathcal{M}_{\text{sec}}(n, -1, \epsilon) \) is a sequence of spaces with \( \beta_1(X_i) \geq n \) that converges in the Gromov–Hausdorff sense to a space \( X \) of Hausdorff dimension \( k \), then \( X \) is bi-Hölder homeomorphic to a flat \( k \)-dimensional torus.

Remark 3. Theorem 1 shows that the first Betti number cannot drop more than the dimension. Contrastingly, the fundamental group can decrease in the limit even if there is no collapse: Otsu has constructed a sequence of metrics in \( S^3 \times \mathbb{R}P^2 \) of positive Ricci curvature that converges in the Gromov–Hausdorff sense to a simply connected 5-dimensional space [11].

Theorem 1 is an improvement of the main result of [15]. On the other hand, the goal of this program is to solve following problem.

Question 4. Assume a sequence \( X_i \in \mathcal{M}_{\text{Ric}}(n, c, D) \) of spaces homeomorphic to the \( n \)-dimensional torus converges in the Gromov–Hausdorff sense to a space \( X \). Is \( X \) necessarily homeomorphic to a torus?

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2 Preliminaries

In this section we recall the required material for Theorem 1 and Corollary 2, which we prove in the following section.

2.1 Gromov–Hausdorff topology

The basics on the subject can be found in ([2], Chapter 7).

Definition 5. We say that a function \( f : X \to Y \) between metric spaces is an \( \varepsilon \)-isometry if for all \( x_1, x_2 \in X \) one has \( |d_X(x_1, x_2) - d_Y(fx_1, fx_2)| \leq \varepsilon \), and \( f(X) \) intersects each closed ball of radius \( \varepsilon \) in \( Y \). We say that a sequence of functions \( f_i : X_i \to Y_i \) are Gromov–Hausdorff approximations if \( f_i \) is an \( \varepsilon_i \)-isometry for some sequence \( \varepsilon_i \to 0 \).

Proposition 6. (Gromov) Let \( X_i \) be a sequence of compact metric spaces, and let \( X \) be a complete metric space. Then the following are equivalent:

- There is a sequence \( f_i : X_i \to X \) of Gromov–Hausdorff approximations.
- There is a sequence \( h_i : X \to X_i \) of Gromov–Hausdorff approximations.

In either case, \( X \) is compact and one says that the sequence \( X_i \) converges to \( X \) in the Gromov–Hausdorff sense. Furthermore, there is a metric on the class of compact metric spaces modulo isometry that yields this topology.

Definition 7. We say that a function \( f : (X, x) \to (Y, y) \) between pointed metric spaces is an \( \varepsilon \)-isometry if \( f(x) = y \), for all \( x_1, x_2 \in B^X(x, 2/\varepsilon) \) one has \( |d_X(x_1, x_2) - d_Y(fx_1, fx_2)| \leq \varepsilon \), and \( f(B^X(x, 2/\varepsilon)) \) intersects each closed ball of radius \( \varepsilon \) in \( B^Y(y, 1/\varepsilon) \). We say that a sequence of functions \( f_i : (X_i, x_i) \to (Y_i, y_i) \) are pointed Gromov–Hausdorff approximations if \( f_i \) is a pointed \( \varepsilon_i \)-isometry for some sequence \( \varepsilon_i \to 0 \).

Proposition 8. (Gromov) Let \( (X_i, x_i) \) be a sequence of proper pointed metric spaces, and let \( (X, x) \) be a complete pointed metric space. Then the following are equivalent:

- There is a sequence \( f_i : (X_i, x_i) \to (X, x) \) of pointed Gromov–Hausdorff approximations.
- There is a sequence \( h_i : (X, x) \to (X_i, x_i) \) of pointed Gromov–Hausdorff approximations.

In either case, \( X \) is proper and one says that the sequence \( (X_i, x_i) \) converges to \( (X, x) \) in the pointed Gromov–Hausdorff sense. Furthermore, there is a metric on the class of proper pointed metric spaces modulo isometry that yields this topology.

For \( n \in \mathbb{N}, c \in \mathbb{R} \), we denote by \( \mathcal{M}_{\text{Ric}}(n, c) \) the class of complete \( n \)-dimensional Riemannian manifolds of Ricci curvature \( \geq c \). One reason we know so much about these families of spaces is because they are pre-compact with respect to the Gromov–Hausdorff topology.

