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On fundamental groups of RCD spaces

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Abstract

We obtain results about fundamental groups of $RCD^*(K,N)$ spaces previously known under additional conditions such as smoothness or lower sectional curvature bounds. For fixed $K\in\mathbb{R}$, $N\in[1,\infty)$, $D>0$, we show the following,

- There is $C>0$ such that for each $RCD^*(K,N)$ space X of diameter $\leq D$, its fundamental group $\pi_1(X)$ is generated by at most C elements.
- There is $\tilde{D}>0$ such that for each $RCD^*(K,N)$ space X of diameter $\leq D$ with compact universal cover $\tilde{X}$, one has $\mathrm{diam}(\tilde{X})\leq\tilde{D}$.
- If a sequence of $RCD^*(0,N)$ spaces X_i of diameter $\leq D$ and rectifiable dimension n is such that their universal covers $\tilde{X}_i$ converge in the pointed Gromov–Hausdorff sense to a space X of rectifiable dimension n, then there is $C>0$ such that for each i, the fundamental group $\pi_1(X_i)$ contains an abelian subgroup of index $\leq C$.
- If a sequence of $RCD^*(K,N)$ spaces X_i of diameter $\leq D$ and rectifiable dimension n is such that their universal covers $\tilde{X}_i$ are compact and converge in the pointed Gromov–Hausdorff sense to a space X of rectifiable dimension n, then there is $C>0$ such that for each i, the fundamental group $\pi_1(X_i)$ contains an abelian subgroup of index $\leq C$.
- If a sequence of $RCD^*(K,N)$ spaces X_i with first Betti number $\geq r$ and rectifiable dimension n converges in the Gromov–Hausdorff sense to a compact space X of rectifiable dimension m, then the first Betti number of X is at least $r+m-n$.

The main tools are the splitting theorem by Gigli, the splitting blow-up property by Mondino–Naber, the semi-locally-simple-connectedness of $RCD^*(K,N)$ spaces by Wang, and the isometry group structure by Guijarro and the first author.

1 Introduction

For $K\in\mathbb{R}$, $N\in[1,\infty)$, the class of $RCD^*(K,N)$ spaces consists of proper metric measure spaces that satisfy a synthetic condition of having Ricci curvature bounded below by K and dimension bounded above by N. This class is closed under measured Gromov–Hausdorff convergence and contains the class of complete Riemannian manifolds of Ricci curvature $\geq K$ and dimension $\leq N$.

Building upon work of Mondino, Pan, Wei, and himself ([29], [30], [31], [37]), Wang recently proved that $RCD^*(K,N)$ spaces are semi-locally-simply-connected [38]. This allows us to identify the fundamental group of an $RCD^*(K,N)$ space with the group of deck transformations of its universal cover.

It was shown by Sosa, and independently by Guijarro and the first author that the isometry group of an arbitrary $RCD^*(K,N)$ space $(X,d,\mathfrak{m})$ is a Lie group. As noted by Cheeger–Colding [8], one could exploit this property to draw results about the groups $\pi_1(X)$. This is what we do throughout this paper.

$RCD^*(K,N)$ spaces have a well defined notion of dimension called *rectifiable dimension* (see Theorem 34), which is always an integer between 0 and N, and is lower semi-continuous with respect to pointed measured Gromov–Hausdorff convergence (see Theorem 40).

For $K\in\mathbb{R}$, $N\in[1,\infty)$, $D>0$, we will denote by $RCD^*(K,N;D)$ the class of compact $RCD^*(K,N)$ spaces of diameter $\leq D$. We write $C(\alpha,\beta,\gamma)$ to denote a constant that depends only on the quantities α,β,γ.

1.1 Finite generatedness and compact universal covers

For a compact semi-locally-simply-connected geodesic space X, its fundamental group is always finitely generated (see [27], Proposition 2.25). For the class of compact N-dimensional smooth Riemannian manifolds of Ricci curvature $\geq K$ and diameter $\leq D$, Kapovitch–Wilking showed that the number of generators of the fundamental group can be controlled in terms of K, N, and D [22]. We extend this result to the non-smooth case.

Theorem 1. Let $(X,d,\mathfrak{m})$ be an $RCD^*(K,N;D)$ space. Then $\pi_1(X)$ can be generated by $\leq C(N,KD^2)$ elements.

In the particular case of spaces with a lower sectional curvature bound, there is an elementary proof of Theorem 1, and one could get a simple explicit expression of C in terms of K, N, and D (see [16], Section 2).

Kapovitch–Wilking showed that if a compact N-dimensional smooth Riemannian manifold has Ricci curvature $\geq K$, diameter $\leq D$, and finite fundamental group, the diameter of its universal cover is bounded above by a constant depending on K, N, and D [22]. We extend this result to the non-smooth case.

Theorem 2. Let $(X,d,\mathfrak{m})$ be an $RCD^*(K,N;D)$ space. If the universal cover $(\tilde{X},\tilde{d},\tilde{\mathfrak{m}})$ is compact, then $\mathrm{diam}(\tilde{X})\leq\tilde{D}(N,KD^2)$.

The proof of Theorem 2 is based on the Malcev embedding theorem (see Theorem 68), and the structure of approximate groups by Breuillard–Green–Tao [5], and is considerably different from the one for the smooth case.

1.2 Uniform virtual abelianness

From the Cheeger–Gromoll splitting theorem [9], one can conclude that the fundamental group of a compact Riemannian manifold of non-negative Ricci curvature is virtually abelian. It is not known whether the index of this abelian subgroup can be bounded by a constant depending only on the dimension. Mazur–Rong–Wang showed that for smooth Riemannian manifolds of non-negative sectional curvature and diameter ≤ 1, this index depends only on the dimension and the volume of a ball of radius 1 in the universal cover [25]. We generalize their results to non-smooth spaces with lower Ricci curvature bounds.

Theorem 3. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(0, N; D)$ spaces of rectifiable dimension n such that the sequence of universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converges in the pointed measured Gromov–Hausdorff sense to some $RCD^*(0, N)$ space of rectifiable dimension n. Then there is $C > 0$ such that for each i there is an abelian subgroup $A_i \leq \pi_1(X_i)$ with index $[\pi_1(X_i) : A_i] \leq C$.

Theorem 4. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(K, N; D)$ spaces of rectifiable dimension n with compact universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$. If the sequence $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converges in the pointed measured Gromov–Hausdorff sense to some $RCD^*(K, N)$ space of rectifiable dimension n, then there is $C > 0$ such that for each i there is an abelian subgroup $A_i \leq \pi_1(X_i)$ with index $[\pi_1(X_i) : A_i] \leq C$.

The proofs of Theorems 3 and 4 are based on Turing and Kazhdan results on discrete approximations of Lie groups ([36], [23]). Just like in the smooth case, the key step consists on showing that non-collapsing sequences of $RCD^*(K, N)$ spaces do not have sequences of non-trivial groups of isometries (see Theorem 89).

1.3 First Betti number

The second author has shown that when a sequence of smooth Riemannian manifolds collapses under a lower Ricci curvature bound, the first Betti number cannot drop more than the dimension [39]. We extend this result to the non-smooth setting.

Theorem 5. Let $(X_i, d_i, \mathfrak{m}_i)$ be $RCD^*(K, N; D)$ spaces of rectifiable dimension n and first Betti number $\beta_1(X_i) \geq r$. If the sequence X_i converges in the measured Gromov–Hausdorff sense to an $RCD^*(K, N; D)$ space X of rectifiable dimension m, then

$$\beta_1(X) \geq r + m - n.$$

1.4 Previously known results

Before it was known that $RCD^*(K, N)$ spaces are semi-locally-simply-connected, Mondino–Wei showed that they always have universal covers. In that same paper they obtain some consequences parallel to classic results. We mention a couple of them here, and refer the reader to [29] for a more complete list. All the results below were originally proved for the revised fundamental group, defined as the group of deck transformations of the universal cover, but thanks to the work of Wang [38], we can state them in terms of fundamental groups.

Theorem 6. (Myers) Let $(X, d, \mathfrak{m})$ be a $RCD^*(K, N)$ space with $K > 0$ and $N > 1$. Then $\pi_1(X)$ is finite.

Theorem 7. (Torus rigidity) Let $(X, d, \mathfrak{m})$ be an $RCD^*(0, N; 1)$ space such that $\pi_1(X)$ has $\lfloor N \rfloor$ independent elements of infinite order. Then a finite sheeted cover of $(X, d, \mathfrak{m})$ is isomorphic as a metric measure space to a flat Riemannian torus.

Theorem 8. (Milnor polynomial growth) Let X be a $RCD^*(0, N)$ space. Then any finitely generated subgroup of $\pi_1(X)$ is virtually nilpotent.

In a more recent work Mondello–Mondino–Perales obtained a first Betti number upper bound and a classification of spaces attaining it [27].

Theorem 9. (Torus stability) Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N; D)$ space with $\beta_1(X) \geq \lfloor N \rfloor$ and $KD^2 \geq -\varepsilon(N)$. Then

- The rectifiable dimension of X is $\lfloor N \rfloor$, and X is $\lfloor N \rfloor$-rectifiable.
- A finite sheeted cover of X is $\delta(KD^2)$-close in the Gromov–Hausdorff sense to a flat torus with $\delta \to 0$ as $KD^2 \to 0$.
- If $N \in \mathbb{N}$, then $\mathfrak{m}$ is a constant multiple of the N-dimensional Hausdorff measure, and X is bi-Hölder homeomorphic to a flat Riemannian torus.

From the work of Breuillard–Green–Tao, one can conclude that the revised fundamental group of almost non-negatively curved $RCD^*(K, N)$ spaces is virtually nilpotent [5].

Theorem 10. (Virtual nilpotency) There is $\varepsilon(N) > 0$, such that for any $RCD^*(-\varepsilon, N; 1)$ space $(X, d, \mathfrak{m})$ of rectifiable dimension m, there is a subgroup $G \leq \pi_1(X)$ and a finite normal subgroup $H \triangleleft G$ such that

- $[\pi_1(X) : G] \leq C(N)$.
- G/H is isomorphic to a lattice in a nilpotent Lie group of dimension $\leq m$.

In particular, $\beta_1(X) \leq m$.

1.5 Open problems

Naturally, it would be interesting to know if the fundamental group of a space of non-negative Ricci curvature is finitely generated.

Conjecture 11. (Milnor) Let $(X, d, \mathfrak{m})$ be an $RCD^*(0, N)$ space. Then $\pi_1(X)$ is finitely generated.

It has been pointed out multiple times (see [12], [21], [22], [25]) that an important problem is to determine if one could remove the non-collapsing hypothesis from Theorems 3 and 4.

Conjecture 12. (Fukaya–Yamaguchi) Let $(X, d, \mathfrak{m})$ be an $RCD^*(0, N; 1)$ space. Then $\pi_1(X)$ contains an abelian subgroup A of index $[\pi_1(X) : A] \leq C(N)$.

Conjecture 13. (Fukaya–Yamaguchi) Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N; D)$ space. If $\pi_1(X)$ is finite, then it contains an abelian subgroup A of index $[\pi_1(X) : A] \leq C(N, KD^2)$.

It is well known that in the class of $RCD^*(K, N; D)$ spaces, the first Betti number is lower semi-continuous with respect to Gromov–Hausdorff convergence (see Theorem 96). It would be interesting to determine if this still holds for their universal covers.

Conjecture 14. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(K, N; D)$ spaces and assume their universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converge in the pointed measured Gromov–Hausdorff sense to an $RCD^*(K, N)$ space X. Then $(X, d, \mathfrak{m})$ is simply connected.

1.6 Overview

In Section 2, we introduce the definitions and background material we will need. In Sections 2.2, 2.3, we review the definition and properties of $RCD^*(K, N)$ spaces. In Sections 2.4, 2.5, we give the definition of Gromov–Hausdorff convergence of metric measure spaces and equivariant convergence of groups of isometries. In Section 2.6 we recall the short basis construction. In Section 2.7 we gather results concerning approximations of Lie groups by discrete groups. Finally in Section 2.8 we discuss the relationship between these discrete approximations and equivariant convergence of groups.

In Section 3, we prove Theorem 1. Our proof is not significantly different in spirit from the one by Kapovitch–Wilking for smooth spaces (see Theorem 77). Assuming the theorem fails one gets a sequence $(X_i, d_i, \mathfrak{m}_i)$ of $RCD^*(K, N)$ spaces contradicting the statement of the theorem. Then by taking a subsequence one can assume the sequence $(X_i, d_i, \mathfrak{m}_i)$ converges to some space of dimension m, and proceed by reverse induction on m. The induction step consists of blowing up a contradictory sequence via a construction of Mondino–Naber in order to obtain another one that converges to a space of strictly higher dimension.

In Section 4, we prove Theorem 2 by via a series of lemmas. Assuming the Theorem fails one gets a sequence of $RCD^*(K, N; D)$ spaces X_i whose universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ are compact but their diameters go to infinity. After taking a subsequence one can assume that the sequence $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converges to a non-compact space $(X, d, \mathfrak{m})$ and the actions of $\pi_1(X_i)$ converge to the one of a non-compact Lie group of isometries $\Gamma \leq Iso(X)$ (see Definition 41). From this we can extract a sequence of approximate morphisms $\pi_1(X_i) \to \Gamma$ (see Definition 57). This ends up being incompatible since the groups $\pi_1(X_i)$ are finite but Γ is not compact (see Theorem 88).

In Section 5, we show Theorems 3 and 4. The key step is showing that when a sequence of $RCD^*(K,N)$ spaces doesn't collapse, it cannot have small groups of measure preserving isometries (see Theorem 89). The results then follow from group theory (see Theorems 65 and 90).

In Section 6, we prove Theorem 5. We first extend the normal subgroup theorem by Kapovitch–Wilking to the non-smooth setting. The normal subgroup theorem provides a subgroup of the first homology group that can be detected essentially everywhere (see Theorem 98). We then apply a Theorem of Sormani–Wei that allows us to compute the first homology of the limit space $(X, d, \mathfrak{m})$ in terms of the first homology groups of the spaces $(X_i, d_i, \mathfrak{m}_i)$ and their subgroups generated by small loops (see Theorem 96 and Corollary 97).

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2 Preamble

2.1 Notation

If (X,d) is a metric space, $p \in X$, and $r > 0$, we denote the open ball of radius r around p in X by $B_p(r,X)$. For $m \in \mathbb{N}$, the m-dimensional Hausdorff measure is denoted by $\mathcal{H}^m$, which we assume normalized so that

$$\int_{B_0(1,\mathbb{R}^m)} (1 - d(0,\cdot)) d\mathcal{H}^m = 1.$$

For a metric space (X,d), we can adjoin a point $*$ at infinite distance from any point of X to get a new space we denote as $X \cup \{*\}$. Similarly, for any group G, we can adjoin an element $*$ whose product with any element of G is defined as $*$ to obtain an algebraic structure on $G \cup \{*\}$.

We write $C(\alpha, \beta, \gamma)$ to denote a constant C that depends only on the quantities α, β, γ.

2.2 Optimal transport

In this section we introduce the basic theory of metric measure spaces we will need. We refer the reader to [1] for proofs and further details.

A *metric measure space* $(X, d, \mathfrak{m})$ consists of a complete, separable, geodesic metric space (X,d) and a Radon measure $\mathfrak{m}$ on the Borel σ-algebra $\mathcal{B}(X)$ with $\operatorname{supp}(\mathfrak{m}) = X$ and

$\mathfrak{m}(X) > 0$. We denote by $\mathbb{P}(X)$ the set of probability measures on $\mathcal{B}(X)$ and by

$$\mathbb{P}_2(X) := \left\{ \mu \in \mathbb{P}(X) \,|\, \int_X d^2(x_0, \cdot) d\mu < \infty \text{ for some (and hence all) } x_0 \in X \right\}$$

the set of probability measures with finite second moments.

