Anderson finiteness for RCD spaces

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1 Introduction

For $K \in \mathbb{R}$, $N \in [1, \infty)$, the class of $RCD^*(K, N)$ spaces consists of proper metric measure spaces that satisfy a synthetic condition of having Ricci curvature bounded below by K and dimension bounded above by N. This class is closed under measured Gromov–Hausdorff convergence and contains the class of complete Riemannian manifolds of Ricci curvature $\geq K$ and dimension $\leq N$.

$RCD^*(K, N)$ spaces have a well defined notion of dimension called *rectifiable dimension* (see Theorem 16), which is always an integer between 0 and N, and is lower semi-continuous with respect to pointed Gromov–Hausdorff convergence (see Theorem 20). This motivates the following definition.

Definition 1. Let $K \in \mathbb{R}$, $N \in [1, \infty)$, and (X, p) a pointed $RCD^*(K, N)$ space. We define the *collapsing volume* of (X, p) as

$$\mathrm{vol}^*_{\mathrm{K,N}}(X, p) := \inf d_{GH}((X, p), (Y, q)),$$

where the infimum is taken among all pointed $RCD^*(K, N)$ spaces (Y, q) whose rectifiable dimension is strictly less than the one of X.

Proposition 2. Let (X_i, p_i) be a sequence of pointed $RCD^*(K, N)$ spaces that converges in the Gromov–Hausdorff sense to a pointed $RCD^*(K, N)$ space (X, p). If the rectifiable dimension of X_i is n for each i, then the following are equivalent:

1. The rectifiable dimension of X is strictly less than n.
2. $\mathrm{vol}^*_{\mathrm{K,N}}(X_i, p_i) \to 0$.
3. $\mathrm{vol}^*_{\mathrm{K,N}}(X_{i_k}, p_{i_k}) \to 0$ for a subsequence.

Proof. ($3 \Rightarrow 1$) By hypothesis, there is a sequence of $RCD^*(K, N)$ spaces (Y_{i_k}, q_{i_k}) of rectifiable dimension strictly less than n and converging to (X, p) as $k \to \infty$. From the fact that rectifiable dimension is lower semi-continuous, we deduce 1. The implications $1 \Rightarrow 2 \Rightarrow 3$ are tautological. □

Corollary 3. Let X_i be a sequence of $RCD^*(K, N)$ spaces, and $p_i, p'_i \in X_i$ pairs of points with $\limsup_{i\to\infty} d(p_i, p'_i) < \infty$. Then $\mathrm{vol}^*_{\mathrm{K,N}}(X_i, p_i) \to 0$ if and only if $\mathrm{vol}^*_{\mathrm{K,N}}(X_i, p'_i) \to 0$.

The main result of this note is a generalization to $RCD^*(K, N)$ spaces of a classical finiteness result of Anderson [1].

Theorem 4. For each $K \in \mathbb{R}$, $N \in [1, \infty)$, $D > 0$, $\nu > 0$, the class of pointed $RCD^*(K,N)$ spaces of diameter $\leq D$ and $\mathrm{vol}^*_{\mathrm{K,N}} \geq \nu$ contains finitely many fundamental group isomorphism types.

Theorem 4 will be obtained from the following result which states that a lower bound on the collapsing volume of the quotient of an $RCD^*(K,N)$ space by a discrete group yields a uniform discreteness gap on the corresponding group (see Equation 1 below for the definition of d_p).

Theorem 5. For each $K \in \mathbb{R}$, $N \in [1, \infty)$, $\nu > 0$, there is $\varepsilon > 0$ such that the following holds. If (X, p) is a pointed $RCD^*(K,N)$ space and $\Gamma \leq Iso(X)$ is a discrete group of measure preserving isometries with $\mathrm{vol}^*_{\mathrm{K,N}}(X/\Gamma, [p]) \geq \nu$, then

$$\{g \in \Gamma | \; d_p(g, Id_X) \leq \varepsilon\} = \{Id_X\}.$$

The remainder of this note contains the proof of theorems 4 and 5. In Section 2 we cover the required material and in Section 3 we present the proofs.

2 Preliminaries

2.1 Notation

For a metric space X, $p \in X$, $r > 0$, the closed ball of radius r centered at p will be denoted as $B(p, r, X)$. We denote by $X \cup \{*\}$ the metric space obtained by adjoining to X a point $*$ with $d(x, *) = \infty$ for all $x \in X$.

2.2 RCD spaces and isometries

A $CD^*(K,N)$ space is a proper metric space (X, d) equipped with a fully supported Radon measure $\mathfrak{m}$ for which an appropriate entropy in its space of probability measures is in a suitable sense concave with respect to the L^2-Wasserstein distance. For a $CD^*(K,N)$ space $(X, d, \mathfrak{m})$, if its Sobolev space $W^{1,2}$ is a Hilbert space, we say that it is an $RCD^*(K,N)$ space. See [7] for a precise definition and different reformulations.