Theorem 9. (Gromov) Let \( (Y_i, y_i) \) be a sequence with \( Y_i \in \mathcal{M}_{\text{Ric}}(n, c) \) for each \( i \). Then one can find a subsequence that converges in the pointed Gromov–Hausdorff sense to some proper metric space \( (Y, y) \).


### 2.2 Equivariant Gromov–Hausdorff convergence

There is a well studied notion of convergence of group actions in this setting.

**Definition 10.** Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space \((Y, q)\). Consider pointed Gromov–Hausdorff approximations \(f_i : (Y_i, q_i) \to (Y, q)\) and \(h_i : (Y, q) \to (Y_i, q_i)\) such that \(d^Y(f_i \circ h_i(y), y) \to 0\) for all \(y \in Y\). Also let \(\Gamma_i \leq Iso(Y_i)\) be a sequence of groups of isometries.

We say that \(\Gamma_i\) converges in the equivariant Gromov–Hausdorff sense to a closed group \(\Gamma \leq Iso(Y)\) if for all \(R, \varepsilon > 0\), one has the following:

- For each \(g \in \Gamma\), there is \(i_0 \in \mathbb{N}\) such that for each \(i \geq i_0\) there is \(g_i \in \Gamma_i\) with \(d^Y(f_i \circ g_i \circ h_i(y), g(y)) \leq \varepsilon\) for all \(y \in B^Y(q, R)\).

- There is \(i_0 \in \mathbb{N}\) such that if \(i \geq i_0\), \(g \in \Gamma_i\) with \(d^Y(gq_i, q_i) \leq R\), then there is \(\gamma \in \Gamma\) such that \(d^Y(f_i \circ g \circ h_i(y), \gamma(y)) \leq \varepsilon\) for all \(y \in B^Y(q, 10R)\).

Although this definition clearly depends on \(f_i\) and \(h_i\), we usually omit this when we state that \(\Gamma_i\) converges to \(\Gamma\).

This definition of equivariant convergence allows one to take limits before or after taking quotients.

**Lemma 11.** Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space \((Y, q)\), and \(\Gamma_i \leq Iso(Y_i)\) a sequence of isometry groups that converges in the equivariant Gromov–Hausdorff sense to a closed group \(\Gamma \leq Iso(Y)\). Then the sequence \((Y_i/\Gamma_i, [q_i])\) converges in the pointed Gromov–Hausdorff sense to \((Y/\Gamma, [q])\).

Since the isometry groups of proper metric spaces are locally compact, one has an Arzelà-Ascoli type result ([5], Proposition 3.6).

**Theorem 12.** Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space \((Y, q)\), and take a sequence \(\Gamma_i \leq Iso(Y_i)\) of groups of isometries. Then there is a subsequence \((Y_{i_k}, q_{i_k}, \Gamma_{i_k})_{k \in \mathbb{N}}\) such that \(\Gamma_{i_k}\) converges in the equivariant Gromov–Hausdorff sense to a closed group \(\Gamma \leq Iso(Y)\).

In [6], Gromov studied which is the structure of discrete groups that act transitively on spaces that look like \(\mathbb{R}^n\). Using the Malcev embedding theorem, he showed that they look essentially like lattices in nilpotent Lie groups. In [1], Breuillard, Green and Tao studied in general what is the structure of discrete groups that have a large portion acting on a space of controlled doubling. It turns out that the answer is still essentially just lattices in nilpotent Lie groups. In [14], the ideas from [6] and [1] are used to obtain the following structure result.

**Theorem 13.** Let \((Z, p)\) be a proper pointed geodesic space of topological dimension \(\ell \in \mathbb{N}\) and let \((D_i, p_i)\) be a sequence of discrete metric spaces converging in the pointed Gromov–Hausdorff sense to \((Z, p)\). Assume there is a sequence of isometry groups \(\Gamma_i \leq Iso(D_i)\)
that act transitively and for each $i$, $\Gamma_i$ is generated by its elements that move $p_i$ at most 10. Then for large enough $i$, there are finite index subgroups $G_i \leq \Gamma_i$ and finite normal subgroups $F_i \triangleleft G_i$ such that $G_i/F_i$ is isomorphic to a lattice in a nilpotent Lie group of dimension $\ell$. In particular, if the groups $\Gamma_i$ are abelian, for large enough $i$ their rank is at most $\ell$.