Given measures $\mu_1, \mu_2 \in \mathbb{P}_2(X)$, a *coupling* between them is a probability measure $\pi \in \mathbb{P}(X \times X)$ with $p_{1\#}\pi = \mu_1$ and $p_{2\#}\pi = \mu_2$, where $p_1, p_2 : X \times X \to X$, are the natural projections. The L^2-*Wassertein distance* in $\mathbb{P}_2(X)$ is defined as:

$$\mathbb{W}_2^2(\mu, \nu) := \inf \left\{ \int_{X \times X} d^2(x, y) d\pi(x, y) \,|\, \pi \text{ is a coupling between } \mu \text{ and } \nu \right\}.$$

It is known that $(\mathbb{P}_2(X), \mathbb{W}_2)$ inherits good properties from (X, d) such as being complete, separable and geodesic.

We denote by $\mathrm{Geo}(X)$ the space of constant speed geodesics $\gamma : [0, 1] \to X$ and equip it with the topology of uniform convergence. For $t \in [0, 1]$ we denote the evaluation map $e_t : \mathrm{Geo}(X) \to X$ as $\gamma \mapsto \gamma_t$.

Given $\mu_0, \mu_1 \in \mathbb{P}_2(X)$ and a geodesic $(\mu_t)_{t \in [0,1]}$ in $\mathbb{P}_2(X)$, there is $\pi \in \mathbb{P}(\mathrm{Geo}(X))$ with $e_{t\#}\pi = \mu_t$ for all $t \in [0, 1]$, and

$$\mathbb{W}_2^2(\mu_0, \mu_1) = \int_{\mathrm{Geo}(X)} \mathrm{length}^2(\gamma) d\pi(\gamma). \tag{1}$$

The collection of all measures $\pi \in \mathbb{P}(\mathrm{Geo}(X))$ with $e_{0\#}\pi = \mu_0$, $e_{1\#}\pi = \mu_1$, and satisfying Equation 1 with will be denoted as $\mathrm{OptGeo}(\mu_0, \mu_1)$.

2.3 Riemannian curvature dimension condition

In [3], Ambrosio–Gigli–Savaré introduced a class of metric probability spaces satisfying a "Riemannian curvature dimension" condition. This class was later extended by Ambrosio–Gigli–Mondino–Rajala to obtain what is known now as $RCD^*(K, N)$ spaces [2]. Multiple equivalent definitions and reformulations have been obtained throughout the years. We give a definition here but refer the reader to ([3], [11]) for many others.

For $K \in \mathbb{R}$, $N \in [1, \infty)$, the distortion coefficients $\sigma_{K,N}^{(\cdot)}(\cdot) : [0, 1] \times \mathbb{R}^+$ are defined as

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 < 0. \end{cases}$$

Definition 15. For $K \in \mathbb{R}$, $N \geq 1$, we say that a metric measure space $(X, d, \mathfrak{m})$ is a $CD^*(K, N)$ space if for any pair of measures $\mu_0, \mu_1 \in \mathbb{P}_2(X)$ with bounded support there is a measure $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that for each $t \in [0, 1]$ and $N' \geq N$, one has

$$\int_X \rho_t^{1-\frac{1}{N'}} d\mathfrak{m} \geq \int_{\mathrm{Geo}(X)} \left(\sigma_{K,N'}^{(1-t)}(d(\gamma_0, \gamma_1)) \rho_0^{-\frac{1}{N'}}(\gamma_0) + \sigma_{K,N'}^{(t)}(d(\gamma_0, \gamma_1)) \rho_1^{-\frac{1}{N'}}(\gamma_1) \right) d\pi(\gamma),$$

where $e_{t\#}\pi = \rho_t \mathfrak{m} + \mu_t^s$ with $\mu_t^s \perp \mathfrak{m}$ for each $t \in [0, 1]$.

We denote by $\mathrm{LIP}(X)$ the set of Lipschitz functions $f : X \to \mathbb{R}$. For $f \in \mathrm{Lip}(X)$, its *local Lipschitz constant* $\mathrm{Lip}\, f(x) : X \to \mathbb{R}$ is defined as:

$$\mathrm{Lip}\, f(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)},$$

and by convention $\mathrm{Lip}\, f(x) = 0$ if x is isolated.

Definition 16. Let $(X, d, \mathfrak{m})$ be a metric measure space. For $f \in L^2(\mathfrak{m})$ we define its *Cheeger energy* as:

$$\mathrm{Ch}_{\mathfrak{m}}(f) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int |\,\mathrm{Lip}\, f_n|^2 d\mathfrak{m} \mid f_n \in \mathrm{LIP}(X), f_n \to f \text{ in } L^2(\mathfrak{m}) \right\}.$$

The *Sobolev space* $W^{1,2}(X, d, \mathfrak{m})$ is defined as the space of L^2-functions $f : X \to \mathbb{R}$ with $\mathrm{Ch}_{\mathfrak{m}}(f) < \infty$. It is equipped with the norm

$$\|f\|_{W^{1,2}} := \|f\|_{L^2(\mathfrak{m})} + 2\, \mathrm{Ch}_{\mathfrak{m}}(f).$$

If this space is a Hilbert space we say that $(X, d, \mathfrak{m})$ is *infinitesimally Hilbertian*.

Definition 17. We say a $CD^*(K, N)$ space $(X, d, \mathfrak{m})$ is an $RCD^*(K, N)$ space if it is infinitesimally Hilbertian.

A well known property of the $RCD^*(K, N)$ condition is that it can be checked locally. We refer the reader to ([11], Section 3) by Erbar–Kuwada–Sturm for details and further discussions.

Theorem 18. Let $(X, d, \mathfrak{m})$ be a metric measure space. If for each $x \in X$ there is $r > 0$, a pointed $RCD^*(K, N)$ space $(Y, d^Y, \mathfrak{m}^Y, y)$, and a measure preserving isometry $\varphi_x : B_x(r, X) \to B_y(r, X)$, then $(X, d, \mathfrak{m})$ is an $RCD^*(K, N)$ space.

Proof. Since $(Y, d^Y, \mathfrak{m}^Y)$ is an $RCD^*(K, N)$ space, it satisfies $CD^e_{loc}(K, N)$ (see ([11], Section 3) for the precise definition). Hence via φ_x we can find a neighborhood of x such that for any two measures supported there and absolutely continuous with respect to $\mathfrak{m}$, there is a Wasserstein geodesic between them satisfying the inequality of the $CD^e(K, N)$

condition (see [11], Equation 3.1). Therefore, from ([11], Theorem 3.14), $(X, d, \mathfrak{m})$ is a $CD^e(K, N)$ space.

Similarly, we can find for every $x \in X$ via φ_x a ball B with $\mathfrak{m}(\partial B) = 0$ and such that $(B, d|_B, \mathfrak{m}\llcorner B)$ is infinitesimally Hilbertian. From ([3], Theorem 6.22) it then follows that $(X, d, \mathfrak{m})$ is infinitesimally Hilbertian.

Finally by ([11], Theorem 7), this is equivalent to being an $RCD^*(K, N)$ space. $\square$

Remark 19. A direct computation shows that if $(X, d, \mathfrak{m})$ is an $RCD^*(K, N)$ space, then for any $c > 0$, $(X, d, c\mathfrak{m})$ is also an $RCD^*(K, N)$ space, and for any $\lambda > 0$, $(X, \lambda d, \mathfrak{m})$ is an $RCD^*(\lambda^{-2}K, N)$ space.

Let $(X, d^X, \mathfrak{m})$ be an $RCD^*(K, N)$ space and $\rho : Y \to X$ a covering space. Y has a natural geodesic structure such that for any curve $\gamma : [0, 1] \to Y$ one has

$$\text{length}(\rho \circ \gamma) = \text{length}(\gamma).$$

Throughout this paper, we will implicitly assume Y carries such geodesic metric. Set

$$\mathcal{W} := \{W \subset Y \text{ open bounded} \mid \rho|_W : W \to \rho(W) \text{ is an isometry}\}$$

and define a measure $\mathfrak{m}^Y$ on Y by setting $\mathfrak{m}^Y(A) := \mathfrak{m}(\rho(A))$ for each Borel set A contained in an element of $\mathcal{W}$. The measure $\mathfrak{m}^Y$ makes $\rho : Y \to X$ a local isomorphism of metric measure spaces, so by Theorem 18 $(Y, d^Y, \mathfrak{m}^Y)$ is an $RCD^*(K, N)$ space, and its group of deck transformations acts by measure preserving isometries. In particular, this holds for the universal cover [37].

Theorem 20. (Wang) Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space. Then X is semi-locally-simply-connected, so its universal cover $\tilde{X}$ is simply connected.

Due to Theorem 20, for an $RCD^*(K, N)$ space $(X, d, \mathfrak{m})$ we can identify its fundamental group $\pi_1(X)$ with the group of deck transformations of the universal cover $\tilde{X}$.

One of the most powerful tools in the study of $RCD^*(K, N)$ spaces is the Bishop–Gromov inequality by Bacher–Sturm. All we will need here is the following version, and refer the reader to [4] for stronger ones.

Theorem 21. (Bishop–Gromov) For each $K \in \mathbb{R}$, $N \geq 1$, $R > r > 0$, there is $C(K, N, R, r) > 0$ such that for any pointed $RCD^*(K, N)$ space $(X, d, \mathfrak{m}, p)$, one has

$$\mathfrak{m}(B_p(R, X)) \leq C \cdot \mathfrak{m}(B_p(r, X)).$$

Theorem 21 implies that each $RCD^*(K, N)$ is proper. For a proper metric space X, the topology that we use on its group of isometries $Iso(X)$ is the compact-open topology, which in this setting coincides with both the topology of pointwise convergence and the topology of uniform convergence on compact sets. This topology makes $Iso(X)$ a locally compact second countable metric group. In the case $(X, d, \mathfrak{m})$ is an $RCD^*(K, N)$ space, Sosa, and independently Guijarro with the first author showed that this group is a Lie group ([18], [34]).

Theorem 22. (Sosa, Guijarro–Santos) Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space. Then $Iso(X)$ is a Lie group.

An important ingredient in the proof of Theorem 22 is the following lemma [18].

Lemma 23. Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space and $f : X \to X$ an isometry. If $\mathfrak{m}(\{x \in X | fx = x\}) \neq 0$, then $f = Id_X$.

Recall that if X is a proper geodesic space and $\Gamma \leq Iso(X)$ is a closed group of isometries, the metric d' on X/Γ defined as $d'([x], [y]) := \inf_{g\in\Gamma}(d(gx, y))$ makes it a proper geodesic space.

Notice that for a metric measure space $(X, d, \mathfrak{m})$, the subgroup of $Iso(X)$ consisting of measure preserving transformation is closed. By the work of Galaz–Kell–Mondino–Sosa, the class of $RCD^*(K, N)$ spaces is closed under quotients by compact groups of measure preserving isometries [13].

Theorem 24. (Galaz–Kell–Mondino–Sosa) Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space and $\Gamma \leq Iso(X)$ a compact group of measure preserving isometries. Then the metric measure space $(X/\Gamma, d', \mathfrak{m}')$ is an $RCD^*(K, N)$ space, where $\mathfrak{m}'$ is the pushforward of $\mathfrak{m}$ under the projection $X \to X/\Gamma$.

For a proper metric space X, a group of isometries $\Gamma \leq Iso(X)$, and $x \in X$, we denote by Γ_x its *isotropy group at* x. That is,

$$\Gamma_x := \{g \in \Gamma \,|\, gx = x\}.$$

Recall that a group $\Gamma \leq Iso(X)$ is discrete with respect to the compact-open topology if and only if its action on X has discrete orbits and the isotropy group Γ_x is finite for all $x \in X$.

Corollary 25. Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space and $\Gamma \leq Iso(X)$ a discrete group of measure preserving isometries. Then the proper geodesic space X/Γ admits a measure that makes it an $RCD^*(K, N)$ space.

Proof of Corollary 25. Let $F := \{y \in X \,|\, \exists g \in \Gamma - \{Id_X\}, gy = y\}$. We claim that F is a closed set, to see this, take a sequence $\{y_i\}_{i=1}^{\infty} \subset F$ such that $y_i \to y$. Now just notice that there exists some $r_y > 0$ that gives us that for all y_i with $d(y_i, y) < r_y$ we have $\Gamma_{y_i} \leq \Gamma_y$, hence $y \in F$.

Furthermore, this set has zero $\mathfrak{m}-$measure, just observe that F is the countable union of the sets $\{z \in X \,|\, gz = z\}$, $g \in \Gamma$ and that by Lemma 23 all of these have zero $\mathfrak{m}-$measure.

Let $\rho : X \to X/\Gamma$ be the quotient map, as $\rho(F)$ is closed we have

$$X/\Gamma - \rho(F) = \bigcup_{j=1}^{\infty} B_{[x_j]}(r_j, X/\Gamma),$$

where $[x_j] := \rho(x_j)$ and the radii $r_j > 0$ are taken so that the balls $B_{[x_j]}(r_j, X/\Gamma)$ and $B_{[x_j]}(r_j, X/\Gamma_{x_j})$ are isometric.

We define the measure $\mathfrak{m}^\Gamma$ in X/Γ as follows: Given a set $A \in \mathcal{B}(X/\Gamma)$ we define

- $A_0 := A \cap \rho(F)$,
- $A_{k+1} := \left(A - \cup_{j=0}^{k} A_j\right) \cap B_{[x_{k+1}]}(r_{k+1}, X/\Gamma)$

which implies that $A = \sqcup_{j=0}^{\infty} A_j$. Now we define

$$\mathfrak{m}^\Gamma(A) := \sum_{j=0}^{\infty} \mathfrak{m}(\tilde{A}_j),$$

where $\tilde{A}_j := \rho^{-1}(A_j) \cap B_{x_j}(r_j, X)$. It is easy to check that this defines a Radon measure.

Now, given a point $z \in X$ take $r_z > 0$ sufficiently small such that there exists an isometry

$$\varphi_z : B_{[z]}(r_z, X/\Gamma) \to B_{[z]}(r_z, X/\Gamma_z)$$

Denote by $\rho_z : X \to X/\Gamma_z$ the quotient map, by a slight abuse of notation we will also denote $\rho_z(z)$ by $[z]$. As the group Γ_z is finite we can use Theorem 24 to get that the quotient space is an $RCD^*(K, N)$ space.

Take $E \in \mathcal{B}(X/\Gamma)$ compact such that $E \subset B_{[z]}(r_z, X/\Gamma)$, then we have that there must exist some $n \in \mathbb{N}$ such that $E = \sqcup_{j=0}^{n} E_j$.

This gives us that $\mathfrak{m}^\Gamma(E) = \sum_{j=0}^{n} \mathfrak{m}(\tilde{E}_j)$. We can now find isometries $g_j \in \Gamma$ such that $g_j \tilde{E}_j \subset B_z(r_z, X)$ for all $j = 0, \cdots, n$. As the action is by measure preserving isometries we obtain that $\mathfrak{m}^\Gamma(E) = \mathfrak{m}(\sqcup_{j=0}^{n} g_j \tilde{E}_j)$, and let us write $\overline{E} := \sqcup_{j=0}^{n} g_j \tilde{E}_j$ Then we have $\sqcup_{g \in \Gamma_z} g\overline{E} = \rho_z^{-1}\left(\varphi_z(E)\right)$, which yields

$$\rho_{z\#}\mathfrak{m}\left(\varphi_z(E)\right) = \mathfrak{m}\left(\sqcup_{g\in\Gamma_z} g\overline{E}\right) = |\Gamma_z|\mathfrak{m}\left(\overline{E}\right) = |\Gamma_z|\mathfrak{m}^\Gamma(E).$$

So from this it follows that

$$\frac{1}{|\Gamma_z|}\rho_{z\#}\mathfrak{m}\llcorner B_{[z]}(r_z, X/\Gamma_z) = \varphi_{z\#}\mathfrak{m}^\Gamma \llcorner B_{[z]}(r_z, X/\Gamma).$$

And we conclude using Theorem 18. □

Due to Theorem 24 (Corollary 25), whenever we have an $RCD^*(K, N)$ space and a compact (discrete) group of measure preserving isometries $\Gamma \leq Iso(X)$, we will assume that the quotient X/Γ is equipped with a measure that makes it an $RCD^*(K, N)$ space. However, notice that if Γ is infinite and discrete then this measure is not the one given by the image of the measure of X under the projection $X \to X/\Gamma$.

