Bounding Diameters of Covering Spaces

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Geometry Luncheon Seminar

Let (X, d) be a compact metric space, and $\pi\colon \hat{X} \to X$ an n-fold covering map.

Is there a natural way to lift the metric d to a metric $\hat{d}$ in $\hat{X}$ such that $\pi\colon (\hat{X}, \hat{d}) \to (X, d)$ is a local isometry?

We say that (X, d) is a *length space* if

$$d(p, q) = \inf\{\ell(\gamma) \mid \gamma\colon [0, 1] \to X, \gamma(0) = p, \gamma(1) = q\}.$$

If (X, d) is a length space, we can define the metric $\hat{d}$ on $\hat{X}$ as

$$\hat{d}(x, y) = \inf\{\ell(\pi \circ \gamma) \mid \gamma : [0, 1], \gamma(0) = x, \gamma(1) = y\}.$$

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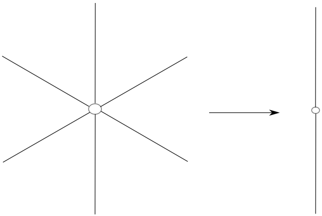
Main Problem

Estimate $diam(\hat{X})$ in terms of X and the number of sheets.

Since $\hat{X}$ as a set is n copies of X one may expect the equality $diam(\hat{X}) = n * diam(X)$, but it is false.



If we re-scale X, then $\hat{X}$ will suffer the same transformation. So we may assume that $diam(X) = 1$.

Theorem (Ivanov, 2010)

Let $\hat{X}$ be the n-fold covering of a compact length space X. If we equip $\hat{X}$ with the lifted metric, then

$$diam(\hat{X}) \leq n.$$

This bound is sharp.

Conjecture (Petrunin, 2009)

If $\hat{X}$ is the n-fold universal cover, and $\alpha > 0$,

$$diam(\hat{X}) = o(n^p) \text{ for any } p > 0.$$

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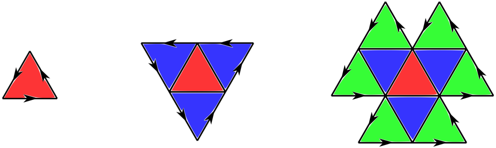
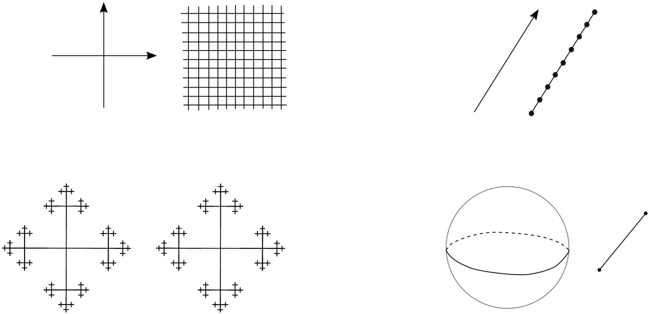
Example (Petrunin)

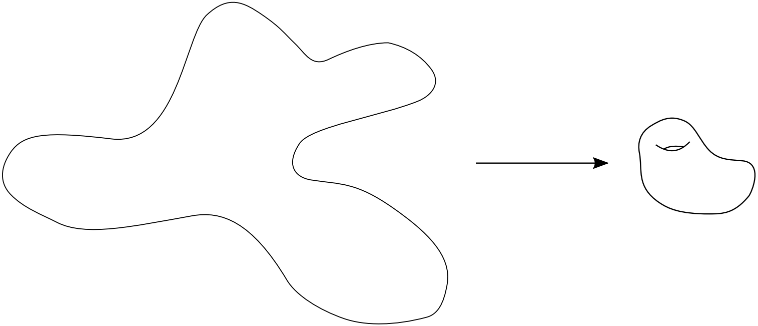
This sequence of spaces have universal coverings of 3, 6, 12, 24, 48, ... sheets (their fundamental groups are $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{24}$, $\mathbb{Z}_{48}$, $\ldots$).

