Non-Smooth Curvature

Sergio Zamora

Penn State University

Student Directed Colloquium
Spring 2018
We say a metric space $(X, d)$ is a length space if for any $x, y$,

$$d(x, y) = \inf \{ \ell(\gamma) \mid \gamma : [0, 1] \to X, \gamma(0) = x, \gamma(1) = y \}.$$ 

For any $x, y, z$ in a metric space $(X, d)$, there is a unique triangle $x', y', z'$ in $\mathbb{R}^2$ with the same distance between their vertices. We define the semiangle $\angle yxz$ as the angle at $x'$ of the triangle $x'y'z'$.

For two curves $\alpha, \beta : [0, 1] \to X$ with $\alpha(0) = \beta(0) = p$, we define the angle between $\alpha$ and $\beta$ as

$$\lim_{s,t \to 0} \angle \alpha(s)p\beta(t).$$
We say a metric space \((X, d)\) is a *length space* if for any \(x, y\),
\[
d(x, y) = \inf \{ \ell(\gamma) \mid \gamma : [0, 1] \to X, \gamma(0) = x, \gamma(1) = y \}.\]

For any \(x, y, z\) in a metric space \((X, d)\), there is a unique triangle \(x', y', z'\) in \(\mathbb{R}^2\) with the same distance between their vertices. We define the semiangle \(\angle yxz\) as the angle at \(x'\) of the triangle \(x'y'z'\).

For two curves \(\alpha, \beta : [0, 1] \to X\) with \(\alpha(0) = \beta(0) = p\), we define the angle between \(\alpha\) and \(\beta\) as
\[
\lim_{s,t \to 0} \angle \alpha(s)p \beta(t).\]
We say a metric space \((X, d)\) is a \textit{length space} if for any \(x, y\),

\[
d(x, y) = \inf \{ \ell(\gamma) \mid \gamma : [0, 1] \to X, \gamma(0) = x, \gamma(1) = y \}.
\]

For any \(x, y, z\) in a metric space \((X, d)\), there is a unique triangle \(x', y', z'\) in \(\mathbb{R}^2\) with the same distance between their vertices. We define the semiangle \(\angle yxz\) as the angle at \(x'\) of the triangle \(x'y'z'\).

For two curves \(\alpha, \beta : [0, 1] \to X\) with \(\alpha(0) = \beta(0) = p\), we define the angle between \(\alpha\) and \(\beta\) as

\[
\lim_{s, t \to 0} \angle \alpha(s)p\beta(t).
\]
1-D Curvature
Boring! Curvature does not give information about the object properties.
C.F. Gauss (1777-1855)
2-D Surfaces
A closed subset $\Sigma \subset \mathbb{R}^3$ is called a surface if it is locally the graph of a smooth function.
Examples:

- Torus with one or more holes.
- Planes.
- Graphs of functions $z = f(x, y)$. 
Distance in $\Sigma$

$$d_\Sigma(x, y) = \inf\{\ell(\gamma) | \gamma : [0, 1] \to \Sigma, \gamma(0) = x, \gamma(1) = y\}.$$ 

We say that a curve $\gamma : [0, 1] \to \Sigma$ is minimizing if

$$\ell(\gamma) = d_\Sigma(\gamma(0), \gamma(1)).$$

A curve $\gamma : [a, b] \to \Sigma$ is called a geodesic if for any sufficiently close $a', b'$ in the domain, $\gamma |_{[a', b']}$ is minimizing.
Distance in $\Sigma$

$$d_\Sigma(x, y) = \inf\{\ell(\gamma)|\gamma : [0, 1] \rightarrow \Sigma, \gamma(0) = x, \gamma(1) = y\}.$$

We say that a curve $\gamma : [0, 1] \rightarrow \Sigma$ is minimizing if

$$\ell(\gamma) = d_\Sigma(\gamma(0), \gamma(1)).$$

A curve $\gamma : [a, b] \rightarrow \Sigma$ is called a geodesic if for any sufficiently close $a', b'$ in the domain, $\gamma |_{[a', b']}$ is minimizing.
Distance in $\Sigma$

$$d_\Sigma(x, y) = \inf\{\ell(\gamma)|\gamma : [0, 1] \to \Sigma, \gamma(0) = x, \gamma(1) = y\}.$$  