For $K \in \mathbb{R}$, $N \in [1, \infty)$, $D > 0$, we will denote by $RCD^*(K, N; D)$ the class of compact $RCD^*(k, N)$ spaces of diameter $\leq D$.

2.4 Gromov–Hausdorff topology

Definition 26. Let (X_i, p_i) be a sequence of pointed proper metric spaces. We say that it *converges in the pointed Gromov–Hausdorff sense* (or pGH sense) to a proper pointed metric space (X, p) if there is a sequence of functions $\phi_i : X_i \to X \cup \{*\}$ with $\phi_i(p_i) \to p$ and such that for each $R > 0$ one has

$$\phi_i^{-1}(B_p(R, X)) \subset B_{p_i}(2R, X_i) \text{ for large enough } i, \tag{2}$$

$$\lim_{i\to\infty} \sup_{x_1,x_2\in B_{p_i}(2R,X_i)} |d(\phi_i(x_1), \phi_i(x_2)) - d(x_1, x_2)| = 0, \tag{3}$$

$$\lim_{i\to\infty} \sup_{y\in B_p(R,X)} \inf_{x\in B_{p_i}(2R,X_i)} d(\phi_i(x), y) = 0. \tag{4}$$

If in addition to that, $(X_i, d_i, \mathfrak{m}_i)$, $(X, d, \mathfrak{m})$ are metric measure spaces, the maps ϕ_i are Borel measurable, and

$$\int_X f \cdot d((\phi_i)_* \mathfrak{m}_i) \to \int_X f \cdot d(\mathfrak{m})$$

for all $f : X \to \mathbb{R}$ bounded continuous with compact support, then we say that $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges to $(X, d, \mathfrak{m}, p)$ in the *pointed measured Gromov–Hausdorff sense* (or pmGH sense).

Remark 27. Whenever we say that a sequence of spaces X_i converges in the pointed (measured) Gromov–Hausdorff sense to some space X, we implicitly assume the existence of the maps ϕ_i satisfying the above conditions, and if a sequence $x_i \in X_i$ is such that $\phi_i(x_i) \to x \in X$, by an abuse of notation we say that x_i *converges* to x.

Remark 28. If there is a sequence of groups Γ_i acting on X_i by (measure preserving) isometries with $\mathrm{diam}(X_i/\Gamma_i) \leq C$ for some $C > 0$, one could ignore the points p_i when one talks about pointed (measured) Gromov–Hausdorff convergence, as any pair of limits are going to be isomorphic as metric (measure) spaces.

A particular instance of this situation is when all the spaces X_i have diameter $\leq C$ for some $C > 0$. In that case, we simply say that the sequence X_i converges in the (measured) Gromov–Hausdorff sense (or (m)GH sense) to X.

Remark 29. If a sequence of pointed proper metric spaces (X_i, p_i) converges in the pGH sense to the pointed proper metric space (X, p), from the functions $\phi_i : X_i \to X \cup \{*\}$ given by Definition 26 one could construct approximate inverses $\psi_i : X \to X_i$ such that for each $R > 0$ one has

$$\sup_{x\in B_p(R,X)} d(\phi_i\psi_i(x), x) \to 0 \text{ as } i \to \infty. \tag{5}$$

One of the main features of the class of $RCD^*(K,N)$ spaces is the compactness property ([17], [4]).

Definition 30. A pointed $RCD^*(K,N)$ space $(X,d,\mathfrak{m},p)$ is said to be *normalized* if

$$\int_{B_p(1,X)} (1-d(p,\cdot))d\mathfrak{m} = 1.$$

Clearly there is a unique $c>0$ such that $(X,d,c\mathfrak{m},p)$ is normalized, and by Remark 19, it is also an $RCD^*(K,N)$ space.

Theorem 31. (Gromov) If $(X_i,d_i,\mathfrak{m}_i,p_i)$ is a sequence of pointed normalized $RCD^*(K,N)$ spaces, then one can find a subsequence that converges in the pmGH sense to some pointed metric measure space $(X,d,\mathfrak{m},p)$.

Theorem 32. (Bacher–Sturm) The class of pointed normalized $RCD^*(K,N)$ spaces is closed under pmGH convergence.

Let $(X,d,\mathfrak{m})$ be an $RCD^*(K,N)$ space, $p\in X$, and $r>0$. We will denote as $\mathfrak{m}_r^p$ the constant multiple of $\mathfrak{m}$ that satisfies

$$\int_{B_p(r,X)} \left(1-\frac{1}{r}d(p,\cdot)\right) d\mathfrak{m}_r^p = 1.$$

That is, the pointed metric measure space $(X,r^{-1}d,\mathfrak{m}_r^p,p)$ is normalized.

Definition 33. Let $(X,d,\mathfrak{m})$ be an $RCD^*(K,N)$ space and $m\in\mathbb{N}$. We say that $p\in X$ is an *m-regular* point if for each $\lambda_i\to\infty$, the sequence $(X,\lambda_i d,\mathfrak{m}^p_{1/\lambda_i},p)$ converges in the pmGH sense to $(\mathbb{R}^m,d^{\mathbb{R}^m},\mathcal{H}^m,0)$.

Mondino–Naber showed that the set of regular points in an $RCD^*(K,N)$ space has full measure [28]. This result was refined by Brué–Semola who showed that most points have the same local dimension [6].

Theorem 34. (Brué–Semola) Let $(X,d,\mathfrak{m})$ be an $RCD^*(K,N)$ space. Then there is a unique $m\in\mathbb{N}\cap[0,N]$ such that the set of m-regular points in X has full measure. This number m is called the *rectifiable dimension* of X.

Remark 35. By Theorem 22, an $RCD^*(K,N)$ space $(X,d,\mathfrak{m})$ whose isometry group acts transitively is isometric to a Riemannian manifold, so its rectifiable dimension coincides with its topological dimension.

The well known Cheeger–Gromoll splitting theorem was extended by Cheeger–Colding for limits of Riemannian manifolds with lower Ricci curvature bounds [8], and later by Gigli to this setting [15].

Theorem 36. (Gigli) For each $i \in \mathbb{N}$, let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a pointed normalized $RCD^*(-\varepsilon_i, N)$ space with $\varepsilon_i \to 0$. Assume the sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges in the pmGH sense to a space $(X, d, \mathfrak{m}, p)$. If (X, d) contains an isometric copy of $\mathbb{R}^m$, then there is $c > 0$ and a metric measure space (Y, d^Y, ν) such that $(X, d, c\mathfrak{m})$ is isomorphic to the product $(\mathbb{R}^m \times Y, d^{\mathbb{R}^m} \times d^Y, \mathcal{H}^m \otimes \nu)$. Moreover, if $N - m \in [0, 1)$ then Y is a point, and in general, (Y, d^Y, ν) is an $RCD^*(0, N - m)$ space.

This condition of Theorem 36 is not stable under blow-up. That is, if one considers a sequence $\lambda_i \to \infty$, then the sequence of normalized spaces $(X_i, \lambda_i d_i, (\mathfrak{m}_i)^{p_i}_{1/\lambda_i}, p_i)$ may fail to converge to a space containing a copy of $\mathbb{R}^m$. However, Mondino–Naber showed that this issue could be fixed by slightly shifting the basepoint [28].

Theorem 37. (Mondino–Naber) For each $i \in \mathbb{N}$, let $(X_i, d_i, \mathfrak{m}_i)$ be a normalized $RCD^*(-\varepsilon_i, N)$ space with $\varepsilon_i \to 0$. Assume that for some choice $p_i \in X_i$, the sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges in the pmGH sense to $(\mathbb{R}^m, d^{\mathbb{R}^m}, \mathcal{H}^m, 0)$. Then there is a sequence of subsets $U_i \subset B_{p_i}(1, X_i)$ with $\mathfrak{m}_i(U_i)/\mathfrak{m}_i(B_{p_i}(1, X_i)) \to 1$ such that for any sequence $\lambda_i \to \infty$ and any sequence $q_i \in U_i$, the sequence of spaces $(X_i, \lambda_i d_i, (\mathfrak{m}_i)^{q_i}_{1/\lambda_i}, q_i)$ converges (up to subsequence) in the pmGH sense to a pointed $RCD^*(0, N)$ space containing an isometric copy of $\mathbb{R}^m$.

Some consequences of Theorems 36 and 37 are the following results, which were proven by Kitabeppu in [24].

Theorem 38. (Kitabeppu) Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed $RCD^*(K, N)$ spaces of rectifiable dimension m converging in the pmGH sense to the space $(X, d, \mathfrak{m}, p)$. Then the rectifiable dimension of X is at most m.

Corollary 39. Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space of rectifiable dimension m. Then

$$m = \sup\{n \in \mathbb{N} | \text{ there is an } n\text{-regular point } x \in X\}.$$

Remark 40. From Theorem 38, it follows that if the spaces X_i in Theorem 36 have rectifiable dimension n, then the rectifiable dimension of the space Y is at most $n - m$.

2.5 Equivariant Gromov–Hausdorff convergence

There is a well studied notion of convergence of group actions in this setting. For a pointed proper metric space (X, p), we define the distance between two functions $h_1, h_2 : X \to X$ as

$$d_0(h_1, h_2) := \inf_{r>0} \left\{ \frac{1}{r} + \sup_{x \in B_p(r,X)} d(h_1 x, h_2 x) \right\}. \tag{6}$$

When we restrict this metric to $Iso(X)$ we get a left invariant (not necessarily geodesic) metric that induces the compact open topology and makes $Iso(X)$ a proper metric space.

However, this distance is defined on the full class of functions $X \to X$, where it is not left invariant nor proper anymore.

Recall that if a sequence of pointed proper metric spaces (X_i, p_i) converges in the pGH sense to the pointed proper metric space (X, p), one has functions $\phi_i : X_i \to X \cup \{*\}$ and $\psi_i : X \to X_i$ with $\phi_i(p_i) \to p$ and satisfying Equations 2, 3, 4, and 5.

Definition 41. Consider a sequence of pointed proper metric spaces (X_i, p_i) that converges in the pGH sense to a pointed proper metric space (X, p) and a sequence of closed groups of isometries $\Gamma_i \leq Iso(X_i)$. We say that the sequence Γ_i *converges equivariantly* to a closed group $\Gamma \leq Iso(X)$ if:

- For each $g \in \Gamma$, there is a sequence $g_i \in \Gamma_i$ with $d_0(\psi_i g_i \phi_i, g) \to 0$ as $i \to \infty$.
- For a sequence $g_i \in \Gamma_i$ and $g \in Iso(X)$, if there is a subsequence g_{i_k} with $d_0(\psi_{i_k} g_{i_k} \phi_{i_k}, g) \to 0$ as $k \to \infty$, then $g \in \Gamma$.

We say that a sequence of isometries $g_i \in \Gamma_i$ *converges* to an isometry $g \in \Gamma$ if $d_0(\psi_i g_i \phi_i, g) \to 0$ as $i \to \infty$.

This definition of equivariant convergence allows one to take limits before or after taking quotients [12].

Lemma 42. Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pGH sense to a proper space (Y, q), and $\Gamma_i \leq Iso(Y_i)$ a sequence of closed groups of isometries that converges equivariantly to a closed group $\Gamma \leq Iso(Y)$. Then the sequence $(Y_i/\Gamma_i, [q_i])$ converges in the pGH sense to $(Y/\Gamma, [q])$.

Since the isometry groups of proper metric spaces are locally compact, one has an Arzelá-Ascoli type result ([12], Proposition 3.6).

Theorem 43. (Fukaya–Yamaguchi) Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pGH sense to a proper space (Y, q), and take a sequence $\Gamma_i \leq Iso(Y_i)$ of closed groups of isometries. Then there is a subsequence $(Y_{i_k}, q_{i_k}, \Gamma_{i_k})_{k \in \mathbb{N}}$ such that Γ_{i_k} converges equivariantly to a closed group $\Gamma \leq Iso(Y)$.

As a consequence of Theorem 36, it is easy to understand the situation when the quotients of a sequence converge to $\mathbb{R}^m$.

Proposition 44. For each $i \in \mathbb{N}$, let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a normalized pointed $RCD^*(-\varepsilon_i, N)$ space of rectifiable dimension n with $\varepsilon_i \to 0$. Assume $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges in the pmGH sense to $(X, d^X, \mathfrak{m}, p)$, there is a sequence of closed groups of isometries $\Gamma_i \leq Iso(X_i)$ that converges equivariantly to $\Gamma \leq Iso(X)$, and the sequence of pointed proper metric spaces $(X_i/\Gamma_i, [p_i])$ converges in the pGH sense to $(\mathbb{R}^m, 0)$. Then there is $c > 0$ and a metric measure space (Y, d^Y, ν) such that $(X, d^X, c\mathfrak{m})$ is isomorphic to the product $(\mathbb{R}^m \times Y, d^{\mathbb{R}^m} \times d^Y, \mathcal{H}^m \otimes \nu)$. Moreover, if $N - m \in [0, 1)$, then Y is a point, and in general, (Y, d^Y, ν) is an $RCD^*(0, N - m)$ space of rectifiable dimension $\leq n - m$. Furthermore, the Γ-orbits coincide with the Y-fibers of the splitting $\mathbb{R}^m \times Y$.

Proof. One can use the submetry $\phi : X \to X/\Gamma = \mathbb{R}^m$ to lift the lines of $\mathbb{R}^m$ to lines in X passing through p. By Theorem 36 and Remark 40, we get the desired splitting $X = \mathbb{R}^m \times Y$ with the property that for some $q \in Y$, we have $\phi(x, q) = x$ for all $x \in \mathbb{R}^m$.

Now we show that the action of Γ respects the Y-fibers. Let $g \in \Gamma$ and assume $g(x_1, q) = (x_2, y)$ for some $x_1, x_2 \in \mathbb{R}^m$, $y \in Y$. Then for all $t \geq 1$, one has

$$\begin{aligned} t|x_1 - x_2| &= |\phi(x_1 + t(x_2 - x_1), q) - \phi(x_1, q)| \\ &= |\phi(x_1 + t(x_2 - x_1), q) - \phi(x_2, y)| \\ &\leq d^X((x_1 + t(x_2 - x_1), q), (x_2, y)) \\ &= \sqrt{|(t-1)(x_2 - x_1)|^2 + d^Y(q, y)^2}. \end{aligned}$$

As $t \to \infty$, this is only possible if $x_1 = x_2$. ☐

In [16], Gromov studied which is the structure of discrete groups that act transitively on spaces that look like $\mathbb{R}^n$. Using the Malcev embedding theorem, he showed that they look essentially like lattices in nilpotent Lie groups. In [5], Breuillard–Green–Tao studied in general what is the structure of discrete groups that have a large portion acting on a space of controlled doubling. It turns out that the answer is still essentially just lattices in nilpotent Lie groups. In ([41], Sections 7-9) the ideas from [16] and [5] are used to obtain the following structure result.

Theorem 45. Let (Z, p) be a proper pointed geodesic space of topological dimension $\ell \in \mathbb{N}$ and let (D_i, p_i) be a sequence of pointed discrete metric spaces converging in the pGH sense to (Z, p). Assume there is a sequence of isometry groups $\Gamma_i \leq Iso(D_i)$ acting transitively and with the property that for each i, Γ_i is generated by its elements that move p_i at most 10. Then for large enough i, there are finite index subgroups $G_i \leq \Gamma_i$ and finite normal subgroups $F_i \triangleleft G_i$ such that G_i/F_i is isomorphic to a quotient of a lattice in a nilpotent Lie group of dimension ℓ. In particular, if the groups Γ_i are abelian, for large enough i their rank is at most ℓ.