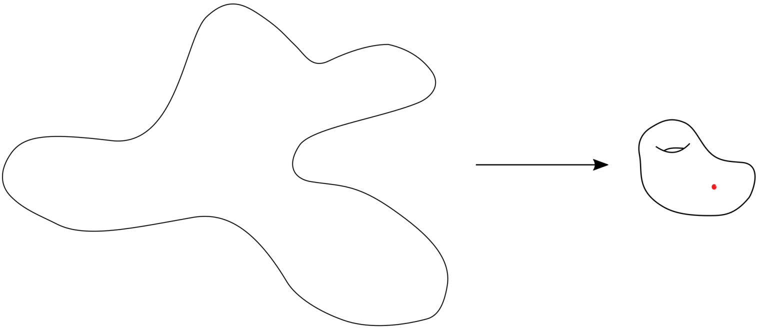


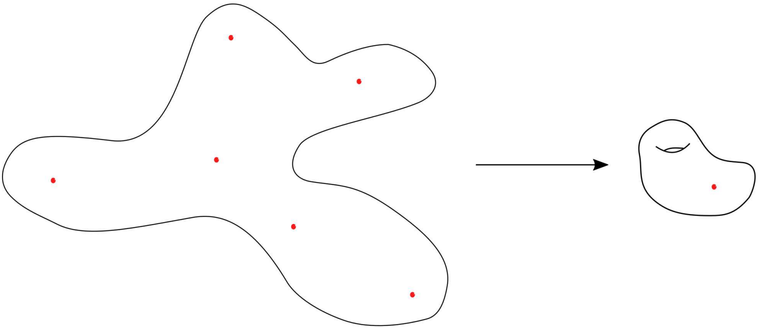
As $n \to \infty$, if we use Reuleaux triangles, we have $diam(\hat{X}) \sim \log(n)$.

Universal Covers look like Cayley graphs of Fundamental Groups.









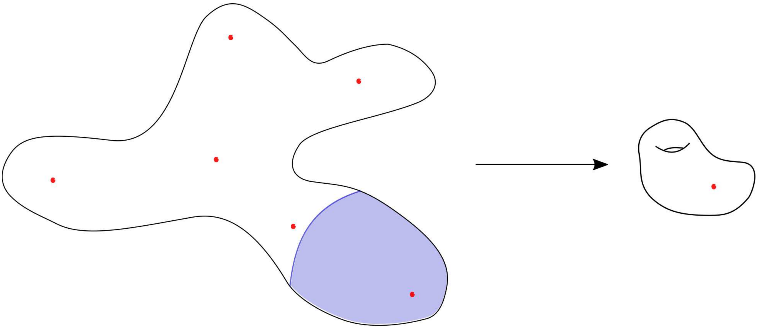
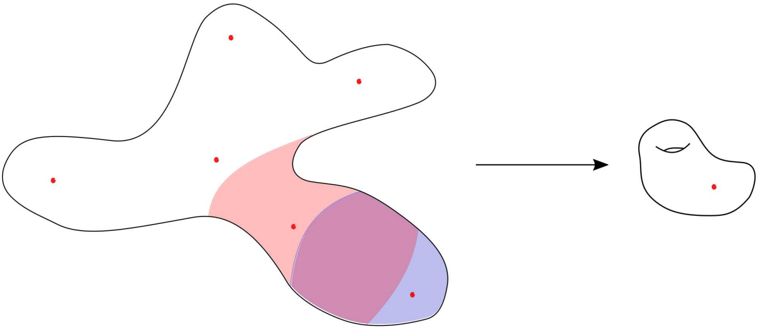


Figure: We obtain an open cover of balls $\{B_i\}_{i=1}^{n}$ of radii $1+\varepsilon$.



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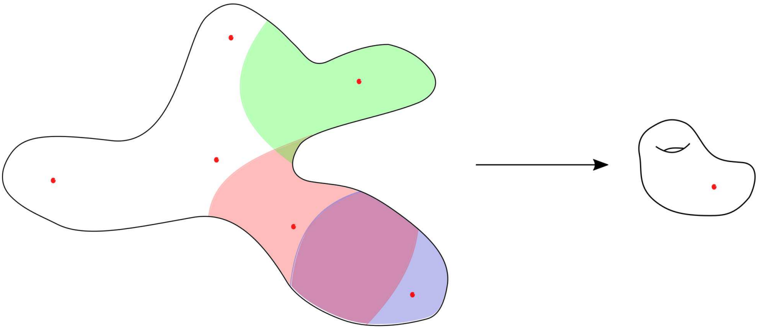