We say that a curve $\gamma : [0, 1] \to \Sigma$ is minimizing if

$$\ell(\gamma) = d_\Sigma(\gamma(0), \gamma(1)).$$

A curve $\gamma : [a, b] \to \Sigma$ is called a geodesic if for any sufficiently close $a', b'$ in the domain, $\gamma |_{[a', b']}$ is minimizing.
Exponential map
Take $p$ in $\Sigma$, and $\alpha$ a direction in $p$ tangent to $\Sigma$. Then there is a unique geodesic in $\Sigma$, starting at $p$ with direction $\alpha$. We are going to call this geodesic $e^\alpha$. 
Gaussian curvature
For Gauss, curvature is a function $K : \Sigma \rightarrow \mathbb{R}$ defined as follows: To compute $K(p)$, choose a coordinate system in which the tangent plane $T_p\Sigma$ is horizontal.
Considering \( \Sigma \) as the graph of \( z = f(x, y) \), its local behavior is ruled by the eigenvalues \( \lambda_1, \lambda_2 \) of the Hessian of \( f \) at \( p \).

\[
\text{Hess}(f, p) = \begin{bmatrix}
  f_{xx} & f_{xy} \\
  f_{xy} & f_{yy}
\end{bmatrix}
\]

If \( \lambda_1, \lambda_2 > 0 \), we have a local minimum.
If \( \lambda_1, \lambda_2 < 0 \), we have a local maximum.
Considering \( \Sigma \) as the graph of \( z = f(x, y) \), its local behavior is ruled by the eigenvalues \( \lambda_1, \lambda_2 \) of the Hessian of \( f \) at \( p \).

\[
Hess(f, p) = \begin{bmatrix}
    f_{xx} & f_{xy} \\
    f_{xy} & f_{yy}
\end{bmatrix}
\]

If \( \lambda_1, \lambda_2 > 0 \), we have a local minimum.
If \( \lambda_1, \lambda_2 < 0 \), we have a local maximum.
If $\lambda_1 < 0$, $\lambda_2 > 0$, we have a saddle point.

If $\lambda_1, \lambda_2 = 0$, the graph looks like a plane.
If $\lambda_1 = 0$, $\lambda_2 \neq 0$, the graph looks like a cylinder.
If $\lambda_1 < 0$, $\lambda_2 > 0$, we have a saddle point.

If $\lambda_1, \lambda_2 = 0$, the graph looks like a plane.

If $\lambda_1 = 0$, $\lambda_2 \neq 0$, the graph looks like a cylinder.
\[ K(p) = \lambda_1 \lambda_2 \]

*Theorema Egregium (Gauss)*

Let \( \Sigma_1, \Sigma_2 \) be two surfaces, and \( \phi : \Sigma_1 \to \Sigma_2 \) an isometry. Then

\[ K_{\Sigma_2}(\phi(p)) = K_{\Sigma_1}(p) \]

for all \( p \) in \( \Sigma_1 \) (in particular, there is no exact map of any region of earth).
Theorema Egregium (Gauss)

Let \(\Sigma_1, \Sigma_2\) be two surfaces, and \(\phi : \Sigma_1 \to \Sigma_2\) an isometry. Then

\[
K_\Sigma_2(\phi(p)) = K_\Sigma_1(p)
\]

for all \(p\) in \(\Sigma_1\) (in particular, there is no exact map of any region of earth).
Gauss-Bonnet Theorem

Let $\Delta$ be a triangle with angles $\theta_1$, $\theta_2$, $\theta_3$ in a surface $\Sigma$. Then

$$\int_{\Delta} KdA = \theta_1 + \theta_2 + \theta_3 - \pi.$$
Other consequences of curvature?
Let $\Sigma$ be a surface, and $K : \Sigma \to \mathbb{R}$ its curvature.