Theorem 46. Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed $RCD^*(-\varepsilon_i, N)$ spaces with $\varepsilon_i \to 0$ admitting a sequence of discrete groups of isometries $\Gamma_i \leq Iso(X_i)$ with $\operatorname{diam}(X_i/\Gamma_i) \to 0$. Assume the spaces $(X_i, d_i, \mathfrak{m}_i)$ are their own universal covers and the sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges in the pGH sense to a space $(X, d, \mathfrak{m}, p)$. Then X is isometric to $\mathbb{R}^m$ for some $m \leq N$.

Proof. By ([41], Theorem 6), X has to be a simply connected nilpotent Lie group. By Theorem 32, X is an $RCD^*(0, N)$ space, so its metric is Riemannian. By the work of Milnor ([26], Theorem 2.4), X is then abelian hence isometric to $\mathbb{R}^m$ for some $m \leq N$. ☐

2.6 Group norms

Let (X, p) be a pointed proper geodesic space and $\Gamma \leq Iso(X)$ a group of isometries. The *norm* $\| \cdot \|_p : \Gamma \to \mathbb{R}$ associated to p is defined as $\|g\|_p := d(gp, p)$. We denote as $\mathcal{G}(\Gamma, X, p, r)$ the subgroup of Γ generated by the elements of norm $\| \cdot \|_p \leq r$. The *spectrum* $\sigma(\Gamma)$ is defined as the set of $r \geq 0$ such that $\mathcal{G}(\Gamma, X, p, r) \neq \mathcal{G}(\Gamma, X, p, r - \varepsilon)$ for all $\varepsilon > 0$. Notice that by definition we always have $0 \in \sigma(\Gamma)$. If we want redundancy we sometimes write $\sigma(\Gamma, X, p)$ to denote the spectrum of the action of Γ on the pointed space (X, p). This spectrum is closely related to the covering spectrum introduced by Sormani–Wei in [33], and it also satisfies a continuity property.

Proposition 47. Let (X_i, p_i) be a sequence of pointed proper metric spaces that converges in the pGH sense to (X, p) and consider a sequence of closed isometry groups $\Gamma_i \leq Iso(X_i)$ that converges equivariantly to a closed group $\Gamma \leq Iso(X)$. Then for any convergent sequence $r_i \in \sigma(\Gamma_i)$, we have $\lim_{i\to\infty} r_i \in \sigma(\Gamma)$.

Proof. Let $r = \lim_{i\to\infty} r_i$ By definition, there is a sequence $g_i \in \Gamma_i$ with $\|g_i\|_p = r_i$, and $g_i \notin \mathcal{G}(\Gamma_i, X_i, p_i, r_i - \varepsilon)$ for all $\varepsilon > 0$. Up to subsequence, we can assume that g_i converges to some $g \in Iso(X)$ with $\|g\|_p = r$.

If $r \notin \sigma(\Gamma)$, it would mean there are $h_1, \ldots, h_k \in \Gamma$ with $\|h_j\|_p < r$ for each $j \in \{1, \ldots, k\}$, and $h_1 \cdots h_k = g$. For each j, choose sequences $h_j^i \in \Gamma_i$ that converge to h_j. As the norm is continuous with respect to convergence of isometries, for i large enough one has $\|h_j^i\|_p < r_i$ for each j.

The sequence $g_i(h_1^i \cdots h_k^i)^{-1} \in \Gamma_i$ converges to $g(h_1 \cdots h_k)^{-1} = e \in \Gamma$, so its norm is less than r_i for i large enough, allowing us to write g_i as a product of $k+1$ elements with norm $< r_i$, thus a contradiction. ☐

Remark 48. It is possible that an element in $\sigma(\Gamma)$ is not a limit of elements in $\sigma(\Gamma_i)$, so this spectrum is not necessarily continuous with respect to equivariant convergence (see [22], Example 1).

Let (X, p) be a pointed proper metric space and $\Gamma \leq Iso(X)$ a group of isometries. A *short basis* of Γ with respect to p is a countable collection $\{\gamma_1, \gamma_2, \ldots\} \subset \Gamma$ such that γ_{j+1} is an element of minimal norm $\| \cdot \|_p$ in $\Gamma \backslash \langle \gamma_1, \ldots, \gamma_j \rangle$. Notice that if $\beta \subset \Gamma$ is a short basis with respect to p, then

$$\sigma(\Gamma, X, p) = \{0\} \cup \{\|\gamma\|_p | \gamma \in \beta\}.$$

If Γ is discrete, a short basis always exists, and if additionally Γ is finitely generated, then any short basis is finite. If a doubling condition is assumed, one can control how many elements of the spectrum lie on a compact subinterval of $(0, \infty)$.

Proposition 49. Let $(X, d, \mathfrak{m}, p)$ be a pointed metric measure space, and assume that for each $R > r > 0$, there is $C(R, r) > 0$ with the property that for each $x \in X$ one has

$$\mathfrak{m}(B_x(R, X)) \leq C \cdot \mathfrak{m}(B_x(r, X)).$$

Then for each $[a,b] \subset (0,\infty)$, any discrete group $\Gamma \leq Iso(X)$, and any short basis $\beta \subset \Gamma$, there are at most $C(3b, a/2)$ elements of β with norm in $[a,b]$.

Proof. Notice that if $g, h \in \Gamma$ are distinct elements of a short basis with $\|g\|_p, \|h\|_p \geq a$, one must have $d(gp, hp) \geq a$. Otherwise, $d(g^{-1}hp, p) < a$ and one must take $g^{-1}h$ as an element of the short basis instead of g or h.

Now assume one has a discrete group of isometries $\Gamma \leq Iso(X)$ having a short basis β with N elements $\{g_1, \ldots, g_N\}$ whose norm is in $[a,b]$. Since the balls of the collection $\{B_{g_jp}(a/2, X)\}_{j=1}^N$ are disjoint and all of them are contained in $B_{g_\ell p}(3b, X)$ for each $\ell \in \{1, \ldots, N\}$, we get

$$\sum_{j=1}^{N} \mathfrak{m}(B_{g_jp}(a/2, X)) \leq \mathfrak{m}(B_{g_\ell p}(3b)) \leq C(3b, a/2) \cdot \mathfrak{m}(B_{g_\ell p}(a/2, X)).$$

Summing over all ℓ, we conclude that $N \leq C(3b, a/2)$. $\square$

It is well known that around a regular point, the spectrum displays a gap phenomenon.

Lemma 50. (Gap Lemma) Let $(Y_i, d_i^Y, \mathfrak{m}_i^Y, p_i)$ be a sequence of pointed $RCD^*(K,N)$ spaces and $\Gamma_i \leq Iso(Y_i)$ a sequence of discrete groups of measure preserving isometries such that the sequence of $RCD^*(K,N)$ spaces $(Y_i/\Gamma_i, [p_i])$ converges in the pmGH sense to an $RCD^*(K,N)$ space (X,q) with q an m-regular point. Then there is $\eta > 0$ and $\eta_i \to 0$ such that for i large enough we have

$$\sigma(\Gamma_i, Y_i, p_i) \cap [\eta_i, \eta] = \emptyset.$$

Proof. Since q is m-regular, we can construct $\eta_i > 0$ converging to 0 so slowly that for any $\lambda_i \to \infty$ with $\lambda_i \eta_i^2 \to 0$, the sequence $(\lambda_i Y_i/\Gamma_i, [p_i])$ converges in the pGH sense to $(\mathbb{R}^m, 0)$. The result then follows from the following claim.

Claim: For each $r_i \in \sigma(\Gamma_i, Y_i, p_i)$ with $r_i \geq \eta_i$, we have $\liminf_{i\to\infty} r_i > 0$.

If after taking a subsequence, $r_i \to 0$, then by our choice of η_i, the sequence $\left(\frac{1}{r_i} Y_i/\Gamma_i, [p_i]\right)$ converges in the pGH sense to $(\mathbb{R}^m, 0)$. Then by Proposition 44, after taking again a subsequence we can assume that $\left(\frac{1}{r_i} Y_i, p_i\right)$ converges in the pGH sense to $(\mathbb{R}^m \times Z, (0,z))$ for some geodesic space Z, and Γ_i converges equivariantly to a closed group $\Gamma \leq Iso(\mathbb{R}^m \times Z)$ in such a way that the Γ-orbits coincide with the Z-fibers. This implies that $\sigma(\Gamma) = \{0\}$, but $1 \in \sigma\left(\Gamma_i, \frac{1}{r_i} Y_i, p_i\right)$ for each i, contradicting Proposition 47. $\square$

2.7 Discrete approximations of Lie groups

This elementary theorem of Jordan is one of the first results in geometric group theory ([35], Section 4.1). We denote the group of unitary operators on a Hilbert space H by $U(H)$, and if $H = \mathbb{C}^m$, we denote $U(H)$ by $U(m)$.

Theorem 51. (Jordan) For each $m \in \mathbb{N}$, there is $C > 0$ such that any finite subgroup $\Gamma \leq U(m)$ contains an abelian subgroup $A \leq \Gamma$ of index $[\Gamma : A] \leq C$.

By Peter–Weyl Theorem, Theorem 51 applies to arbitrary compact Lie groups [42].

Theorem 52. (Peter–Weyl) If G is a compact Lie group, then it has a faithful unitary representation $\rho : G \to U(H)$ for some finite dimensional Hilbert space H.

A torus $\mathbb{R}^m/\mathbb{Z}^m$ has arbitrarily dense finite subgroups. The following result generalizes this fact to non-connected Lie groups ([14], Lemma 3.5).

Theorem 53. (Gelander, Kazhdan, Margulis, Zassenhaus) Let G be a compact Lie group, whose identity component is abelian. Then there is a dense subgroup which is a direct limit of finite subgroups.

Corollary 54. Let G be a compact Lie group. Then there is $C > 0$ such that any virtually abelian subgroup $H \leq G$ has an abelian subgroup $A \leq H$ of index $[H, A] \leq C$.

Proof. Since H is virtually abelian, then so is its closure $\overline{H} \leq G$. This means that the identity connected component of $\overline{H}$ is abelian. By Theorem 53, there is a dense subgroup $H' \leq \overline{H}$ that is a direct limit of finite subgroups. By Theorems 51 and 52, there is $C > 0$, depending only on G, such that all those finite groups have abelian subgroups of index $\leq C$, and hence the same holds for H' and $\overline{H'} = \overline{H}$. Since this property passes to subgroups, H satisfies it too. □

One of the most important applications of Theorem 51 is the structure of crystallographic groups [7].

Theorem 55. (Bieberbach) There is $C(m) > 0$ such that any discrete group of isometries $\Gamma \leq Iso(\mathbb{R}^m)$ has an abelian subgroup $A \leq \Gamma$ of index $[\Gamma : A] \leq C$.

For amenable groups, approximate representations are close to actual representations ([23], Theorem 1). In the statement of Theorem 56, $\| \cdot \|$ stands for the operator norm.

Theorem 56. (Kazhdan) Let Γ be an amenable group, H a Hilbert space, $\varepsilon \in [0, 1/100]$, and $\rho : \Gamma \to U(H)$ a map such that for all $g, h \in \Gamma$, one has $\|\rho(g)\rho(h) - \rho(gh)\| \leq \varepsilon$. Then there is a representation $\rho_0 : \Gamma \to U(H)$ such that for all $g \in \Gamma$ one has $\|\rho(g) - \rho_0(g)\| \leq \varepsilon$.

The following notion of discrete approximation is closely related to the concepts of *approximable groups* studied by Turing in [36] and *good model* studied by Hrushovski in [20] and by Breuillard–Green–Tao in [5].

Definition 57. Let Γ be a Lie group with a left invariant metric and Γ_i a sequence of discrete groups. We say that a sequence of functions $f_i : \Gamma_i \to \Gamma$ is a *discrete approximation* of Γ if there is $R_0 > 0$ such that $f_i^{-1}(B_e(R_0, \Gamma)) \subset \Gamma_i$ is a generating set containing the identity for large enough i, and for each $R > 0$, $\varepsilon > 0$ there is $i_0(R, \varepsilon) \in \mathbb{N}$ such that for all $i \geq i_0$ the following holds:

- the preimage $f_i^{-1}(B_e(R, \Gamma))$ is finite.
- $f_i(\Gamma_i)$ intersects each ball of radius ε in $B_e(R, \Gamma)$.
- For each $g_1, g_2 \in f_i^{-1}(B_e(R, \Gamma))$, one has

$$d(f_i(g_1 g_2^{-1}), f_i(g_1) f_i(g_2^{-1})) \leq \varepsilon.$$

An immediate consequence of the above definition is that if a Lie group Γ with a left invariant metric d^Γ admits a discrete approximation $f_i : \Gamma_i \to \Gamma$ then (Γ, d^Γ) is a proper metric space.

Lemma 58. Let Γ be a Lie group with a left invariant metric, $\Gamma_0 \triangleleft \Gamma$ the connected component of the identity, Γ_i a sequence of discrete groups, and $f_i : \Gamma_i \to \Gamma$ a discrete approximation. Let $r > 0$ be such that $B_e(r, \Gamma) \subset \Gamma_0$. For each $i \in \mathbb{N}$ let $G_i \leq \Gamma_i$ denote the subgroup generated by $f_i^{-1}(B_e(r, \Gamma))$. Then for i large enough, G_i is a normal subgroup of Γ_i.

Proof. First we show that for any fixed $\delta \in (0, r]$, the subgroup of Γ_i generated by $f_i^{-1}(B_e(\delta, \Gamma))$ in Γ_i coincides with G_i for large enough i. To see this, first take a collection $y_1, \ldots, y_n \in B_e(r, \Gamma)$ with

$$B_e(r, \Gamma) \subset \bigcup_{j=1}^{n} B_{y_j}(\delta/10, \Gamma).$$

By connectedness, for each $j \in \{1, \ldots, n\}$ we can construct a sequence $e = z_{j,0}, \ldots, z_{j,k_j} = y_j$ in Γ with $d(z_{j,\ell-1}, z_{j,\ell}) \leq \delta/10$ for each $\ell \in \{1, \ldots, k_j\}$. If $R > 0$ is such that each $z_{j,\ell}$ lies in $B_e(R/2, \Gamma)$, then for each $i \geq i_0(R, \delta/10)$ and any any element $x \in f_i^{-1}(B_e(r, \Gamma))$ we can find y_j with $d(y_j, f_i(x)) \leq \delta/10$, and $e = x_0, \ldots, x_{k_j} = x$ in Γ_i with $d(z_{j,\ell}, f_i(x_\ell)) \leq \delta/10$ for each ℓ. This allows us to write $x = (x_1)(x_1^{-1}x_2) \cdots (x_{k_j-1}^{-1} x_{k_j})$ as a product of k_j elements in $f_i^{-1}(B_e(\delta, \Gamma))$, proving our claim.

Let $R_0 > 0$ be given by the definition of discrete approximation, and choose $\delta > 0$ small enough such that for all $x \in B_e(2R_0, \Gamma)$, $y \in B_e(\delta, \Gamma)$ one has $xyx^{-1} \in B_e(r/2, \Gamma)$. Then for large enough i, the conjugate of an element in $f_i^{-1}(B_e(\delta, \Gamma))$ by an element in $f_i^{-1}(B_e(R_0, \Gamma))$ lies in $f_i^{-1}(B_e(r, \Gamma))$. Since $f_i^{-1}(B_e(R_0, \Gamma))$ generates Γ_i, this shows that G_i is normal in Γ_i. $\square$

Definition 59. Let G be a group and $S \subset G$ a generating subset containing the identity. We say that S is a *determining set* if G has a presentation $G = \langle S | R \rangle$ with R consisting of words of length 3 using as letters the elements of $S \cup S^{-1}$.

Example 60. The subset $\{[-k], [1-k], \dots, [k-1], [k]\} \subset \mathbb{Z}/n\mathbb{Z}$ is a determining set if and only if $k \geq n/3$.

Example 61. If X is a compact Riemannian manifold, then

$$\{[\gamma] \in \pi_1(X) | \ \mathrm{length}(\gamma) \leq 2 \cdot \mathrm{diam}(X)\}$$

is a determining set of $\pi_1(X)$ (see [17], Proposition 5.28).