For $k \in \mathbb{N}$, a proper metric space $X$, we say that $x \in X$ is a $k$-regular point if for any sequence $\lambda_i \to \infty$, the sequence $(\lambda_i X, x)$ converges in the pointed Gromov–Hausdorff sense to $\mathbb{R}^k$. For limits of sequences in $\mathcal{M}_{Ric}(n, c)$, almost all points are regular [3].

**Theorem 14.** (Cheeger–Colding) Let $X_i \in \mathcal{M}_{Ric}(n, c)$ converge in the pointed Gromov–Hausdorff sense to a space $X$. If $R\kappa$ denotes the set of $k$-regular points of $X$, then $R\kappa \neq \emptyset$ implies $k \leq n$, and $\bigcup_{j=0}^{n}R\kappa$ is dense in $X$.

Arguably the most used tool in the theory of Riemannian manifolds of non-negative Ricci curvature is the Cheeger–Gromoll splitting theorem. It was later generalized by Cheeger and Colding to limits of Riemannian manifolds [4]. Using this, one could understand how $\mathbb{R}^k$ arises as a quotient of such spaces.

**Theorem 15.** (Cheeger–Colding) Let $\varepsilon_i \to 0$ and $(Y_i, q_i) \in \mathcal{M}_{Ric}(n, -\varepsilon_i)$ a sequence that converges in the pointed Gromov–Hausdorff sense to $(Y, q)$. If $Y$ contains an isometric copy of $\mathbb{R}^k$, then $Y$ splits as a metric space as $\mathbb{R}^k \times Z$ for some proper geodesic space $Z$ of Hausdorff dimension $\leq n - k$.

**Corollary 16.** Let $\varepsilon_i \to 0$ and $(Y_i, q_i) \in \mathcal{M}_{Ric}(n, -\varepsilon_i)$ a sequence that converges in the pointed Gromov–Hausdorff sense to $(Y, q)$. Assume there is a sequence of groups of isometries $\Gamma_i \leq Iso(Y_i)$ such that $(Y_i/\Gamma_i, [q_i])$ converges in the pointed Gromov–Hausdorff sense to $\mathbb{R}^k$ and $\Gamma_i$ converges in the equivariant Gromov–Hausdorff sense to a group $\Gamma \leq Iso(Y)$. Then $Y$ splits as a metric space as $\mathbb{R}^k \times Z$ for some proper geodesic space $Z$ of Hausdorff dimension $\leq n - k$, and the $Z$-fibers given by this product coincide with the orbits of $\Gamma$.

*Proof.* One can use the submetry $\phi : Y \to Y/\Gamma = \mathbb{R}^k$ to lift the lines of $\mathbb{R}^k$ to lines in $Y$ passing through $q$. By Theorem 15, we get the desired splitting $Y = \mathbb{R}^k \times Z$ with $\phi(z_0, x) = x$ for all $x \in \mathbb{R}^k$ and some $z_0 \in Z$.

Let $g \in \Gamma$ and assume $g(z_0, x) = (z, y)$ for some $z_0, z \in Z$, $x, y \in \mathbb{R}^k$. Then for all $t \geq 1$, one has

$$t|y - x| = |\phi(z_0, x + t(y - x)) - \phi(z_0, x)|$$

$$= |\phi(z_0, x + t(y - x)) - \phi(z, y)|$$

$$\leq d^Y((z_0, x + t(y - x)), (z, y))$$

$$= \sqrt{d^Z(z_0, z)^2 + |(t - 1)(y - x)|^2}.$$ 

As $t \to \infty$, this is only possible if $x = y$, and we conclude that the action of $\Gamma$ respects the splitting $Y = \mathbb{R}^k \times Z$. \qed
2.3 Homology and Ricci curvature bounds

We define the content of a map $A \to X$ between topological spaces to be the image of the natural map $H_1(A) \to H_1(X)$. If $\mathcal{U}$ is a family of subsets of $X$, we denote by $H_1(\mathcal{U} < X) \leq H_1(X)$ the subgroup generated by the contents of the inclusions $U \to X$ with $U \in \mathcal{U}$. This group satisfies a natural monotonicity property.

