Fundamental Groups and Limits of Almost Homogeneous Spaces.

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Group actions in metric spaces.

$\delta$-transitive actions

$x \bullet \quad \bullet y$
Group actions in metric spaces.

$\delta$-transitive actions

$x, y$
$\delta$-transitive actions
Group actions in metric spaces.

$\delta$-transitive actions

Discrete Actions
Group actions in metric spaces.

δ-transitive actions

Discrete Actions
Group actions in metric spaces.

Example

\[ \pi_1(Y) \text{ acts } \delta \text{-transitively and discretely on } \tilde{Y}. \]
**Definition**

Let $X_n$ be a sequence of proper metric spaces. We say that a sequence of groups $\Gamma_n$ acts almost transitively on the sequence $X_n$ if there are isometric actions $\Gamma_n \rightarrow Iso(X_n)$ which are $\delta_n$-transitive and discrete, with $\delta_n \rightarrow 0$. 
Main Problem

Let \((X_n, x_n)\) be a sequence of pointed proper length spaces converging to \((X, x)\) in the pointed Gromov Hausdorff sense. If we have a sequence of groups \(\Gamma_n\) acting almost transitively on the sequence \(X_n\), what can we say about the limit \(X\)?

Example:
Main Problem.

Example:
Let $G$ be a nilpotent Lie group that admits lattices. Fix any left invariant metric in $G$

$$X_n := G, \ X = G, \ \Gamma_n \leq G \text{ lattices.}$$

Example:
Let $Y_n$ be compact length spaces with $\text{diam}(Y_n) \to 0$.

$$\Gamma_n := \pi_1(Y_n, y_n), \ X_n := \tilde{Y}_n.$$ 

Some conditions on $Y_n$ may ensure the existence of limit $X$, like $Y_n \in \text{Alex}^d(k)$, or $Y_n \in \text{CD}(k, d)$. 
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**Main Problem.**

*Example (Gromov-Pansu, 1983):* Let $\Gamma$ be a finitely generated group of polynomial growth and $Y$ is its Cayley graph.

$$\Gamma_n := \Gamma, \ X_n := Y/n.$$ 

Then the limit is a simply connected nilpotent Lie group with a sub-Finsler Carnot metric.
Compact case.

**Theorem (Turing, 1938)**

If $X$, the limit, is a compact Lie group, then it is a finite dimensional torus.

**Proof:**

- By Peter-Weyl, $X$ has a finite dimensional faithful representation $\rho : X \to GL(N, \mathbb{C})$.
- If GH approximations are good enough, we can extract from $\rho$, finite dimensional faithful representations $\rho_n : \Gamma_n \to GL(N, \mathbb{C})$.
- By Jordan’s Theorem, $\Gamma_n$ are uniformly virtually abelian.
Compact case.

Theorem (Jordan, 1878)
If $\Gamma \leq GL(N, \mathbb{C})$ is discrete, generated by $\Gamma \cap B(Id, 1/10)$, then $\Gamma$ is nilpotent.

Proof:
For $A, B \in B(Id, 1/10) \subset GL(N, \mathbb{C})$,

$$
\| [A, B] - 1 \| \leq \| AB - BA \| \| A^{-1} \| \| B^{-1} \|
\leq 4 \| A - 1 \| \| B - 1 \|.
$$

$$
0 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_k = \Gamma.
$$
Compact case.

**Theorem (Gelander, 2012)**
If $X$, the limit, is compact, then it is a torus, i.e. homeomorphic to a finite or infinite product of circles.

**Corollary**
If $X$ is a compact topological manifold, then it homeomorphic to a finite dimensional torus.
Non-compact case.

Theorem ♣, 2020
If $X$, the limit, is a topological manifold, then $X$ is a nilpotent Lie group with a sub-Finsler metric. For large enough $n$, there are subgroups $\Lambda_n \leq \pi_1(X_n, x_n)$ with surjective homomorphisms

$$\Lambda_n \to \pi_1(X, x).$$

Remark
$X$ is automatically a topological manifold if:
- $X$ has finite topological dimension (Montgomery-Zippin, 1939).
- $X$ is locally contractible (Berestovskii, 1990).
Lower Semicontinuity of $\pi_1$.

**Theorem (Gromov?, 1970’s)**

If $Y_n$ is a sequence of compact length spaces converging to a compact semilocally simply connected length space $Y$, then for large enough $n$, there are surjective morphisms

$$\pi_1(Y_n, y_n) \rightarrow \pi_1(Y, y).$$

**Example:**

![Diagram](image)
Lower Semicontinuity of $\pi_1$.

Non-Example:
Semilocal simple connectedness is necessary:
Lower Semicontinuity of $\pi_1$.

Non-Example:
Compactness is necessary:
Lower Semicontinuity of $\pi_1$.