Definition 62. A discrete approximation is said to be *clean* if there is an open generating set $S_0 \subset \Gamma$, a compact subset $K_0 \subset \Gamma$, and for large enough i symmetric determining sets $S_i \subset \Gamma_i$ with

$$f_i^{-1}(S_0) \subset S_i \subset f_i^{-1}(K_0).$$

If Γ is compact, any discrete approximation is clean with $S_0 = K_0 = \Gamma$.

The following proposition is a simple exercise in real analysis.

Definition 63. Let Γ be a Lie group with a left invariant metric, Γ_i a sequence of discrete groups, and $f_i : \Gamma_i \to \Gamma$ a discrete approximation. We say a sequence of subgroups $H_i \leq \Gamma_i$ consists of *small subgroups* if for any $h_i \in H_i$, we have $f_i(h_i) \to e$.

Proposition 64. Let Γ be a Lie group with a left invariant metric, Γ_i a sequence of discrete groups, $f_i : \Gamma_i \to \Gamma$ a discrete approximation, $H_i \triangleleft \Gamma_i$ a sequence of small normal subgroups, and $\theta_i : \Gamma_i / H_i \to \Gamma_i$ a sequence of sections. Then the functions $\tilde{f}_i : \Gamma_i / H_i \to \Gamma$ defined as $\tilde{f}_i = f_i \circ \theta_i$ form a discrete approximation. Moreover, the sequence $\tilde{f}_i$ is clean if and only if the sequence f_i is clean.

Turing determined which compact Lie groups were approximable by finite groups [36].

Theorem 65. (Turing) Let Γ be a compact Lie group with a left invariant metric, Γ_i a sequence of discrete groups, and $f_i : \Gamma_i \to \Gamma$ a discrete approximation. Then there is $C > 0$ and a sequence of small normal subgroups $H_i \triangleleft \Gamma_i$ such that Γ_i / H_i has an abelian subgroup A_i of index $[\Gamma_i / H_i : A_i] \leq C$ for all i. In particular, the connected component of the identity in Γ is abelian.

A non-compact and considerably deeper version of Turing Theorem was obtained by Breuillard–Green–Tao [5].

Definition 66. Let G be a group, $u_1, \dots, u_r \in G$, $N_1, \dots, N_r \in \mathbb{R}^+$, and $C > 0$. The set $P = P(u_1, \dots, u_r; N_1, \dots, N_r) \subset G$ is defined as the set of elements of G that can be written as words in the u_j's and their inverses such that the number of appearances of u_j and u_j^{-1} is not more than N_j for each j. We say that P is a *nilprogression* of rank r in C-regular form if

- For each $1 \leq j \leq k \leq r$ and any choice of signs,

$$[u_j^{\pm 1}, u_k^{\pm 1}] \in P\left(u_{k+1}, \dots, u_r; \frac{CN_{k+1}}{N_j N_k}, \dots, \frac{CN_r}{N_j N_k}\right)$$

- The expressions $u_1^{n_1} \dots u_r^{n_r}$ represent distinct elements as $n_1, \dots, n_r$ range over the integers with $|n_j| \leq N_j/C$ for each j.

The minimum of $N_1, \dots, N_r$ is called the *thickness* of P and is denoted as thick(P). The set $G(P) := \{u_1^{n_1} \dots u_r^{n_r} \mid |n_j| \leq N_j/C\}$ is called the *grid part* of P. For $\varepsilon \in (0,1]$, the set $P(u_1, \dots, u_r; \varepsilon N_1, \dots, \varepsilon N_r)$ is also a nilprogression in C-regular form and will be denoted as εP.

Theorem 67. (Breuillard–Green–Tao) Let Γ be an r-dimensional Lie group with a left invariant metric, Γ_i a sequence of discrete groups, and $f_i : \Gamma_i \to \Gamma$ a discrete approximation. Then there is $C > 0$, a sequence of small normal subgroups $H_i \triangleleft \Gamma_i$, and rank r nilprogressions $P_i \subset \Gamma_i / H_i$ in C-regular form with thick$(P_i) \to \infty$ and satisfying that for each $\varepsilon \in (0,1]$ there is $\delta > 0$ with

- $\tilde{f}_i^{-1}(B_e(\delta, \Gamma)) \subset G(\varepsilon P_i)$ for i large enough,
- $\tilde{f}_i(G(\delta P_i)) \subset B_e(\varepsilon, \Gamma)$ for i large enough,

where $\tilde{f}_i : \Gamma_i / H_i \to \Gamma$ is the discrete approximation given by Proposition 64. In particular, the connected component of the identity in Γ is nilpotent.

Lattices in simply connected nilpotent Lie groups have large portions that look like nilprogressions. The converse is known as the Malcev embedding theorem ([5], Lemma C.3).

Theorem 68. (Malcev) There is $N(C, r) > 0$ such that for any nilprogression $P(u_1, \dots, u_r; N_1, \dots, N_r)$ in C-regular form with thickness $\geq N$, there is a unique polynomial nilpotent group structure $*_P : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^r$ of degree $\leq N$ that restricts to a group structure on the lattice $\mathbb{Z}^r$ and such that the map $(n_1, \dots, n_r) \to u_1^{n_1} \dots , u_r^{n_r}$ is a group morphism from $(\mathbb{Z}^r, *_P)$ onto the group generated by P, injective in the box $\mathbb{Z}^r \cap \left(\left[-\frac{N_1}{C}, \frac{N_1}{C} \right] \times \dots \times \left[-\frac{N_r}{C}, \frac{N_r}{C} \right] \right)$, which is a determining set of $(\mathbb{Z}^r, *_P)$.

2.8 Equivariant convergence and discrete approximations

Let (X_i, p_i) be a sequence of pointed proper geodesic spaces that converges in the pGH sense to a space (X, p) whose isometry group $Iso(X)$ is a Lie group. Assume there is a sequence of discrete groups $\Gamma_i \leq Iso(X_i)$ that converges equivariantly to a closed group $\Gamma \leq Iso(X)$ and such that diam$(X_i / \Gamma_i) \leq C$ for some $C > 0$. In this section we discuss how to build a discrete approximation $f_i : \Gamma_i \to \Gamma$ out of this situation.

Recall that from the definition of pGH convergence one has functions $\phi_i : X_i \to X \cup \{*\}$ and $\psi_i : X \to X_i$ with $\phi_i(p_i) \to p$ and satisfying Equations 2, 3, 4, and 5.

With these functions, one could define $f_i' : \Gamma_i \to \Gamma \cup \{*\}$ in a straightforward way: for each $g \in \Gamma_i$, if there is an element $\gamma \in \Gamma$ with $d_0(\phi_i \circ g \circ \psi_i, \gamma) \leq 1$, choose $f_i'(g)$ to be an element of Γ that minimizes $d_0(\phi_i \circ g \circ \psi_i, f_i'(g))$. Otherwise, set $f_i'(g) = *$. Then we divide the situation in two cases to obtain $f_i : \Gamma_i \to \Gamma$ out of f_i'. In either case, it is straightforward to verify that f_i is a discrete approximation.

Case 1: X is compact.

The sequence $\operatorname{diam}(X_i)$ is bounded, and there is no need to use $*$ to define f_i' for i large enough. Then the functions $f_i = f_i' : \Gamma_i \to \Gamma$ as defined above will be a discrete approximation. Notice that in this case, Γ is compact and Γ_i is finite for large enough i.

Case 2: X is non-compact.

The group Γ is non-compact and we can find a sequence $g_i \in \Gamma$ with $d(e, g_i) \to \infty$. Then define $f_i : \Gamma_i \to \Gamma$ as

$$f_i(g) := \begin{cases} f_i'(g) & \text{if } f_i'(g) \in \Gamma, \\ g_i & \text{if } f_i(g) = *. \end{cases}$$

Notice that we need the fact that the sequence $\operatorname{diam}(X_i/\Gamma_i)$ is bounded to guarantee that Γ is non-compact.

Remark 69. In case the groups Γ_i were not discrete, the construction of the functions $f_i : \Gamma_i \to \Gamma$ described above still works, but the finiteness condition in the defintion of discrete approximation (and hence Theorems 65 and 67) may fail to hold.

Due to our construction, the maps $f_i : \Gamma_i \to \Gamma$ are actually Gromov–Hausdorff approximations when we equip the groups with the metric d_0 from Equation 6. In particular, we have the following.

Proposition 70. For each $R > 0$, and $\varepsilon > 0$, there is $i_0(R, \varepsilon) \in \mathbb{N}$ such that for $i \geq i_0$ we have

- $f_i(B_e(R, \Gamma_i)) \subset B_e(R + \varepsilon, \Gamma)$.
- $f_i^{-1}(B_e(R, \Gamma)) \subset B_e(R + \varepsilon, \Gamma_i)$.

Corollary 71. A sequence of subgroups $H_i \leq \Gamma_i$ consists of small subgroups in the sense of Definition 63 if and only if $d_0(h_i, Id_{X_i}) \to 0$ for any choice of $h_i \in H_i$.

Remark 72. Due to Corollary 71, if we have a sequence of pointed proper geodesic spaces (X_i, p_i), we will say that a sequence of groups of isometries $H_i \leq Iso(X_i)$ consists of *small subgroups* if $d_0(h_i, Id_{X_i}) \to 0$ for any choice of $h_i \in H_i$.

In order to obtain clean discrete approximations, we consider universal covers. The following lemmas can be found in ([41], Sections 2.4 and 2.12).

Lemma 73. Let $D > 0$, (Y, q) a pointed proper geodesic space, and $G \leq Iso(Y)$ a closed group of isometries with $\text{diam}(Y/G) \leq D$. Then $\{g \in G | \|g\|_q \leq 3D\}$ is a generating set of G.

Remark 74. Notice that in the particular case when G acts transitively, Lemma 73 implies that $\sigma(G) = \{0\}$.

Lemma 75. Let $D > 0$, (Y, q) a pointed proper geodesic space and $G \leq Iso(Y)$ a closed group of isometries with $\text{diam}(Y/G) \leq D$. If $\{g \in G | \|g\|_q \leq 10D\}$ is not a determining set, then there is a non-trivial covering map $\tilde{Y} \to Y$.

Corollary 76. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(K, N; D)$ spaces and $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ their universal covers. Assume for some choice of points $p_i \in \tilde{X}_i$, the sequence $(\tilde{X}_i, p_i)$ converges in the pmGH sense to a pointed $RCD^*(K, N)$ space (X, p) and the sequence $\pi_1(X_i)$ converges equivariantly to a closed group $\Gamma \leq Iso(X)$. Then the discrete approximation $f_i : \pi_1(X_i) \to \Gamma$ constructed above is clean.

Proof. Let $S_i := \{g \in \pi_1(X_i) | \|g\|_{p_i} \leq 10D\}$. Then for each sequence $g_i \in S_i$, the sequence $f_i(g_i) \in \Gamma$ is bounded, so the sets $f_i(S_i)$ are contained in a compact set $K_0 \subset \Gamma$. On the other hand, by Lemma 73 the sequence $T_i := \{g \in \pi_1(X_i) | \|g\|_{p_i} \leq 4D\}$ converges to a compact generating set $T \subset \Gamma$ containing a neighborhood of the identity. Hence $S_0 := \text{int}(T^2)$ is an open generating set of Γ and for large enough i the set $f_i^{-1}(S_0)$ is contained in S_i. By Lemma 75, S_i is determining and the sequence f_i is clean. ∎

3 Bounded generation

In this section we prove Theorem 1 following the lines of [22], but with tools adapted to the non-smooth setting.

Theorem 77. There is $C(K, N, R, r) > 0$ such that the following holds. Let $(X, d, \mathfrak{m}, p)$ be a pointed $RCD^*(K, N)$ space, and $\Gamma \leq Iso(X)$ a discrete group of measure preserving isometries. Then there is a point $q \in B_p(r, X)$ with the property that any short basis of Γ with respect to q has at most C elements of norm $\| \cdot \|_q \leq 3R$.

Theorem 1. Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N; D)$ space. Then $\pi_1(X)$ can be generated by $\leq C(N, KD^2)$ elements.

Proof of Theorem 1: Let $\tilde{X}$ be the universal cover of X. By Theorem 77 there is $x \in \tilde{X}$ such that any short basis of $\pi_1(X)$ with respect to x has at most $\leq C(K, N, D)$ elements of norm $\| \cdot \|_x \leq 3D$. On the other hand, by Lemma 73 no element of such short basis can have norm $\| \cdot \|_x > 3D$ and hence $C(K, N, D)$ elements generate $\pi_1(X)$. $\square$

Proof of Theorem 77: Since for $K > 0$ any $RCD^*(K, N)$ space is automatically an $RCD^*(0, N)$ space, it is enough to assume $K \leq 0$. As in [22], we call a sequence of quadruples $(N_i, X_i, p_i, \Gamma_i)$ a *contradicting sequence* if for each i, $N_i \in \mathbb{N}$, (X_i, p_i) is a pointed $RCD^*(K, N)$ space, and $\Gamma_i \leq Iso(X_i)$ is a discrete group of measure preserving isometries such that

- For each $q \in B_{p_i}(r, X_i)$, Γ_i has a short basis with respect to q with at least N_i elements of norm $\| \cdot \|_q \leq 3R$.
- $N_i \to \infty$.

The first crucial observation is that we can blow up such a contradicting sequence to obtain another one. That is, if $\lambda_i \to \infty$ slowly enough, then for any $q \in B_{p_i}(r, X_i)$ the short basis of the action of Γ_i on X_i with respect to q will have more than $N_i/2$ elements of length $\| \cdot \|_q \leq 3R/\lambda_i$. This is because by Theorem 21 and Proposition 49, the number of elements of such short basis with norm $\| \cdot \|_q$ in $[3R/\lambda_i, 3R]$ is controlled, so we can arrange for most of the elements to have norm $\| \cdot \|_q$ in $[0, 3R/\lambda_i]$.

By Corollary 25, given any contradicting sequence $(N_i, X_i, p_i, \Gamma_i)$, we can always assume after taking a subsequence and renormalizing the measure (notice that the measure does not play any role in the definition of contradicting sequence so we can renormalize it at will) that the quotient $(X_i/\Gamma_i, [p_i])$ converges in the pmGH sense to a pointed $RCD^*(K, N)$ space (X, p) of rectifiable dimension m. The proof is done by reverse induction on m, which ends in finitely many steps since necessarily $m \leq N$.

Let $p_0 \in B_p(r/2, X)$ be an m-regular point and $p_i' \in B_{p_i}(3r/4, X_i)$ be such that $[p_i']$ converges to p_0. By the observation above, if $\lambda_i \to \infty$ slowly enough $(N_i/2, \lambda_i X_i, p_i', \Gamma_i)$ is a contradicting sequence with the additional property that $(\lambda_i X_i/\Gamma_i, [p_i'])$ converges in the pmGH sense (after renormalizing the measure around $[p_i']$) to $(\mathbb{R}^m, d^{\mathbb{R}^m}, \mathcal{H}^m, 0)$. By Theorem 37, there is a sequence $q_i \in B_{p_i'}(r/2, \lambda_i X_i)$ such that for any sequence $\alpha_i \to \infty$, the sequence $(\alpha_i \lambda_i X_i/\Gamma_i, [q_i])$ converges in the pGH sense to a space containing an isometric copy of $\mathbb{R}^m$. Since this holds for any sequence α_i, we choose it as follows:

- We make it diverge so slowly that $(N_i/4, \alpha_i \lambda_i X_i, q_i, \Gamma_i)$ is a contradicting sequence.
- For i large enough, $1/\alpha_i \in \sigma(\Gamma_i, \lambda_i X_i, q_i')$ for some $q_i' \in B_{q_i}(r, \alpha_i \lambda_i X_i)$.

The following claim will finish the induction step, constructing a contradicting sequence whose quotient converges to an $RCD^*(K, N)$ space of rectifiable dimension strictly greater than m.

Claim: The sequence $(\alpha_i\lambda_i X_i/\Gamma_i, [q_i])$ converges in the pmGH sense (up to subsequence and after renormalizing the measure) to an $RCD^*(0,N)$ space of rectifiable dimension strictly greater than m.