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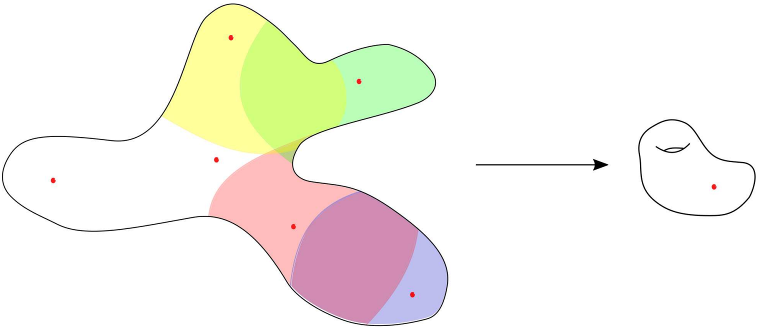

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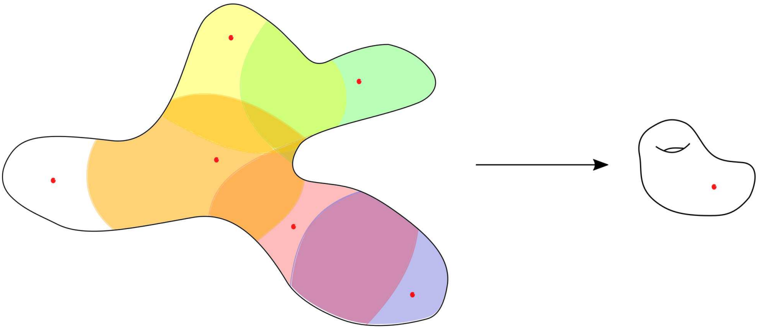

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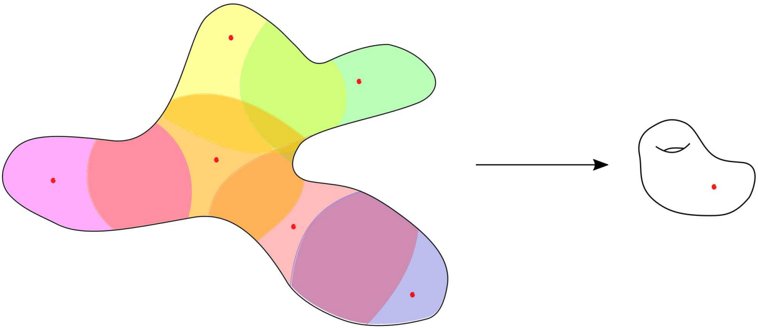

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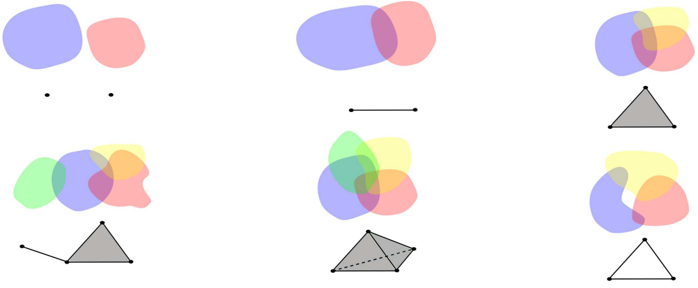
The *nerve* of an open cover $\{U_i\}_{i=1}^n$ is the simplicial complex $N \leq \Delta^{n-1} \subset \mathbb{R}^n$ with the property that a k-dimensional face $Conv(e_{i_0}, \ldots, e_{i_k})$ is in N if and only if $U_{i_0} \cap \ldots \cap U_{i_k} \neq \phi$.



The 1-skeleton N_1 of the nerve associated to the cover $\{B_i\}_{i=1}^n$ is the Cayley graph of the fundamental group $\pi_1(X, p)$ with generators the classes with loops of length less than $2 + 2\varepsilon$.