**Liebman Theorem**
If $K \equiv c > 0$, then $\Sigma$ is a sphere of radius $1/\sqrt{c}$.

**Bonnet Theorem**
If $K \geq c > 0$, then $diam(\Sigma) \leq diam_\pi / \sqrt{c}$.

**Hadamard Theorem**
If $K \leq 0$, then the universal cover of $\Sigma$ is a plane.

**Hilbert Theorem**
The case $K \equiv c < 0$ is impossible.
Two things Riemann noticed:

- Theorema Egregium lets you define curvature in an abstract surface with a metric in it.
- In higher dimensions, curvature at each point is not just a number (Gauss also did know that).

An $n$ dimensional Riemannian manifold (of codimension $k$) is a closed subset $M \subset \mathbb{R}^{n+k}$ which locally is the graph of a smooth function of the form

$$(y_1, y_2, \ldots, y_k) = f(x_1, x_2, \ldots, x_n).$$
Two things Riemann noticed:

- Theorema Egregium lets you define curvature in an abstract surface with a metric in it.
- In higher dimensions, curvature at each point is not just a number (Gauss also did know that).

An $n$ dimensional Riemannian manifold (of codimension $k$) is a closed subset $M \subset \mathbb{R}^{n+k}$ which locally is the graph of a smooth function of the form

$$(y_1, y_2, \ldots, y_k) = f(x_1, x_2, \ldots, x_n).$$
We also say that an abstract length space \((X, d)\) is a Riemannian manifold if it is isometric to one defined above.

Examples:

- Surfaces are examples with \(n = 2\) and \(k = 1\).
- Spheres \(S^{n-1} \subset \mathbb{R}^n\).
- Hyperbolic space.
- Finite dimensional Hilbert spaces.
- Graphs of any smooth function.
- Products of these.
We also say that an abstract length space \((X, d)\) is a Riemannian manifold if it is isometric to one defined above.

Examples:

- Surfaces are examples with \(n = 2\) and \(k = 1\).
- Spheres \(S^{n-1} \subset \mathbb{R}^n\).
- Hyperbolic space.
- Finite dimensional Hilbert spaces.
- Graphs of any smooth function.
- Products of these.
Understanding curvature in dimensions \( > 2 \) is essentially impossible. Instead, one studies curvature of 2-D surfaces inside the manifold \( M \).

Define \( P_M \) as the set of pairs \((p, \Pi)\) with \( p \in M \) and \( \Pi \subset \mathbb{R}^{n+k} \) a 2-dimensional plane tangent to \( M \) at \( p \). The sectional curvature is a function \( \sec : P_M \to \mathbb{R} \) defined as follows:

For \((p, \Pi)\), the geodesics \( e^{\alpha} \) with \( \alpha \) in \( \Pi \) span a surface. \( \sec(p, \Pi) \) will be the Gaussian curvature of that surface at \( p \).
Understanding curvature in dimensions $> 2$ is essentially impossible. Instead, one studies curvature of 2-D surfaces inside the manifold $M$.

Define $P_M$ as the set of pairs $(p, \Pi)$ with $p \in M$ and $\Pi \subset \mathbb{R}^{n+k}$ a 2-dimensional plane tangent to $M$ at $p$. The sectional curvature is a function $sec : P_M \to \mathbb{R}$ defined as follows:

For $(p, \Pi)$, the geodesics $e^\alpha$ with $\alpha$ in $\Pi$ span a surface. $sec(p, \Pi)$ will be the Gaussian curvature of that surface at $p$. 
Let $M$ be an $n$ dimensional manifold and $\text{sec} : P_M \to \mathbb{R}$ its sectional curvature.

**Liebman Theorem**
If $\text{sec} \equiv c > 0$, then the universal cover of $M$ is a sphere of radius $1/\sqrt{c}$.

**Bonnet Theorem**
If $\text{sec} \geq c > 0$, then $\text{diam}(M) \leq \pi/\sqrt{c}$.

**Hadamard Theorem**
If $\text{sec} \leq 0$, then the universal cover of $M$ is homeomorphic to $\mathbb{R}^n$. 
Ricci curvature of a Riemannian manifold $M$. Let $UM$ be the set of pairs $(p, \alpha)$ with $p \in M$ and $\alpha$ a direction at $p$ tangent to $M$. The Ricci curvature is a function $Ric : UM \rightarrow \mathbb{R}$.