By Theorem 36 any partial limit splits as $Y = \mathbb{R}^m \times W$, so all we need is to rule out W being a point. If $Y = \mathbb{R}^m$, then by Proposition 44 after taking a subsequence we can assume that the sequence $(\alpha_i\lambda_i X_i, q_i')$ converges in the pGH sense to a space of the form $(\mathbb{R}^m \times Z, (0,z))$ with (Z,z) a pointed proper geodesic space and Γ_i converges equivariantly to a closed group $\Gamma \leq Iso(\mathbb{R}^m \times Z)$ in such a way that the Γ-orbits coincide with the Z-fibers. Notice however, that $1 \in \sigma(\Gamma_i, \alpha_i\lambda_i X_i, q_i')$ for each i by construction, but by Remark 74, $1 \notin \sigma(\Gamma, \mathbb{R}^m \times Z, (0,z))$, contradicting Proposition 47. ☐

4 Diameter of Compact Universal Covers

In this Section we prove Theorem 2 through a series of lemmas. Our proof is significantly different from the one for the smooth case by Kapovitch–Wilking and is mostly geometric-group-theoretical based on the structure of approximate groups by Breuillard–Green–Tao.

Lemma 78. Let Γ be a Lie group with a left invariant metric, $\Gamma_0 \triangleleft \Gamma$ the connected component of the identity, Γ_i a sequence of discrete groups, and $f_i : \Gamma_i \to \Gamma$ a clean discrete approximation. Let $r > 0$ be such that $B_e(r,\Gamma) \subset \Gamma_0$. For each $i \in \mathbb{N}$ let $G_i \leq \Gamma_i$ denote the subgroup generated by $f_i^{-1}(B_e(r,\Gamma))$. Then for large enough i, there is a surjective morphism $\Gamma_i/G_i \to \Gamma/\Gamma_0$.

For the proof of Lemma 78 we will need the fact below which follows immediately from the definition of determining set.

Proposition 79. Let G, H be groups, $S \subset G$ a determining set, and a function

$$\varphi : S \cup S^{-1} \to H$$

such that $\varphi(s_1s_2) = \varphi(s_1)\varphi(s_2)$ for all $s_1, s_2 \in S \cup S^{-1}$ with $s_1s_2 \in S \cup S^{-1}$. Then φ extends to a unique group morphism $\tilde{\varphi} : G \to H$.

Proof of Lemma 78: Let $S_0 \subset K_0 \subset \Gamma$, $S_i \subset \Gamma_i$ be the sets given by the definition of clean discrete approximation. We then define a map $\varphi_i : S_i \cup S_i^{-1} \to \Gamma/\Gamma_0$ as the composition of f_i with the projection $\Gamma \to \Gamma/\Gamma_0$. Certainly, if i is large enough, $\varphi_i(s_1s_2) = \varphi_i(s_1)\varphi_i(s_2)$ for all $s_1, s_2 \in S_i \cup S_i^{-1}$ with $s_1s_2 \in S_i \cup S_i^{-1}$, as the space Γ/Γ_0 is discrete, and f_i is uniformly close to being a morphism when restricted to $(S_i \cup S_i^{-1})^2 \subset f_i^{-1}((K_0 \cup K_0^{-1})^3)$. By Proposition 79, we then obtain a group morphism $\tilde{\varphi}_i : \Gamma_i \to \Gamma/\Gamma_0$ for all large i.

For large enough i,

$$S_0 \subset \bigcup_{x \in f_i(S_i)} B_x(r,\Gamma),$$

so $f_i(S_i)$ intersects each component of $S_0\Gamma_0$. Since the latter by hypothesis generates Γ, we conclude that $\varphi_i(S_i)$ generates Γ/Γ_0 and $\tilde{\varphi}_i$ is surjective. On the other hand, G_i is clearly in the kernel of $\tilde{\varphi}_i$, so one gets a surjective morphism $\Gamma_i/G_i \to \Gamma/\Gamma_0$. $\square$

Proposition 81 below characterizes determining sets in terms of discrete homotopies ([40], Section 4).

Definition 80. Let G be a group and $S \subset G$ a symmetric generating set. An S-*curve* in G is a sequence of elements $(g_0, \dots, g_n)$ with $g_j^{-1}g_{j-1} \in S$ for each $j \in \{1, \dots, n\}$. If $g_0 = g_n$, the S-curve $(g_0, \dots, g_n)$ is said to be *closed*.

For two S-curves $g = (g_0, \dots, g_n)$, $h = (h_0, \dots, h_m)$ in G, their *uniform distance* $d_U^S(g, h)$ is defined as the infimum $r \in \mathbb{N}$ such that there is a relation $R \subset \{0, \dots, n\} \times \{0, \dots, m\}$ such that:

- For all $i \in \{0, \dots, n\}$, $j \in \{0, \dots, m\}$, there are $i' \in \{0, \dots, n\}$, $j' \in \{0, \dots, m\}$ with $(i, j'), (i', j) \in R$.
- If $(i_1, j_1), (i_2, j_2) \in R$ and $i_1 < i_2$, then $j_1 \leq j_2$.
- $d(g_i, h_j) \leq r$ for all $(i, j) \in R$.

Proposition 81. Let G be a group, $S \subset G$ a symmetric generating set containing the identity, then

- If S is a determining set, then for any closed S-curve γ_0, there is a sequence $\gamma_1, \dots, \gamma_n$ of closed S-curves with $d_U^S(\gamma_{j-1}, \gamma_j) \leq 1$ for each $j \in \{1, \dots, n\}$ and $\gamma_n = (e)$.
- If for each closed S-curve γ_0, there is a sequence of closed S-curves $\gamma_1, \dots, \gamma_n$ with $d_U^S(\gamma_{j-1}, \gamma_j) \leq N$ for each $j \in \{1, \dots, n\}$ and $\gamma_N = (e)$, then S^{2N} is a determining set.

Proposition 82. Let G be a group, $S \subset G$ a symmetric determining set, and $H \triangleleft G$ a normal subgroup with $[G : H] < \infty$. If S intersects each H-coset in G, then $(S^3 \cap H)^2$ is a determing set of H.

Proof. It is easy to see that $T := S^3 \cap H$ generates H ([19], Lemma 7.2.2). Choose $\{x_1, \dots, x_M\} \subset S$ be a set of H-coset representatives with $x_1 = e$ and let $\eta : G \to H$ be the map that sends g to gx_j^{-1}, where x_j is the only element of $\{x_1, \dots, x_M\}$ with $gx_j^{-1} \in H$. By our construction of T, if $g_0, g_1 \in G$ are such that $g_0^{-1}g_1 \in S$, then $\eta(g_0)^{-1}\eta(g_1) \in T$.

This implies that for each S-curve $w = (g_0, \dots, g_n)$ in G, the sequence $\tilde{\eta}(w) := (\eta(g_0), \dots, \eta(g_n))$ is a T-curve in H, and if two S-curves w_0, w_1 have the same endpoints and $d_U^S(w_0, w_1) \leq 1$, then $\tilde{\eta}(w_0), \tilde{\eta}(w_1)$ also have the same endpoints and $d_U^T(\tilde{\eta}(w_0), \tilde{\eta}(w_1)) \leq 1$.

On the other hand, for each T-curve $(h_0, \dots, h_n)$ in H there is an S-curve $(g_0, \dots, g_{3n})$ in G with $g_{3j} = h_j$ for each $j \in \{0, \dots, n\}$. Then it is easy to check that the d_U^T-distance between $\tilde{\eta}(g_0, \dots, g_{3n})$ and $(h_0, \dots, h_n)$ is ≤ 1.

From the above, for any closed T-curve γ_0 in H, there is a closed S-curve α_0 with $d_U^T(\tilde{\eta}(\alpha_0), \gamma_0) \leq 1$. Then by Proposition 81, there is a sequence of closed S-curves $\alpha_1 \dots, \alpha_n$ in G with $d_U^S(\alpha_{j-1}, \alpha_j) \leq 1$ for each $j \in \{1, \dots, n\}$ and $\alpha_n = (e)$.

Applying $\tilde{\eta}$ we obtain a sequence of closed T-curves $\tilde{\eta}(\alpha_0), \dots, \tilde{\eta}(\alpha_n)$ in H with $d_U^T(\tilde{\eta}(\alpha_0), \gamma_0) \leq 1$,

$$d_U^T(\tilde{\eta}(\alpha_{j-1}), \tilde{\eta}(\alpha_j)) \leq 1 \text{ for each } j \in \{1, \dots, n\},$$

and $\tilde{\eta}(\alpha_n) = (e)$, which by Proposition 81 implies that T^2 is a determining set in H. □

Lemma 83. Under the conditions of Lemma 78, if Γ / Γ_0 is finite then there is a clean discrete approximation $h_i : G_i \to \Gamma_0$.

Proof. We split the situation in two cases. In either case, it is straightforward to check that h_i is a discrete approximation.

Case 1: Γ_0 is compact.

For i large enough, G_i is finite and $G_i = f_i^{-1}(\Gamma_0)$, so we define h_i as $f_i|_{G_i}$.

Case 2: Γ_0 is non-compact.

Consider a sequence $g_i \in \Gamma_0$ with $d(e, g_i) \to \infty$. Then define $h_i : G_i \to \Gamma_0$ as

$$h_i(g) := \begin{cases} f_i(g) & \text{if } f_i(g) \in \Gamma_0, \\ g_i & \text{if } f_i(g) \notin \Gamma_0. \end{cases}$$

Let $S_0 \subset K_0 \subset \Gamma$, $S_i \subset \Gamma_i$ be the sets given by the definition of clean discrete approximation. Then by Proposition 82, $S_i' := (S_i^3 \cap G_i)^2$ is a determining set. Since the maps h_i are approximate morphisms, we have

$$h_i^{-1}(S_0 \cap \Gamma_0) \subset S_i' \subset h_i^{-1}(K_0^7 \cap \Gamma_0)$$

for large enough i, showing that the discrete approximation $h_i : G_i \to \Gamma_0$ is clean. □

Remark 84. Notice that if Γ is a connected Lie group with a proper left invariant metric d, then for any left invariant geodesic metric d_1 in Γ and any $R > 0$ there is $R' > 0$ with $B_e^{d_1}(R, \Gamma) \subset B_e^d(R', \Gamma)$. Hence if one has a discrete approximation $f_i : \Gamma_i \to (\Gamma, d)$, the maps $f_i : \Gamma_i \to (\Gamma, d_1)$ also form a discrete approximation.

Lemma 85. Let Γ be a connected Lie group with a left invariant geodesic metric, Γ_i a sequence of discrete groups, $f_i : \Gamma_i \to \Gamma$ a clean discrete approximation, and $K_0 \leq \Gamma$ a compact subgroup. Then K_0 is contained in the center of Γ, and there is a clean discrete approximation $h_i : \Gamma_i \to \Gamma / K_0$.

Proof. By Theorem 67, Γ is nilpotent, and since compact subgroups of connected nilpotent Lie groups are central ([41], Section 2.6), K_0 is. As K_0 is compact normal, the metric d' in Γ/K_0 given by

$$d'([a],[b]) := \inf_{k\in K_0} d(a,kb)$$

is proper, geodesic, and left invariant. Then define $h_i : \Gamma_i \to \Gamma/K_0$ as the composition of f_i with the projection $\Gamma \to \Gamma/K_0$. All conditions in the definition of clean discrete approximation then follow for h_i from the corresponding ones of f_i, as the projection $\Gamma \to \Gamma/K_0$ is 1-Lipschitz and proper. $\square$

We will also need the following elementary fact about nilpotent Lie groups ([10], Section 1.2).

Proposition 86. Let Γ be a connected nilpotent Lie group. Then there is a unique normal compact subgroup $K_0 \leq \Gamma$ such that the quotient Γ/K_0 is simply connected.

Proposition 87. Let G_0, G_1 be groups, $S_0 \subset G_0$, $S_1 \subset G_1$ determining sets, and $\theta : G_0 \to G_1$ a morphism with

- $S_1 \subset \theta(S_0)$.
- $Ker(\theta) \cap (S_0 \cup S_0^{-1})^2 = \{e\}$.

Then θ is an isomorphism.

Proof. Construct $\psi : S_1 \to S_0$ as $\psi(s) = \theta^{-1}(s)$. By the first condition such element exists, and by the second condition the map is well defined. Since S_1 is determining in G_1, there is an inverse map $\tilde{\psi} : G_1 \to G_0$. $\square$

Theorem 88. Let Γ be a non-compact r-dimensional Lie group with a left invariant metric, Γ_i a sequence of discrete groups, and $f_i : \Gamma_i \to \Gamma$ a clean discrete approximation. Then Γ_i is infinite for i large enough.

Proof. If Γ has infinitely many connected components, then by Lemma 78, Γ_i is infinite for large enough i, so we can assume Γ has finitely many connected components. By Lemma 83, we can assume that Γ is connected. By Remark 84 we can assume its metric is geodesic, and by Lemma 85 and Proposition 86, we can further assume Γ is simply connected.

By Theorem 67, there is $C > 0$, a sequence of small subgroups $H_i \triangleleft \Gamma_i$, and rank r nilprogressions $P_i \subset \Gamma_i' := \Gamma_i/H_i$ in C-regular form with thick$(P_i) \to \infty$ and satisfying that for each $\varepsilon \in (0,1]$ there is $\delta > 0$ with

- $\tilde{f}_i^{-1}(B_e(\delta,\Gamma)) \subset G(\varepsilon P_i)$ for i large enough,
- $\tilde{f}_i(G(\delta P_i)) \subset B_e(\varepsilon,\Gamma)$ for i large enough,

where $\tilde{f}_i : \Gamma_i' \to \Gamma$ is the discrete approximation given by Proposition 64.

By iterated applications of Theorem 68, for i large enough there are infinite groups $(\mathbb{Z}^r, *_{P_i})$, symmetric determining sets $T_i \subset T_i' \subset (\mathbb{Z}^r, *_{P_i})$, $\delta' > \delta > 0$, and surjective morphisms $\theta_i : (\mathbb{Z}^r, *_{P_i}) \to \Gamma_i'$ such that $Ker(\theta_i) \cap (T_i')^2 = \{e\}$ and

$$\tilde{f}_i^{-1}(B_e(\delta, \Gamma)) \subset \theta_i(T_i) \subset \tilde{f}_i^{-1}(B_e(\delta'/8, \Gamma)) \subset \tilde{f}_i^{-1}(B_e(\delta', \Gamma)) \subset \theta_i(T_i').$$

It then follows by induction on $N \in \mathbb{N}$ that if $a_{1,i}, \ldots, a_{N,i} \in \theta_i^{-1}(\tilde{f}_i^{-1}(B_e(\delta'/3, \Gamma))) \cap T_i'$ and $b_i \in T_i$ are sequences with $\tilde{f}_i(\theta_i(b_i)) \to e$, then

$$\begin{gathered} a_{1,i} \cdots a_{N,i} b_i (a_{1,i} \cdots a_{N,i})^{-1} \in T_i \text{ for large enough } i, \\ \tilde{f}_i(\theta_i(a_{1,i} \cdots a_{N,i} b_i (a_{1,i} \cdots a_{N,i})^{-1})) \to e. \end{gathered} \tag{7}$$

Let $S_i \subset \Gamma_i'$ be the sets given by the definition of clean discrete approximation. Then by connectedness of Γ, there is $M \in \mathbb{N}$ such that for i large enough, $S_i \cup S_i^{-1} \subset \theta_i(T_i)^M$.

Claim: $Ker(\theta_i) \cap T_i^{2M} = \{e\}$ for i large enough.

Working by contradiction, after taking a subsequence we can find sequences $x_j^i \in T_i$ with $j \in \{1, \ldots, 2M\}$, $i \in \mathbb{N}$, such that $x_1^i \cdots x_{2M}^i \in Ker(\theta_i) \backslash \{e\}$ for each i. After further taking a subsequence, we can assume there are elements $y_j \in B_e(\delta'/4, \Gamma)$ with $\tilde{f}_i(\theta_i(x_j^i)) \to y_j$ as $i \to \infty$ for $j \in \{1, \ldots, 2M\}$.