**Theorem ♢**
Let $X_n \to X$, and $\Gamma_n \to Iso(X_n)$ acting almost transitively. If $X$ is a topological manifold, then $X$ is a nilpotent Lie group with a sub-Finsler metric. For large enough $n$, there are subgroups $\Lambda_n \leq \pi_1(X_n, x_n)$ with surjective homomorphisms

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**Non-Example**
Discreteness of $\Gamma_n$ is necessary: Define $Y_n := n \mathbb{S}^3$, $Z = \mathbb{S}^1$. Take

$$X_n := (Y_n \times Z)/\mathbb{S}^1.$$

Then $X_n \cong \mathbb{S}^3$, but $X_n \to \mathbb{S}^1 \times \mathbb{R}^2$. 
Proof that $X$ is nilpotent Lie group.

- $\Gamma_\alpha$, the ultralimit of $\Gamma_n$, acts transitively by isometries on $X$.
- $X$ has a sub-Finsler metric (Berestovskii, 1990).
- $X$ has uniformly bounded doubling at small scales (Nagel-Stein-Wainger, 1985).
- $\Gamma_n$ are uniformly virtually nilpotent (Breuillard-Green-Tao, 2011).
- $\text{Iso}(X)$ is a Lie group (Montgomery-Zippin, 1939).
- Isometries of $X$ are smooth (LeDonne-Ottazzi, 2016).
- $\Gamma_\alpha$ is connected and acts freely, hence $\Gamma_\alpha \cong X$. 
Approximations.

Construct $\phi_n : \Gamma_n \to \text{Iso}(X)$. For $g \in \Gamma_n$, define $\phi_n(g)$.
Discrete Subgroups of Lie Groups.

We have small groups $H_n \triangleleft \Gamma_n$. Very informally:

$$H_n \approx \text{Ker}(\phi_n).$$

$\Gamma_n/H_n$ near the identity are $C$-regular nilprogressions:

$$\star(u_1, \ldots, u_d; N_1, \ldots, N_d) := \{ u_1^{n_1} \ldots u_d^{n_d} \mid |n_i| \leq N_i \},$$

where $u_i \in \Gamma_n$, $N_i \in \mathbb{N}$, $d = \text{dim}(X)$.

$$(n_1, \ldots, n_d) \to u_1^{n_1} \ldots u_d^{n_d} \text{ is injective for } |n_i| \leq N_i.$$ 

For $i \leq j$, 

$$[u_i, u_j] = u_{j+1}^{\beta_{i,j}^{j+1}} \ldots u_d^{\beta_{i,j}^d}, \quad |\beta_{i,j}^l| \leq C \frac{N_i N_j}{N_l}.$$ 

$$N_1 \ll \ldots \ll N_d.$$
$\mathbb{Z}^2$

$u_1 = (1, 0)$

$u_2 = (0, 1)$

$[u_1, u_2] = 0$

$H(\mathbb{Z})$

$u_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$u_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$u_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$[u_1, u_2] = u_3^{-1}$
**Malcev Theorem.**

*Theorem (Malcev, 1962)*

If $N_i$’s are large enough, depending on $d$, $C$, then the $C$-regular nilprogression

$$\star(u_1, \ldots, u_d; N_1, \ldots, N_d)$$

can be embedded in a lattice of a nilpotent Lie group.

*Corollary*

The groups $\Gamma_n/H_n$ are isomorphic to lattices of nilpotent Lie groups.
Informally:

\((\pi_1(X) \neq 0) \Rightarrow (\Gamma_\alpha \text{ has torsion}) \Rightarrow (\Gamma_n/H_n \text{ has torsion}) \Rightarrow (\pi_1(X_n) \neq 0)\).
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End of the proof.

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\[ \pi_1(X) \neq 0 \Rightarrow \Gamma_\alpha \text{ has torsion} \Rightarrow \Gamma_n/H_n \text{ has torsion} \Rightarrow \pi_1(X_n) \neq 0. \]
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**Theorem ♣**

Let $X_n \to X$, and $\Gamma_n \to Iso(X_n)$ acting almost transitively. If $X$ is a topological manifold, then $X$ is a nilpotent Lie group with a sub-Finsler metric. For large enough $n$, there are subgroups $\Lambda_n \leq \pi_1(X_n, x_n)$ with surjective homomorphisms

$$\Lambda_n \to \pi_1(X, x).$$

**Corollary**

If $Y_n$ is a sequence of $d$-dimensional manifolds with $Ric(Y_n) \geq -1$ and $diam(Y_n) \to 0$, then, up to subsequence, the sequence of universal covers converge to a simply connected nilpotent Lie group of dimension $\leq d$, equipped with a left invariant Riemannian metric.
“Easy” questions.

**Theorem ♠**
Let $X_n \to X$, and $\Gamma_n \to \text{Isom}(X_n)$ acting almost transitively. If $X$ is a topological manifold, then $X$ is a nilpotent Lie group with a sub-Finsler metric. For large enough $n$, there are subgroups $\Lambda_n \leq \pi_1(X_n, x_n)$ with surjective homomorphisms

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**Question**
Which nilpotent Lie groups can arise as limits of almost homogenous spaces?

**Question**
Can we replace $\Lambda_n$ in Theorem ♠ by $\pi_1(X_n, x_n)$, just like in the compact case?
Non-manifold case.

**Theorem**

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$$\Lambda_n \to \pi_1(X, x).$$

**Hard Question?**

Does the lower semicontinuity of $\pi_1$ hold when $X$ is semilocally simply connected, but not a manifold?