$Ric(\alpha)$ equals $(n - 1)$ times the average of $\sec(\Pi)$ among all planes $\Pi$ containing $\alpha$. 
Ricci curvature of a Riemannian manifold $M$. Let $UM$ be the set of pairs $(p, \alpha)$ with $p \in M$ and $\alpha$ a direction at $p$ tangent to $M$. The Ricci curvature is a function $Ric : UM \to \mathbb{R}$.

$Ric(\alpha)$ equals $(n - 1)$ times the average of $\sec(\Pi)$ among all planes $\Pi$ containing $\alpha$. 
Let $M$ be an $n$ dimensional manifold and $Ric : UM \to \mathbb{R}$ its Ricci curvature.

**Myers Theorem**
If $Ric \geq (n - 1)c > 0$, then $diam(M) \leq \pi/\sqrt{c}$.

**Splitting Theorem**
If $Ric \geq 0$ and there is a subspace $L \subset M$ isometric to a line, then $M = N \times \mathbb{R}$ for some $n - 1$ dimensional manifold.

**Laplace Eigenvalue Bound**
If $Ric \geq (n - 1)c > 0$ and $M$ is compact, then $\lambda_1 \geq nc$ where $\lambda_1$ is the smallest eigenvalue of the Laplacian in $M$. 
A. D. Alexandrov (1912-1999)
Convex polyhedra look much like surfaces with $K \geq 0$.

Triangles are fat!
Geodesics spread very little.
We say that a complete length space $(X, d)$ is an Alexandrov space of curvature $\geq c$ if geodesics emanating from the same point spread less than geodesics in the surface of curvature $c$.

Riemannian manifolds with sectional curvature $\geq c$ are Alexandrov spaces of curvature $\geq c$. 
Examples:

- The boundary of a convex subset in a Riemannian manifold of sectional curvature $\geq c$ is an Alexandrov space of curvature $\geq c$ (this generalizes the convex polyhedra above).
- Quotients of spaces of curvature $\geq c$ under group actions are spaces of curvature $\geq c$.
- Limits of spaces of curvature $\geq c$ are spaces of curvature $\geq c$. 
Let $X$ be an Alexandrov space of curvature $\geq c$.

**Bonnet Theorem**
If $c > 0$, then $diam(X) \leq \pi / \sqrt{c}$.

**Splitting Theorem (Milka)**
If $c \geq 0$ and there is a subspace $L \subset X$ isometric to a line, then $X = Y \times \mathbb{R}$ for some Alexandrov space $Y$ of curvature $\geq c$.

**Stratification Theorem (Perelman)**
If $X$ is not infinite dimensional, $X$ has a stratification as a union of manifolds.
Dual concept, Alexandrov spaces of curvature $\leq k$ (often called CAT($k$) spaces) also exist. In here, geodesics spread more than in the space of curvature $k$.

Examples

- Manifolds of sectional curvature $\leq k$.
- Convex sets of CAT($k$) spaces (not their boundaries).
- Gluing two CAT($k$) spaces among a common convex set, we obtain another CAT($k$) space.
- Graphs (by previous bullet).
**Hadamard Theorem**

If $X$ is a CAT($k$) space, for $k \leq 0$, the universal cover of $X$ is contractible.

If $k < 0$ and is simply connected its behavior at infinity looks pretty much like the hyperbolic plane.
Mikhail Gromov
Gauss-Bonnet formula

\[ \theta_1 + \theta_2 + \theta_3 = \pi + \int_{\Delta} K dA \]

If \( K \leq -1 \), then \( \text{Area}(\Delta) \leq \pi \). So very large triangles have to be extremely thin.
In trees all triangles are infinitely thin. In fact, they have area 0.
In trees we have the inequality for any vertices $A, B, C, D$.

$$|AB| + |CD| \leq \max\{|AC| + |BD|, |AD| + |BC|\}$$

This inequality characterizes trees.