Since Γ is simply connected, there is a finite sequence of chains $\{y_{\ell,1}, \ldots, y_{\ell,k_\ell}\} \subset B_e(\delta'/4, \Gamma)$ with $\ell \in \{0, \ldots, n\}$ such that:

- $k_0 = 2M$ and $y_{0,j} = y_j$ for all $j \in \{1, \ldots, 2M\}$.
- $k_n = 1$ and $y_{n,1} = e$.
- $\{y_{\ell,1}, \ldots, y_{\ell,k_\ell}\}$ is obtained from $\{y_{\ell-1,1}, \ldots, y_{\ell-1,k_{\ell-1}}\}$ by one substitution of the form
$$\{s\} \mapsto \{s_1, s_2\} \text{ or } \{s_1, s_2\} \mapsto \{s\}$$
where $s, s_1, s_2 \in B_e(\delta'/4, \Gamma)$ and $s = s_1 s_2$ in Γ.

For each $j \in \{1, \ldots, 2M\}$, set $x_{0,j}^i := x_j^i$, and for each $\ell \in \{1, \ldots, n\}$, $j \in \{1, \ldots, k_j\}$, choose a sequence $x_{\ell,j}^i \in T_i'$ such that $\tilde{f}_i(\theta_i(x_{\ell,j}^i)) \to y_{\ell,j}$ as $i \to \infty$ (if $y_{\ell,j} = y_{\ell',j'}$ for some $\ell, \ell' \in \{0, \ldots, n\}$, $j \in \{1, \ldots, k_\ell\}$, $j' \in \{1, \ldots, k_{\ell'}\}$, we can assume $x_{\ell,j}^i = x_{\ell',j'}^i$ for all i). Notice that

$$x_{n,1}^i \in T_i \text{ for large enough } i \text{ and } \tilde{f}_i(\theta_i(x_{n,1}^i)) \to e. \tag{8}$$

By Equation 7, working by reverse induction on ℓ (see Figure 1), this implies that

$$w_\ell^i := x_{\ell,1}^i \cdots x_{\ell,k_\ell}^i \in T_i \text{ for large enough } i \text{ and } \tilde{f}_i(\theta_i(x_{\ell,1}^i \cdots x_{\ell,k_\ell}^i)) \to e. \tag{9}$$

Therefore, $x_{0,1}^i \cdots x_{0,2M}^i \in Ker(\theta_i) \cap T_i = \{e\}$. However, this contradicts our choice of $x_j^i = x_{0,j}^i$, proving our claim.

By Proposition 87, θ_i is an isomorphism and Γ_i' is infinite for i large enough. $\square$

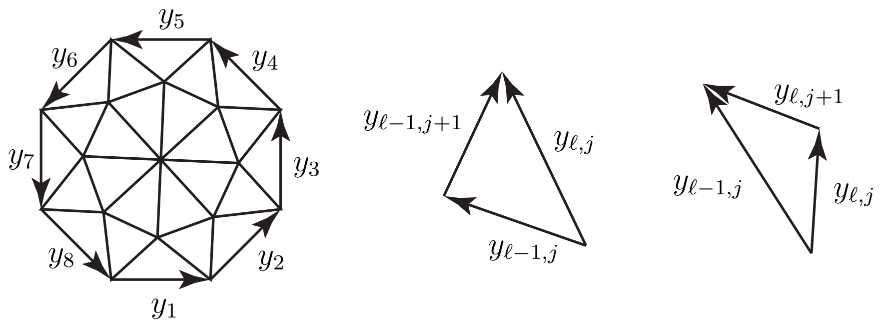

Figure 1: For each ℓ, there is j such that either $y_{\ell-1,j}y_{\ell-1,j+1} = y_{\ell,j}$ or $y_{\ell-1,j} = y_{\ell,j}y_{\ell,j+1}$. In the former case, $k_{\ell-1} = k_\ell + 1$ and $w^i_{\ell-1}$ is obtained from w^i_ℓ by multiplying it by $(x^i_{\ell,j+1}\cdots x^i_{\ell,k_\ell})^{-1}\left((x^i_{\ell,j})^{-1}x^i_{\ell-1,j}x^i_{\ell-1,j+1}\right)(x^i_{\ell-1,j+2}\cdots x^i_{\ell-1,k_{\ell-1}})$. In the latter, $k_{\ell-1} = k_\ell - 1$ and $w^i_{\ell-1}$ is obtained from w^i_ℓ by multiplying it by $(x^i_{\ell,j+2}\cdots x^i_{\ell,k_\ell})^{-1}\left((x^i_{\ell,j}x^i_{\ell,j+1})^{-1}x^i_{\ell-1,j}\right)(x^i_{\ell-1,j+1}\cdots x^i_{\ell-1,k_{\ell-1}})$.

Theorem 2. Let $(X,d,\mathfrak{m})$ be an $RCD^*(K,N;D)$ space. If the universal cover $(\tilde{X},\tilde{d},\tilde{\mathfrak{m}})$ is compact, then $\operatorname{diam}(\tilde{X}) \leq \tilde{D}(N,KD^2)$.

Proof of Theorem 2: Assuming the contrary, we get a sequence X_i of compact $RCD^*(K,N;D)$ spaces such that their universal covers $\tilde{X}_i$ satisfy $\operatorname{diam}(\tilde{X}_i)\to\infty$. After taking points $p_i\in\tilde{X}_i$ and a subsequence, we can assume that the sequence $(\tilde{X}_i,p_i)$ converges in the pGH sense to a pointed $RCD^*(K,N)$ space (X,p) and the sequence $\pi_1(X_i)$ converges equivariantly to a closed group $\Gamma\leq Iso(X)$. Since X is non-compact, and X/Γ is compact, we conclude that Γ is non-compact. By Corollary 76, we have a clean discrete approximation $f_i:\pi_1(X_i)\to\Gamma$. However, the groups $\pi_1(X_i)$ are finite, contradicting Theorem 88. □

5 Uniform Virtual Abelianness

In this section we prove Theorems 3 and 4. The main ingredient is the following result, which implies that non-collapsing sequences of $RCD^*(K,N)$ spaces cannot have small groups of measure preserving isometries.

Theorem 89. Let $(X_i,d_i,\mathfrak{m}_i,p_i)$ be a sequence of pointed $RCD^*(K,N)$ spaces of rectifiable dimension n and let $H_i\leq Iso(X_i)$ be a sequence of small subgroups in the sense of Remark 72 acting by measure preserving isometries. Assume the sequence $(X_i,d_i,\mathfrak{m}_i,p_i)$ converges in the pmGH sense to an $RCD^*(K,N)$ space $(X,d,\mathfrak{m},p)$ of rectifiable dimension n. Then H_i is trivial for large enough i.

Proof. Working by contradiction, after taking a subsequence and replacing H_i by its closure, we can assume that H_i is non-trivial and compact for each i.

Let $q \in X$ be an n-regular point and $p_i' \in X_i$ be such that p_i' converges to q. Notice that if $\lambda_i \to \infty$ slowly enough, then $(\lambda_i X_i, p_i')$ converges (after renormalizing the measure) in the pmGH sense to $(\mathbb{R}^n, d^{\mathbb{R}^n}, \mathcal{H}^n, 0)$ and the groups H_i are still small subgroups when acting on $(\lambda_i X_i, p_i')$, so by Theorem 24 the sequence $(\lambda_i X_i / H_i, [p_i'])$ converges (after renormalizing the measure on $\lambda_i X_i / H_i$) to $(\mathbb{R}^n, d^{\mathbb{R}^n}, \mathcal{H}^n, 0)$ as well.

Define $\eta_i : X_i \to \mathbb{R}$ as

$$\eta_i(x) := \sup_{h \in H_i} d^{\lambda_i X_i}(hx, x).$$

Then by Lemma 23, $\mathfrak{m}_i(\eta_i^{-1}(0)) = 0$, and by Theorem 37 there are sequences $q_i \in X_i$ such that $\eta_i(q_i) \neq 0$ and for any $\alpha_i \to \infty$, the sequence $(\alpha_i \lambda_i X_i / H_i, [q_i])$ converges (possibly after taking a subsequence) in the pGH sense to a space containing an isometric copy of $\mathbb{R}^n$. Set $\alpha_i := 1/\eta_i(q_i)$.

By Proposition 44 after taking a subsequence we can assume that $(\alpha_i \lambda_i X_i, q_i)$ converges in the pGH sense to a pointed space $(\mathbb{R}^n \times Z, (0, z))$ and H_i converges equivariantly to a group $H \leq Iso\,(\mathbb{R}^n \times Z)$ in such a way that the H-orbits coincide with the Z-fibers. By Remark 40, Z is a point and H is trivial, but by our choice of α_i, H is non-trivial which is a contradiction. $\square$

Theorem 4. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(K, N; D)$ spaces of rectifiable dimension n with compact universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$. If the sequence $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converges in the pointed measured Gromov–Hausdorff sense to some $RCD^*(K, N)$ space of rectifiable dimension n, then there is $C > 0$ such that for each i there is an abelian subgroup $A_i \leq \pi_1(X_i)$ with index $[\pi_1(X_i) : A_i] \leq C$.

Proof of Theorem 4: Arguing by contradiction, after taking a subsequence we can assume that $\pi_1(X_i)$ does not admit an abelian subgroup of index $\leq i$. By Theorem 2, X is compact, and after taking a subsequence we can assume the groups $\pi_1(X_i)$ converge equivariantly to a compact group $\Gamma \leq Iso(X)$. By Corollary 76, we have a clean discrete approximation $f_i : \pi_1(X_i) \to \Gamma$. By Theorem 65, there are small subgroups $H_i \triangleleft \pi_1(X_i)$ for which the quotients $\pi_1(X_i)/H_i$ are uniformly virtually abelian. On the other hand, by Theorem 89 the groups H_i are eventually trivial, showing the existence of a constant C for which $\pi_1(X_i)$ admits an abelian subgroup $A_i \leq \pi_1(X_i)$ of index $[\pi_1(X_i) : A_i] \leq C$. $\square$

For the proof of Theorem 3 we require the following intermediate step.

Theorem 90. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(0, N; D)$ spaces and $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ their universal covers. Then one of the following holds:

- There is a sequence $H_i \leq Iso(\tilde{X}_i)$ of small subgroups of measure preserving isometries such that $H_i \neq 0$ for infinitely many i's.
- There is $C > 0$ and abelian subgroups $A_i \leq \pi_1(X_i)$ with index $[\pi_1(X_i), A_i] \leq C$.

Proof. By contradiction, we can assume that any sequence $H_i \leq Iso(\tilde{X}_i)$ of small subgroups of measure preserving isometries is eventually trivial, and for each i, the group $\pi_1(X_i)$ does not admit an abelian subgroup of index $\leq i$. Notice that these properties are passed to subsequences.

By Theorem 36, each universal cover $\tilde{X}_i$ splits as a product $Y_i \times \mathbb{R}^{m_i}$ with Y_i a compact $RCD^*(0, N - m_i)$ space, and after passing to a subsequence, we can assume that the sequence $m = m_i$ is constant. The action of $\pi_1(X_i)$ respects such splitting, so one has natural maps $\phi_i : \pi_1(X_i) \to Iso(Y_i)$ and $\psi_i : \pi_1(X_i) \to Iso(\mathbb{R}^m)$ whose images consist of measure preserving isometries.

Claim: There is $C_1 > 0$ and abelian subgroups $Z_i \leq \phi_i(\pi_1(X_i))$ of index $[\phi_i(\pi_1(X_i)) : Z_i] \leq C_1$.

By Theorem 46, the sequence $\mathrm{diam}(Y_i)$ is bounded, and after taking a subsequence we can assume the spaces Y_i converge in the mGH sense to a compact $RCD^*(0, N-m)$ space Y and the groups $Iso(Y_i)$ converge equivariantly to a compact subgroup $\Gamma \leq Iso(Y)$, so we have almost morphisms $f_i : Iso(Y_i) \to \Gamma$ (see Remark 69).

By Theorem 22, $Iso(Y)$ is a compact Lie group, so by Theorem 52, it has a faithful unitary representation $\rho_0 : Iso(Y) \to U(H)$ for some finite dimensional Hilbert space H. Since the maps f_i are almost morphisms, the compositions

$$\eta_i := \rho_0 \circ f_i : \phi_i(\pi_1(X_i)) \to U(H)$$

are almost representations. The groups $\phi_i(\pi_1(X_i))$, being virtually abelian, are amenable, so by Theorem 56, the maps η_i are close to actual representations $\rho_i : \phi_i(\pi_1(X_i)) \to U(H)$.

Since ρ_0 is faithful, the kernels $H_i := Ker(\rho_i)$ form a sequence of small subgroups of measure preserving isometries of Y_i. From the splitting $\tilde{X}_i = Y_i \times \mathbb{R}^m$, one obtains inclusions $H_i \to Iso(\tilde{X}_i)$ and hence by our no-small-subgroup assumption, the representations ρ_i are faithful for large enough i. This implies that $\phi_i(\pi_1(X_i))$ is isomorphic to a subgroup of $U(H)$. The claim then follows from Corollary 54.

Since Y_i is compact, the map ψ_i is proper and the group $\psi_i(\pi_1(X_i))$ is discrete. Hence by Theorem 55, there is $C_2 > 0$ and abelian subgroups $B_i \leq \psi_i(\pi_1(X_i))$ of index $[\psi_i(\pi_1(X_i)), B_i] \leq C_2$. Finally, the group

$$A_i := \left(\phi_i^{-1}(Z_i) \cap \psi_i^{-1}(B_i)\right) \leq \pi_1(X_i) \cap (Z_i \times B_i)$$

is abelian of index $[\pi_1(X_i) : A_i] \leq C_1 C_2$. This contradicts our initial assumption. $\square$

Theorem 3. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(0, N; D)$ spaces of rectifiable dimension n such that the sequence of universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converges in the pointed measured Gromov–Hausdorff sense to some $RCD^*(0, N)$ space of rectifiable dimension n. Then there is $C > 0$ such that for each i there is an abelian subgroup $A_i \leq \pi_1(X_i)$ with index $[\pi_1(X_i) : A_i] \leq C$.

Proof of Theorem 3: Combine Theorem 89 and Theorem 90. $\square$

6 First Betti number

To prove Theorem 5 we require an extension to $RCD^*(K,N)$ spaces of the normal subgroup theorem by Kapovitch–Wilking. With the tools we have so far our proof is not much different from the one for smooth Riemannian manifolds in [22], but we include it here for completeness and convenience for the reader.

Recall that if (X,p) is a pointed proper metric space, $\Gamma \leq Iso(X)$ is a closed group of isometries, and $r>0$, we denote by $\mathcal{G}(\Gamma, X, p, r) \leq \Gamma$ the subgroup generated by the elements $\gamma \in \Gamma$ with norm $\|\gamma\|_p \leq r$.

Theorem 91. (Normal Subgroup Theorem) For each $K \in \mathbb{R}$, $N \geq 1$, $D > 0$, $\varepsilon_1 > 0$, there are $\varepsilon_0 > 0$, $C_0 \in \mathbb{N}$ such that the following holds. For each $RCD^*(K,N;D)$ space $(X,d,\mathfrak{m})$, there is $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ and a normal subgroup $G \triangleleft \pi_1(X)$ such that for all x in the universal cover $\tilde{X}$ we have

- $G \leq \mathcal{G}(\pi_1(X), \tilde{X}, x, \varepsilon/1000)$.
- $[\mathcal{G}(\pi_1(X), \tilde{X}, x, \varepsilon) : G] \leq C_0$.

For the proof of Theorem 91 the following elementary observations are required.

Proposition 92. Let Y be a proper geodesic space, $\Gamma \leq Iso(Y)$ a closed group of isometries, $y \in Y$, $g \in \Gamma$, and $r > 0$. Then $\mathcal{G}(\Gamma, Y, gy, r) = g \cdot \mathcal{G}(\Gamma, Y, y, r) \cdot g^{-1}$.