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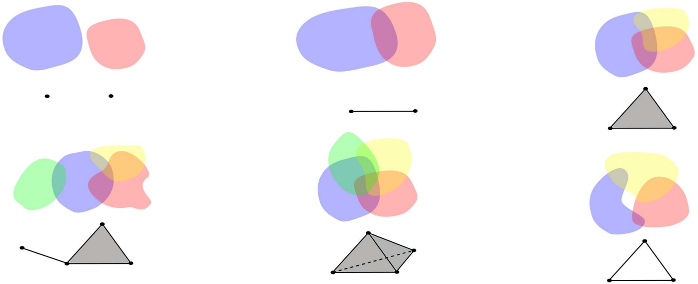

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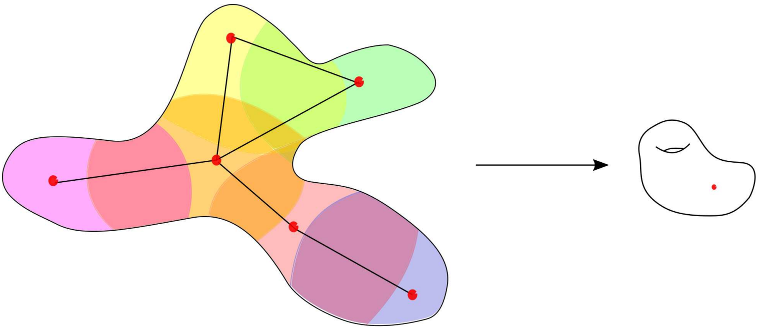

Figure: $diam(\hat{X}) \leq (2 + 2\varepsilon)(diam(N_1) + 1)$.

Associated to the cover $\{B_i\}_{i=1}^n$, we have a partition of unity $\{f_i \colon \hat{X} \to \mathbb{R}\}_{i=1}^n$ with $f_i(x) \neq 0$ if and only if $x \in B_i$.

Nerve Lemma

The map $F \colon \hat{X} \to N \subset \mathbb{R}^n$ given by

$$F(x) = (f_1(x), \ldots, f_n(x))$$

induces an isomorphism of fundamental groups.

Since N is simply connected, its 2-skeleton is simply connected.

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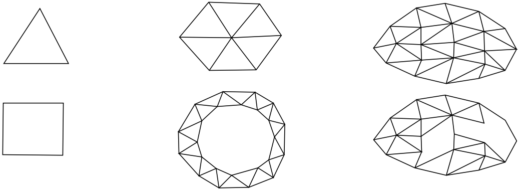
Then N_1 is the 1-skeleton of a 2-dimensional simply connected simplicial complex. That means we can “patch” each hole with triangles. We call this the *triangle patch property*.



New Problem

Estimate $diam(G)$, in terms of $|G|$, where G is a finite Cayley graph with the triangle patch property.

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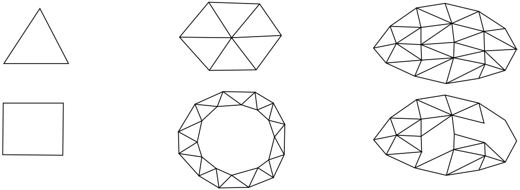

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Estimate *diam*(G), in terms of $|G|$, where G is a finite Cayley graph with the triangle patch property.

Proposition (Z, 2017)

Let G be a Cayley graph with the triangle patch property. Then

$$diam(G) \leq 2\sqrt{|G|}.$$

Corollary

For an n-sheeted universal cover $\hat{X}$ of a compact length space X with $diam(X) = 1$

$$diam(\hat{X}) \leq 4\sqrt{n} + 2.$$

Theorem (Benjamini, Finucane, Tessera, 2012)

If $\{G_k\}_{k\in\mathbb{N}}$ is a sequence of Cayley graphs with the triangle patch property and $diam(G_k) \to \infty$, then

$$diam(G_k) = o(|G_k|^p) \text{ for any } p > 0.$$

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Problems

- Give elementary proof of BFT.
- Describe concentration of measure in large Cayley graphs with the triangle patch property.
- Coarse Ricci curvature?
- Spectral gap, isoperimetric constants?

References

https://mathoverflow.net/questions/7732/diameter-of-m-fold-cover.

https://mathoverflow.net/questions/8534/diameter-of-universal-cover

I. Benjamini, H. Finucane and R. Tessera. On the scaling limit of finite vertex transitive graphs with large diameter. arXiv:1203.5624.