**Lemma 17.** Let $X$ be a topological space, and $U$, $V$ two families of subsets of $X$. If for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ with $U \subset V$, then $H_1(\mathcal{U} < X) \leq H_1(\mathcal{V} < X)$.

If $\varepsilon > 0$, $X$ is a metric space, and $\mathcal{U}$ is the family of balls of radius $\varepsilon$ in $X$, then we denote $H_1(\mathcal{U} < X)$ simply by $H_1^\varepsilon(X)$. It has been recently shown that limits of sequences in $\mathcal{M}_{\text{Rec}}(n, c, D)$ are semi-locally-simply-connected [13].

**Theorem 18.** (Pan–Wang) Let $X_i \in \mathcal{M}_{\text{Rec}}(n, c, D)$ converge in the Gromov–Hausdorff sense to a space $X$. Then $X$ is semi-locally-simply-connected. In particular, $H_1^\varepsilon(X)$ is trivial for small enough $\varepsilon$.

**Theorem 19.** (Sormani–Wei) Let $X$ be a compact geodesic space. Assume there is $\varepsilon > 0$ such that $H_1^\varepsilon(X)$ is trivial, and let $Y$ be a compact geodesic space with $f : Y \to X$ an $\varepsilon/100$-approximation. Then there is a surjective morphism $H_1(Y) \to H_1(X)$ (independent of $\varepsilon$) whose kernel is precisely $H_1^\varepsilon(Y)$.

**Proof sketch:** We follow the lines of ([12], Theorem 2.1), where they prove this result for $\pi_1$ instead of $H_1$. Each 1-cycle in $X$ can be thought as a family of loops $S^1 \to Y$ with integer multiplicity. For each map $\gamma : S^1 \to Y$, by uniform continuity one could pick finitely many cyclically ordered points $\{z_1, \ldots, z_m\} \subset S^1$ such that $\gamma([z_{j-1}, z_j])$ is contained in a ball of radius $\varepsilon/10$ for each $j$. Then set $\phi(\gamma) : S^1 \to X$ to be the loop with $\phi(\gamma)(z_j) = f(\gamma(z_j))$ for each $j$, and $\phi(\gamma)|_{[z_{j-1}, z_j]}$ a minimizing geodesic from $\phi(\gamma)(z_{j-1})$ to $\phi(\gamma)(z_j)$. Clearly, $\phi(\gamma)$ depends on the choice of the points $z_j$ and the minimizing paths $\phi(\gamma)|_{[z_{j-1}, z_j]}$. However, the homology class of $\phi(\gamma)$ in $H_1(X)$ does not depend on these choices, since different choices yield curves that are $\varepsilon$-uniformly close, which by hypothesis are homologous.

Assume that a 1-cycle $c$ in $Y$ is the boundary $\partial \sigma$ of a 2-chain $\sigma$. After taking iterated barycentric subdivision, one could assume that each simplex of $\sigma$ is contained in a ball of radius $\varepsilon/10$. By recreating $\sigma$ in $X$ via $f$ simple by simplex, one could find a 2-chain whose boundary is $\phi(c)$. This means that $\phi$ induces a map $\phi : H_1(Y) \to H_1(X)$.

In a similar fashion, if a 1-cycle $c$ in $Y$ is such that $\phi(c)$ is the boundary of a 2-chain $\sigma$, one could again apply iterated barycentric subdivision to obtain a 2-chain $\sigma'$ in $X$ whose boundary is $\phi(c)$ and such that each simplex is contained in a ball of radius $\varepsilon/10$. Using $f$ one could recreate the 1-skeleton of $\sigma'$ in $Y$ in such a way that expresses $c$ as a linear combination with integer coefficients of 1-cycles contained in balls of radius $\varepsilon$ in $Y$. This implies that the kernel of $\hat{\phi}$ is contained in $H_1^\varepsilon(Y)$.

If a 1-cycle $c$ in $Y$ is contained in a ball of radius $\varepsilon$, then $\phi(c)$ is contained in a ball of radius $2\varepsilon$ and then by hypothesis, $\phi(c)$ is a boundary. This shows that the kernel of $\hat{\phi}$ is precisely $H_1^\varepsilon(Y)$.