Remark 6. If $(X, d, \mathfrak{m})$ is an $RCD^*(K,N)$ space, then for any $c > 0$, $(X, d, c\mathfrak{m})$ is also an $RCD^*(K,N)$ space, and for any $\lambda > 0$, $(X, \lambda d, \mathfrak{m})$ is an $RCD^*(\lambda^{-2}K, N)$ space. Also, if a metric measure space $(X, d, \mathfrak{m})$ is an $RCD^*(K - \varepsilon, N)$ space for all $\varepsilon > 0$, then it is also an $RCD^*(K,N)$ space.

Along this note we are interested only in topological properties of $RCD^*(K,N)$ spaces, so by an abuse of notation, we say that a proper metric space (X, d) is an $RCD^*(K,N)$ space if it admits a Radon measure that makes it an $RCD^*(K,N)$ space. Any two such measures are equivalent, so we can still talk about *full (or zero) measure sets* even after this abuse.

Even though we don't know much about the global topology of $RCD^*(K,N)$ spaces, we know they are semi-locally-simply-connected [16], and their universal cover is still an $RCD^*(K,N)$ space [14].

Theorem 7. (Wang) Let X be an $RCD^*(K,N)$ space. Then X is semi-locally-simply-connected, so for any $p \in X$ we can identify the fundamental group $\pi_1(X,p)$ with the group of deck transformations of the universal cover $\tilde{X}$.

Theorem 8. (Mondino–Wei) If X is an $RCD^*(K,N)$ space, then its universal cover $\tilde{X}$ admits a $\pi_1(X)$-invariant measure that makes it an $RCD^*(K,N)$ space.

Recall that for Riemannian manifolds, if an isometry coincides with the identity up to first order at a point, then it is necessarily the identity. An analogue of this statement for $RCD^*(K,N)$ spaces if the following [11].

Lemma 9. Let X be an $RCD^*(K,N)$ space and $f : X \to X$ a non-trivial isometry. Then the set of fixed points of f has zero measure.

For a pointed proper metric space (X,p), we define the compact-open distance between two functions $h_1, h_2 : X \to X$ as

$$d_p(h_1,h_2) := \inf_{r>0} \left\{ \frac{1}{r} + \sup_{x \in B(p,r,X)} d(h_1 x, h_2 x) \right\}. \tag{1}$$

When we restrict this metric to the group of isometries $Iso(X)$, we get a left invariant (not necessarily geodesic) metric that induces the compact open topology (independent of p) and makes $Iso(X)$ a proper metric group. However, this distance is defined on the full class of functions $X \to X$, where it is not left invariant nor proper anymore.

Recall that if X is a proper geodesic space and $\Gamma \leq Iso(X)$ is a closed group of isometries, the metric d' on X/Γ defined as $d'([x],[y]) := \inf_{g\in\Gamma}(d(gx,y))$ makes it a proper geodesic space. By the work of Galaz–Kell–Mondino–Sosa, the class of $RCD^*(K,N)$ spaces is closed under quotients by discrete groups [9].

Theorem 10. (Galaz–Kell–Mondino–Sosa) Let $(X,d,\mathfrak{m})$ be an $RCD^*(K,N)$ space and $\Gamma \leq Iso(X)$ a discrete group of measure preserving isometries. Then the metric space $(X/\Gamma, d')$ admits a measure $\mathfrak{m}'$ that makes it an $RCD^*(K,N)$ space. Moreover, if $\rho : X \to X/\Gamma$ denotes the projection, $\mathfrak{m}'$ can be taken so that $\mathfrak{m}(A) = \mathfrak{m}'(\rho(A))$ for all Borel subsets A of X sent isometrically to X/Γ by ρ.

2.3 Gromov–Hausdorff topology

The Gromov–Hausdorff topology in the class of pointed proper metric spaces quantifies how much two spaces are from being isometric.