We say that a metric space $X$ is $\delta$-hyperbolic if it satisfies the inequality up to a constant $\delta \geq 0$. That is, for any $A, B, C, D$ in $X$

$$|AB| + |CD| \leq \max\{|AC| + |BD|, |AD| + |BC|\} + 2\delta.$$

We say that a metric space is hyperbolic if it is $\delta$-hyperbolic for some $\delta$. 
In trees we have the inequality for any vertices $A, B, C, D$.

$$|AB| + |CD| \leq \max\{|AC| + |BD|, |AD| + |BC|\}$$

This inequality characterizes trees.

We say that a metric space $X$ is $\delta$-hyperbolic if it satisfies the inequality up to a constant $\delta \geq 0$. That is, for any $A, B, C, D$ in $X$

$$|AB| + |CD| \leq \max\{|AC| + |BD|, |AD| + |BC|\} + 2\delta.$$

We say that a metric space is hyperbolic if it is $\delta$-hyperbolic for some $\delta$. 
A finitely generated group \((G, S)\) is called *hyperbolic* if its Cayley graph is hyperbolic.

Examples

- Simply connected CAT(k) spaces with \(k < 0\) are hyperbolic.
- Spaces with finite diameter are hyperbolic.
- Free groups are hyperbolic.
- Fundamental groups of CAT(k) spaces with \(k < 0\) are hyperbolic.
- Small cancellation groups.
Karl-Theodor Sturm, Cedric Villani, John Lott
Ricci curvature

But we can push to the other side too!

When we displace one shape to another, in positive Ricci curvature, the ’middle shape’ has larger area
We say that a metric measure space $(X, d, \mu)$ has Ricci curvature $\geq 0$ in the sense of Lott-Sturm-Villani if whenever we have two stains $\mu_0$ and $\mu_1$, and we move one to another in the optimal way, the middle stain $\mu_{1/2}$ is more spread than the average of the other two.

How to formalize it? Put distance in the space of measures and define 'spread'.
We say that a metric measure space \((X, d, \mu)\) has Ricci curvature \(\geq 0\) in the sense of Lott-Sturm-Villani if whenever we have two stains \(\mu_0\) and \(\mu_1\), and we move one to another in the optimal way, the middle stain \(\mu_{1/2}\) is more spread than the average of the other two.

How to formalize it? Put distance in the space of measures and define ’spread’.
The $L^2$-Wasserstein distance in the space of Borel probability measures is defined as

$$d_W(\nu_1, \nu_2) := \inf \sqrt{\int_X (d(x, f(x)))^2 d\nu_1(x)}$$

Where the inf is taken along the functions $f : X \rightarrow X$ with $f_*\nu_1 = \nu_2$.

How much does it cost to move $\nu_1$ to $\nu_2$?
Let $X$ be a metric space with a fixed finite Borel measure $\mu$.

For an absolutely continuous probability measure $\nu \ll \mu$ with density $\rho : X \to \mathbb{R}$, we define its entropy as

$$S(\nu) := -\int_X \rho(x) \log \rho(x) d\mu(x) = -\int_X \log \rho(x) d\nu(x).$$

If $\nu$ is not absolutely continuous, we define $S(\nu)$ as $-\infty$. 

We say that a compact length metric measure space \((X, d, \mu)\) is a \(CD(0, \infty)\) space if for any two Borel probability measures \(\mu_0, \mu_1\) on \(X\), there is a midpoint \(\mu_{1/2}\) with

\[
S(\mu_{1/2}) \geq \frac{S(\mu_0) + S(\mu_1)}{2}.
\]

We say that it is a \(CD(c, \infty)\) space, if we replace the above inequality by

\[
S(\mu_{1/2}) \geq \frac{S(\mu_0) + S(\mu_1)}{2} + \frac{cd_W(\mu_0, \mu_1)}{8}.
\]