Proof of Proposition 92: The result follows immediately from the identity

$$\|ghg^{-1}\|_{gy} = \|h\|_y \text{ for all } h \in \Gamma.$$

□

Proposition 93. Let G be a group, $H \triangleleft G$ a normal subgroup, $S \subset G$ a symmetric generating set, and $M \in \mathbb{N}$ such that for any $s_1, \ldots, s_M \in S$ there are $1 \leq j_0 \leq j_1 \leq M$ with $s_{j_0} \cdots s_{j_1} \in H$. Then $[G:H] \leq |S|^M$

Proof of Proposition 93: By hypothesis, each element of G/H can be written as a product of at most M elements of $\{sH | s \in S\}$. There are $|S|^M$ such words, so the result follows. □

Proof of Theorem 91: Working by contradiction, we assume there is a sequence of $RCD^*(K,N;D)$ spaces X_i and sequences $\delta_i \to 0$ with the property that for any normal subgroup $G \triangleleft \pi_1(X_i)$ and any $\varepsilon \in [\delta_i, \varepsilon_1]$ there is x in the universal cover $\tilde{X}_i$ such that either G is not contained in $\mathcal{G}(\pi_1(X_i), \tilde{X}_i, x, \varepsilon/1000)$ or the index of G in $\mathcal{G}(\pi_1(X), \tilde{X}, x, \varepsilon)$ is greater than i.

After choosing $p_i \in \tilde{X}_i$ and passing to a subsequence, we can assume $(\tilde{X}_i, p_i)$ converges in the pmGH sense to a pointed $RCD^*(K,N)$ space (X,p), and $\pi_1(X_i)$ converges equivariantly to a closed group $\Gamma \leq Iso(X)$. By Corollary 76, the discrete approximation $f_i : \pi_1(X_i) \to \Gamma$ constructed in Section 2.8 is clean.

Let $r \in (0, 1/2D]$ be such that $B_e(r, \Gamma)$ is contained in the connected component of Γ, and set $G_i \leq \pi_1(X_i)$ as the subgroup generated by $f_i^{-1}(B_e(r, \Gamma))$. By Lemma 78, for i large enough, G_i is a normal subgroup of $\pi_1(X_i)$. Also, by Proposition 70, if $\delta > 0$ is small enough, for large enough i we have

- $f_i(B_e(\delta, \pi_1(X_i))) \subset B_e(r, \Gamma)$.
- $f_i^{-1}(B_e(\delta/2, \Gamma)) \subset B_e(\delta, \pi_1(X_i))$.

From the proof of Lemma 58, this implies that if i is large enough, then

$$G_i = \langle B_e(\delta, \pi_1(X_i)) \rangle .$$

This implies, using Equation 6, that

$$G_i \leq \bigcap_{x \in B_{p_i}(1/\delta, \tilde{X}_i)} \mathcal{G}(\pi_1(X_i), \tilde{X}_i, x, \delta).$$

If $\delta < 1/2D$, then any element x in $\tilde{X}_i$ is sent by some $g \in \pi_1(X_i)$ to an element gx of $B_{p_i}(1/\delta, \tilde{X}_i)$, and from the fact that G_i is normal and Proposition 92 we have

$$G_i = g^{-1} G_i g \subset g^{-1} \cdot \mathcal{G}(\pi_1(X_i), \tilde{X}_i, gx, \delta) \cdot g = \mathcal{G}(\pi_1(X_i), \tilde{X}_i, x, \delta).$$

The contradiction is then attained with the following claim by setting $\delta = \varepsilon/1000$.

Claim: There is $C_0 > 0$ such that if $\varepsilon > 0$ is small enough, then for i large enough, $[\mathcal{G}(\pi_1(X_i), \tilde{X}_i, x, \varepsilon) : G_i] \leq C_0$ for all $x \in \tilde{X}_i$.

By Theorem 21, there is $m \in \mathbb{N}$ such that any ball of radius $10/r$ in an $RCD^*(K, N)$ space can be covered by m balls of radius $r/10$. Set $\varepsilon := r/10m^m$ and $M := m^m$. Then for any $x \in \tilde{X}_i$ and elements $g_1, \ldots, g_M \in \{g \in \pi_1(X_i) | \|g\|_x \leq \varepsilon\}$, by the pigeonhole principle, there are $1 \leq j_0 \leq j_1 \leq M$ with

$$g_{j_0} \cdots g_{j_1} \in \{g \in \pi_1(X_i) | d(gy, y) \leq r/2 \text{ for all } y \in B_x(2/r, \tilde{X}_i)\} \subset G_i.$$

By Theorem 77, there is a set $S \subset \{g \in \pi_1(X_i) | \|g\|_x \leq \varepsilon\}$ with $\langle S \rangle = \mathcal{G}(\pi_1(X_i), \tilde{X}_i, x, \varepsilon)$ of size $|S| \leq C(k, N, D)$, so by Proposition 93, we get

$$[\mathcal{G}(\pi_1(X_i), \tilde{X}_i, x, \varepsilon) : G_i] \leq C_0 := C^M.$$

$\square$

For a geodesic space X and a collection $\mathcal{U}$ of subsets of X, we denote by $H_1^{\mathcal{U}}(X)$ the subgroup of the first homology group $H_1(X)$ generated by the images of the maps $H_1(U) \to H_1(X)$ induced by the inclusions $U \to X$ with $U \in \mathcal{U}$. If $\mathcal{U}$ is the collection of balls of radius δ in X, we denote $H_1^{\mathcal{U}}(X)$ by $H_1^{\delta}(X)$. This group satisfies a natural monotonicity property.

Lemma 94. Let X be a geodesic space and $\mathcal{U}, \mathcal{V}$ two families of subsets of X. If for each $U \in \mathcal{U}$, there is $V \in \mathcal{V}$ with $U \subset V$, then $H_1^{\mathcal{U}}(X) \leq H_1^{\mathcal{V}}(X)$.

Definition 95. For metric spaces X and Y, a function $\phi : Y \to X$ iss said to be an ε-isometry if

- For each $y_1, y_2 \in Y$, one has $|d(fy_1, fy_2) - d(y_1, y_2)| \leq \varepsilon$.
- For each $x \in X$, there is $y \in Y$ with $d(fy, x) \leq \varepsilon$.

Theorem 96. (Sormani–Wei) Let X be a compact geodesic space and assume there is $\varepsilon_0 > 0$ such that $H_1^{\varepsilon_0}(X)$ is trivial. If Y is a compact geodesic space and $f : Y \to X$ a δ-isometry with $\delta \leq \varepsilon_0/100$, then there is a surjective morphism

$$\tilde{\phi} : H_1(Y) \to H_1(X)$$

whose kernel is $H_1^{10\delta}(Y) = H_1^{\varepsilon_0/10}(Y)$.

Proof. We follow the lines of ([32], Theorem 3.4), where they prove this result for π_1 instead of H_1. Each 1-cycle in Y can be thought as a family of loops $\mathbb{S}^1 \to Y$ with integer multiplicity. For each map $\gamma : \mathbb{S}^1 \to Y$, by uniform continuity one could pick finitely many cyclically ordered points $\{z_1, \ldots, z_m\} \subset \mathbb{S}^1$ such that $\gamma([z_{j-1}, z_j])$ is contained in a ball of radius $\varepsilon/10$ for each j. Then set $\phi(\gamma) : \mathbb{S}^1 \to X$ to be the loop with $\phi(\gamma)(z_j) = f(\gamma(z_j))$ for each j, and $\phi(\gamma)|_{[z_{j-1}, z_j]}$ a minimizing geodesic from $\phi(\gamma)(z_{j-1})$ to $\phi(\gamma)(z_j)$.

Clearly, $\phi(\gamma)$ depends on the choice of the points z_j and the minimizing paths $\phi(\gamma)|_{[z_{j-1}, z_j]}$. However, the homology class of $\phi(\gamma)$ in $H_1(X)$ does not depend on these choices, since different choices yield curves that are ε-uniformly close, which by hypothesis are homologous.

Assume that a 1-cycle c in Y is the boundary $\partial\sigma$ of a 2-chain σ. After taking iterated barycentric subdivision, one could assume that each simplex of σ is contained in a ball of radius $\varepsilon/10$. By recreating σ in X via f simplex by simplex, one could find a 2-chain whose boundary is $\phi(c)$. This means that ϕ induces a map $\tilde{\phi} : H_1(Y) \to H_1(X)$.

In a similar fashion, if a 1-cycle c in Y is such that $\phi(c)$ is the boundary of a 2-chain σ, one could again apply iterated barycentric subdivision to obtain a 2-chain σ' in X whose boundary is $\phi(c)$ and such that each simplex is contained in a ball of radius $\varepsilon/10$. Using f one could recreate the 1-skeleton of σ' in Y in such a way that expresses c as a linear combination with integer coefficients of 1-cycles contained in balls of radius ε in Y. This implies that the kernel of $\tilde{\phi}$ is contained in $H_1^{\varepsilon}(Y)$.

If a 1-cycle c in Y is contained in a ball of radius ε, then $\phi(c)$ is contained in a ball of radius 2ε and then by hypothesis, $\phi(c)$ is a boundary. This shows that the kernel of $\tilde{\phi}$ is precisely $H_1^{\varepsilon}(Y)$.

Lastly, for any loop $\gamma : \mathbb{S}^1 \to X$, one can create via f a loop $\gamma_1 : \mathbb{S}^1 \to Y$ such that $\phi(\gamma_1)$ is uniformly close (and hence homologous) to γ, so $\tilde{\phi}$ is surjective. $\square$

Clearly, a sequence of compact metric spaces X_i converges in the GH sense to a compact metric space X if and only if there is a sequence of δ_i-isometries $\phi_i : X_i \to X$ with $\delta_i \to 0$. Hence by Theorem 96 one gets the following.

Corollary 97. Let X be a pointed compact geodesic space and assume there is $\varepsilon_0 > 0$ such that $H_1^{\varepsilon_0}(X)$ is trivial. If a sequence of compact geodesic spaces X_i converges to X in the GH sense, then there is a sequence $\rho_i \to 0$ such that $H_1^{\rho_i}(X_i) = H_1^{\varepsilon_0/10}(X_i)$ for all i.

For the proof of Theorem 5, it is convenient to reformulate Theorem 91 in the abelian setting.

Theorem 98. For each $K \in \mathbb{R}$, $N \geq 1$, $D > 0$, $\varepsilon_1 > 0$, there are ε_0, $C_0 \in \mathbb{N}$ such that the following holds. For each $RCD^*(K,N;D)$ space $(X,d,\mathfrak{m})$, there is $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ and a subgroup $G \leq H_1(X)$ such that for all x in the covering space $X' \to X$ with Galois group $H_1(X)$ one has

- $G \leq \mathcal{G}(H_1(X), X', x, \varepsilon/1000)$.
- $[\mathcal{G}(H_1(X), X', x, \varepsilon) : G] \leq C_0$.

Theorem 5. Let $(X_i, d_i, \mathfrak{m}_i)$ be $RCD^*(K,N;D)$ spaces of rectifiable dimension n and first Betti number $\beta_1(X_i) \geq r$. If the sequence X_i converges in the measured Gromov–Hausdorff sense to an $RCD^*(K,N;D)$ space X of rectifiable dimension m, then

$$\beta_1(X) \geq r + m - n.$$

Proof of Theorem 5: Let $p \in X$ be an m-regular point and choose $p_i \in X_i$ converging to p. For each i, let $X_i' := \tilde{X}_i/[\pi_1(X_i), \pi_1(X_i)]$ denote the regular cover of X_i with Galois group $H_1(X_i)$, and choose $q_i \in X_i'$ in the preimage of p_i. Then by Lemma 50, there is $\eta > 0$ and a sequence $\eta_i \to 0$ such that

$$\mathcal{G}(H_1(X_i), X_i', q_i, \eta_i) = \mathcal{G}(H_1(X_i), X_i', q_i, \eta) \text{ for all } i.$$

By Theorem 20, there is $\varepsilon_1 \in (0, \eta]$ such that $H^{\varepsilon_1}(X)$ is trivial. By Theorem 96, all we need to show is that for i large enough, $H_1^{\varepsilon_1/10}(X_i)$ has rank $\leq n - m$. By Corollary 97, there is a sequence $\rho_i \to 0$ with $H_1^{\rho_i}(X_i) = H_1^{\varepsilon_1/10}(X_i)$ for all i. By Theorem 98, there are $\varepsilon_0 > 0$, $C_0 \in \mathbb{N}$, subgroups $G_i \leq H_1(X_i)$, and a sequence $\delta_i \in [3\varepsilon_0, \varepsilon_1/10]$ with the property that for each $x \in X_i'$,

$$\mathcal{G}(H_1(X_i), X_i', x, \delta_i) \text{ contains } G_i \text{ as a subgroup of index} \leq C_0.$$

Let $x_1, \ldots, x_m \in X$ be such that

$$X = \bigcup_{j=1}^{m} B_{x_j}(\varepsilon_0/3, X),$$

and for each j, choose $x_j^i \in X_i$ converging to x_j. Then for large enough i, the balls $B_{x_j^i}(\varepsilon_0/2, X_i)$ cover X_i. This implies that for large enough i, each ball of radius ρ_i in X_i is contained in a ball of the form $B_{x_j^i}(\varepsilon_0, X_i)$. Hence if we let $\mathcal{U}_i$ denote the family $\{B_{x_j^i}(\delta_i/3, X_i)\}_{j=1}^m$ for each i, by Lemma 94 we get

$$H_1^{\rho_i}(X_i) \leq H_1^{\mathcal{U}_i}(X_i) \leq H_1^{\varepsilon_1/10}(X_i) = H_1^{\rho_i}(X_i).$$

If we choose $y_j^i \in X_i'$ in the preimage of x_j^i, then by construction of $H_1^{\mathcal{U}_i}(X_i)$ we have

$$H_1^{\mathcal{U}_i}(X_i) \leq \left\langle \bigcup_{j=1}^m \mathcal{G}(H_1(X_i), X_i', y_j^i, \delta_i) \right\rangle .$$

Since $H_1^{\mathcal{U}_i}(X_i)$ is abelian, the index of G_i in $H_1^{\mathcal{U}_i}(X_i)$ is at most C_0^m. Therefore, the rank of $H_1^{\varepsilon_1/10}(X_i)$ equals the rank of G_i for i large enough. Set $\Gamma_i := \mathcal{G}(H_1(X_i), X_i', q_i, \eta)$.

Since $\eta \geq \varepsilon_1$, Γ_i contains G_i, and since $\Gamma_i = \mathcal{G}(H_1(X_i), X_i', p_i, \eta_i)$ and $\eta_i \leq \varepsilon_0$ for large enough i, the index of G_i in Γ_i is finite. Hence the following claim implies Theorem 5.

Claim: For i large enough, Γ_i has rank $\leq n - m$.

Let $\lambda_i \to \infty$ be a sequence that diverges so slowly that $\lambda_i \eta_i \to 0$ and the sequence $(\lambda_i X_i, p_i)$ converges in the pGH sense to $(\mathbb{R}^m, 0)$. We can achieve this since p is m-regular and $\eta_i \to 0$. After taking a subsequence, we can assume the sequence $(\lambda_i X_i', q_i)$ converges in the pmGH sense to an $RCD^*(K, N)$ space (Y, q), and the groups $H_1(X_i)$ converge equivariantly to some group $\Gamma \leq Iso(Y)$. Since all elements of $H_1(X_i) \backslash \Gamma_i$ move q_i at least $\eta \lambda_i$ away, Γ_i converges equivariantly to Γ as well. From the definition of equivariant convergence, it follows that the Γ_i-orbits of q_i converge in the pGH sense to the Γ-orbit of q.

By Proposition 44, Y splits isometrically as a product $\mathbb{R}^m \times Z$ with Z an $RCD^*(0, N - m)$ space of rectifiable dimension $\leq n - m$, such that the Z-fibers coincide with the Γ-orbits. By Proposition 35, Z has topological dimension $\leq n - m$, and by Theorem 45, the rank of Γ_i is at most $n - m$ for large enough i. $\square$

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