Lastly, for any loop $\gamma : S^1 \to X$, one can create via $f$ a loop $\gamma_1 : S^1 \to Y$ such that $\phi(\gamma_1)$ is uniformly close (and hence homologous) to $\gamma$, so $\hat{\phi}$ is surjective. \qed

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Corollary 20. Let \( X \) be a compact geodesic space. Assume there is \( \rho > 0 \) such that \( H_1^{2\rho}(X) \) is trivial, and consider a sequence \( X_i \) of compact geodesic spaces that converges to \( X \) in the Gromov–Hausdorff sense. Then there is a sequence \( \rho_i \to 0 \) such that \( H_1^{\rho_i}(X_i) = H_1^0(X_i) \) for each \( i \).

Proof. For large enough \( i \), let \( \rho_i \in (0, \rho] \) be such that \( \rho_i \to 0 \) and there is a \( \rho_i/100\)-approximation \( X_i \to X \). One could then apply Theorem 19 for \( \varepsilon \in [\rho_i, \rho] \) to get a map \( H_1(X_i) \to H_1(X) \) whose kernel equals both \( H_1^{\rho_i}(X_i) \) and \( H_1^0(X_i) \). For small \( i \), simply set \( \rho_i = \rho \).

The following results were obtained in [8], and are stated in terms of \( \pi_1 \). The first one states that for \( M \in \mathcal{M}_{\text{Ric}}(n, c, D) \), there is a subgroup \( N \leq H_1(M) \) that can be detected anywhere. The second one states that at regular points, there is a gap phenomenon.

Theorem 21. (Kapovitch–Wilking) For each \( n \in \mathbb{N}, c \in \mathbb{R}, D > 0, \varepsilon_1 > 0 \), there are \( \varepsilon_0 > 0 \), \( C \in \mathbb{N} \), such that the following holds. For each \( M \in \mathcal{M}_{\text{Ric}}(n, c, D) \), there is \( \varepsilon \in [\varepsilon_0, \varepsilon_1] \) and a subgroup \( N \leq H_1(M) \) such that for all \( x \in M \),

- \( N \) lies in the content of the inclusion \( B^M(x, \varepsilon/1000) \to M \).
- The index of \( N \) in the content of the inclusion \( B^M(x, \varepsilon) \to M \) is \( \leq C \).

Lemma 22. (Kapovitch–Wilking) Let \( X_i \in \mathcal{M}_{\text{Ric}}(n, c, D) \) converge in the Gromov–Hausdorff sense to a space \( X \). Consider a \( k \)-regular point \( x \in X \), and \( h_i : X \to X_i \) a sequence of Gromov–Hausdorff approximations. Then there is \( \eta > 0 \) and a sequence \( \eta_i \to 0 \) such that the contents of the inclusions \( B^{X_i}(h_i(x), \eta_i) \to X_i \), \( B^{X_i}(h_i(x), \eta) \to X_i \) coincide.

For the proof of Corollary 2 we require the following result from [10].

Theorem 23. (Mondello–Mondino–Perales) For each \( n \in \mathbb{N} \) there is \( \varepsilon > 0 \) such that if \( X_i \in \mathcal{M}_{\text{sec}}(n, -1, \varepsilon) \) converges in the Gromov–Hausdorff sense to a space \( X \) of Hausdorff dimension \( k \) and \( \beta_1(X) \geq k \), then \( X \) is bi-H"older homeomorphic to a flat \( k \)-dimensional torus.

3 Proof of the main results

Proof of Theorem 1: Let \( p \in X \) be a \( k \)-regular point, \( h_i : X \to X_i \) a sequence of Gromov–Hausdorff approximations, and set \( p_i := h_i(p) \). Then by Theorem 22, there is \( \varepsilon_2 > 0 \) and a sequence \( \eta_i \to 0 \) such that the contents of the maps \( B^{X_i}(p_i, \eta_i) \to X_i \), \( B^{X_i}(p_i, \varepsilon_2) \to X_i \) coincide.