This can be seen as a convexity condition of \(S\) in the space of measures.
We say that a compact length metric measure space \((X, d, \mu)\) is a \(CD(0, \infty)\) space if for any two Borel probability measures \(\mu_0, \mu_1\) on \(X\), there is a midpoint \(\mu_{1/2}\) with

\[
S(\mu_{1/2}) \geq \frac{S(\mu_0) + S(\mu_1)}{2}.
\]

We say that it is a \(CD(c, \infty)\) space, if we replace the above inequality by

\[
S(\mu_{1/2}) \geq \frac{S(\mu_0) + S(\mu_1)}{2} + \frac{cd_W(\mu_0, \mu_1)}{8}.
\]

This can be seen as a convexity condition of \(S\) in the space of measures.
We say that a compact length metric measure space \((X, d, \mu)\) is a \(CD(0, \infty)\) space if for any two Borel probability measures \(\mu_0, \mu_1\) on \(X\), there is a midpoint \(\mu_{1/2}\) with

\[
S(\mu_{1/2}) \geq \frac{S(\mu_0) + S(\mu_1)}{2}.
\]

We say that it is a \(CD(c, \infty)\) space, if we replace the above inequality by

\[
S(\mu_{1/2}) \geq \frac{S(\mu_0) + S(\mu_1)}{2} + \frac{c d_{W}(\mu_0, \mu_1)}{8}.
\]

This can be seen as a convexity condition of \(S\) in the space of measures.
Examples:

- If $\mu \equiv 0$, no probability measure is absolutely continuous, $S \equiv -\infty$, and $(X, d, \mu)$ is an $CD(0, \infty)$ space trivially.
- A Riemannian manifold with $Ric \geq c$ is a $CD(c, \infty)$ space.

**Compatibility Theorem (Petrunin)**

If $(X, d)$ is an n-dimensional Alexandrov space of curvature $\geq 0$, then $(X, d, \mu^H_n)$ is an $CD(0, \infty)$ space.

**Stability Theorem**

If a sequence of $CD(c, \infty)$ spaces $(X_n, d_n, \mu_n)$ converges to $(X, d, \mu)$, the limit is an $CD(c, \infty)$ space.
One can also formulate (in a similar, but more technical way) the $CD(c, N)$ condition (Ricci curvature $\geq c$ and dimension $\leq N$).

**Myers Theorem**

If $(X, d, \mu)$ is a $CD(c, N)$ space, and $c > 0$. Then $diam(X) \leq \sqrt{N - 1}/\sqrt{c}$.

**Brunn-Minkowski Inequality**

Let $(X, d, \mu)$ be a $CD(0, N)$ space. For $A, B \subset X$, form the set $C = \{z \mid z$ is a midpoint of $x$ and $y, x \in A, y \in B\}$.

Then $[\mu(C)]^{\frac{1}{N-1}} \geq ([\mu(A)]^{\frac{1}{N-1}} + [\mu(B)]^{\frac{1}{N-1}})/2$. 
One can also formulate (in a similar, but more technical way) the $CD(c, N)$ condition (Ricci curvature $\geq c$ and dimension $\leq N$).

**Myers Theorem**
If $(X, d, \mu)$ is a $CD(c, N)$ space, and $c > 0$. Then $\text{diam}(X) \leq \sqrt{N - 1}\pi / \sqrt{c}$.

**Brunn-Minkowski Inequality**
Let $(X, d, \mu)$ be a $CD(0, N)$ space. For $A, B \subset X$, form the set

$$C = \{ z \mid z \text{ is a midpoint of } x \text{ and } y, x \in A, y \in B \}.$$ 

Then $[\mu(C)]^{\frac{1}{N-1}} \geq ([\mu(A)]^{\frac{1}{N-1}} + [\mu(B)]^{\frac{1}{N-1}}) / 2$. 
References

A visual introduction to Riemannian curvatures and some discrete generalizations, Yann Ollivier

A course in metric geometry, Dmitri Burago, Yuri Burago, Sergei Ivanov.

Optimal transport and (Ricci) curvature, Cedric Villani.