By Theorem 18, there is \( \varepsilon_1 \in (0, \varepsilon_2] \) such that for each \( x \in X \), the content of the inclusion \( B^X(x, 2\varepsilon_1) \to X \) is trivial. By Theorem 19, all we need to show is that for large enough \( i \), \( H_1^{\varepsilon_1}(X_i) \) has rank \( \leq n - k \). By Corollary 20, there is a sequence \( \rho_i \to 0 \) with the property that \( H_1^{\rho_i}(X_i) = H_1^{\varepsilon_1}(X_i) \) for each \( i \).
By Theorem 21, there are $\varepsilon_0 > 0$, $C \in \mathbb{N}$, subgroups $N_i \leq H_1(X_i)$, and a sequence $\delta_i \in [\varepsilon_0, \varepsilon_1]$ with the property that for each $x \in X_i$, the content of the map $B^{X_i}(x, \delta_i) \to X_i$ contains $N_i$ as a subgroup of index $\leq C$.

Let $x_1, \ldots, x_m \in X$ be such that $X = \bigcup_{j=1}^m B^{X_j}(x_j, \varepsilon_0/3)$, and set $x'_i := h_i(x_j)$. Then for large enough $i$, the balls $B^{X_i}(x'_j, \varepsilon_0/2)$ cover $X_i$. This implies that for large enough $i$, each ball of radius $\rho_i$ in $X_i$ is contained in a ball of the form $B^{X_i}(x'_j, \varepsilon_0)$. Hence if we let $U_i$ denote the family $\{B^{X_i}(x'_j, \delta_i)\}_{j=1}^m$, then by Lemma 17 we get

$$H^0_i(X_i) \leq H_1(U_i \prec X_i) \leq H^{\tilde{\Gamma}}_1(X_i) = H^0_i(X_i).$$

Since $H^0_1(X_i)$ is generated by the contents of the inclusions $B^{X_i}(x'_j, \delta_i) \to X_i$, the index of $N_i$ in $H^0_1(X_i)$ is at most $C^m$. Therefore, the rank of $H^0_1(X_i)$ equals the rank of $N_i$ for all large enough $i$.

Let $\Gamma_i \leq H_1(X_i)$ denote the content of the inclusion $B^{X_i}(p_i, \varepsilon_2) \to X_i$. Since $\varepsilon_2 \geq \varepsilon_1$, $\Gamma_i$ contains $N_i$, and since $\Gamma_i$ equals the content of the inclusion $B^{X_i}(p_i, \eta_i) \to X_i$, and $\eta_i \leq \varepsilon_0$ for large enough $i$, the index of $N_i$ in $\Gamma_i$ is finite. Hence Theorem 1 will follow from the following claim.

**Claim:** For large enough $i$, $\Gamma_i$ has rank $\leq n - k$.

Let $\lambda_i \to \infty$ be a sequence that diverges so slowly that $\lambda_i \eta_i \to 0$ and the sequence $(\lambda_i X_i, p_i)$ converges in the pointed Gromov–Hausdorff sense to $\mathbb{R}^k$. We can achieve this since $p$ is $k$-regular and $\eta_i \to 0$.

Let $(Y, q_i)$ denote the regular cover of $(\lambda X_i, p_i)$ with Galois group $H_1(X_i)$. By Theorem 9 and Theorem 12, we can assume that the sequence $(Y_i, q_i)$ converges in the pointed Gromov–Hausdorff sense to a proper geodesic space $(Y, q)$, and the groups $H_1(X_i)$ converge in the equivariant Gromov–Hausdorff sense to some group $\Gamma \leq Iso(Y)$. Since all elements of $H_1(X_i) \backslash \Gamma_i$ move $q_i$ at least $\varepsilon_2 \lambda_i$ away, the equivariant Gromov–Hausdorff limit of $\Gamma_i$ equals $\Gamma$ as well. Note that from the definition of equivariant Gromov–Hausdorff convergence, it follows that the $\Gamma_i$-orbits of $q_i$ converge in the pointed Gromov–Hausdorff sense to the $\Gamma$-orbit of $q$.

By Corollary 16, $Y$ splits isometrically as a product $\mathbb{R}^k \times Z$ with $Z$ a proper geodesic space of Hausdorff dimension $\leq n - k$, such that the $Z$-fibers coincide with the $\Gamma$-orbits. Since the topological dimension is always dominated by the Hausdorff dimension ([7], Chapter 7), the topological dimension of $Z$ is at most $n - k$. Then by Theorem 13, the rank of $\Gamma_i$ is at most $n - k$ for large enough $i$.

**Proof of Corollary 2:** Let $\varepsilon > 0$ be given by Theorem 23. By Theorem 1, $\beta_1(X) \geq k$, and the result follows.

**References**