Definition 11. Let $(X,p),(Y,q)$ be pointed proper metric spaces and $\varepsilon > 0$. We say that a function $f : X \to Y \cup \{*\}$ is an *ε-approximation* if $d(f(p),q) \leq \varepsilon$ and for $R = 1/\varepsilon$ one has

$$f^{-1}(B(q,R,Y)) \subset B(p,2R,X), \tag{2}$$

$$\sup_{x_1,x_2 \in B(p,2R,X)} |d(f(x_1),f(x_2)) - d(x_1,x_2)| \leq \varepsilon, \tag{3}$$

$$\sup_{y \in B(q,R,Y)} \inf_{x \in B(p,2R,X)} d(f(x),y) \leq \varepsilon. \tag{4}$$

The *pointed Gromov–Hausdorff distance* between (X,p) and (Y,q) is defined as

$$d_{GH}((X,p),(Y,q)) := \inf\{\varepsilon > 0 | \text{ there is an } \varepsilon\text{-approximation } f : X \to Y \cup \{*\}\}.$$

Strictly speaking, d_{GH} as defined above is not a distance as it is not symmetric. However, it still generates a first countable Hausdorff topology in the class of pointed proper metric spaces (see [4], Chapter 8). This is called the *Gromov–Hausdorff topology.*

Proposition 12. (Gromov) For pointed proper metric spaces (X,p) and (X_i,p_i), we have $d_{GH}((X_i,p_i),(X,p)) \to 0$ as $i \to \infty$ if and only if $d_{GH}((X,p),(X_i,p_i)) \to 0$ as $i \to \infty$. Moreover, in either case there are sequences $\phi_i : X_i \to X \cup \{*\}$ and $\psi_i : X \to X_i \cup \{*\}$ of ε_i-approximations with $\varepsilon_i \to 0$ and such that

$$\lim_{i \to \infty} d_p(\phi_i \circ \psi_i, Id_X) = 0. \tag{5}$$

Corollary 13. Let (X_i,p_i) be a sequence of pointed proper metric spaces that converges in the Gromov–Hausdorff sense to a pointed proper metric space (X,p). Then for each $R > 0$, $\varepsilon > 0$, there is $M \in \mathbb{N}$ such that any set $S \subset B(p_i,R,X_i)$ with $d(s_1,s_2) \geq \varepsilon$ for each $s_1,s_2 \in S$, one has $|S| \leq M$.

One of the main features of the class of $RCD^*(K,N)$ spaces is the Gromov–Hausdorff compactness property [2].

Theorem 14. (Bacher–Sturm) If (X_i,p_i) is a sequence of pointed $RCD^*(K,N)$ spaces, then one can find a subsequence that converges in the Gromov–Hausdorff sense to a pointed $RCD^*(K,N)$ space (X,p).

Definition 15. Let X be an $RCD^*(K,N)$ space and $n \in \mathbb{N}$. We say that $p \in X$ is an *n-regular* point if for each $\lambda_i \to \infty$, the sequence $(\lambda_i X, p)$ converges in the Gromov–Hausdorff sense to $(\mathbb{R}^n, 0)$.

Mondino–Naber showed that the set of regular points in an $RCD^*(K,N)$ space has full measure [13]. This result was refined by Brué–Semola who showed that almost all points have the same local dimension [3].

Theorem 16. (Brué–Semola) Let X be an $RCD^*(K,N)$ space. Then there is a unique $n \in \mathbb{N} \cap [0,N]$ such that the set of n-regular points in X has full measure. This number n is called the *rectifiable dimension* of X.

Definition 17. Let X_i be a sequence of $RCD^*(K,N)$ spaces of rectifiable dimension n. A choice of points $x_i \in X_i$ is said to be a *Reifenberg sequence* if for any $\lambda_i \to \infty$, the sequence $(\lambda_i X_i, x_i)$ converges in the Gromov–Hausdorff sense to $(\mathbb{R}^n, 0)$.

Theorem 18. (Mondino–Naber) For each $i \in \mathbb{N}$, let $(X_i, d_i, \mathfrak{m}_i)$ be an $RCD^*(-\varepsilon_i, N)$ space with $\varepsilon_i \to 0$ of rectifiable dimension n. Assume that for some choice $p_i \in X_i$, the sequence (X_i, p_i) converges in the Gromov–Hausdorff sense to $(\mathbb{R}^n, 0)$. Then there is a sequence of subsets $U_i \subset B(p_i, 1, X_i)$ with $\mathfrak{m}_i(U_i)/\mathfrak{m}_i(B(p_i, 1, X_i)) \to 1$ such that any sequence $x_i \in U_i$ is a Reifenberg sequence.

From the work of Mondino–Naber and Brué–Semola, we can conclude that the rectifiable dimension is a topological invaraint.

Corollary 19. Let X be an $RCD^*(K,N)$ space of rectifiable dimension n. There is an open dense subset $U \subset X$ locally bi-Lipschitz homeomorphic to $\mathbb{R}^n$.

Using the results above, Kitabeppu showed that the rectifiable dimension is lower semi-continuous with respect to Gromov–Hausdorff convergence [12].

Theorem 20. (Kitabeppu) Let (X_i, p_i) be a sequence of pointed $RCD^*(K,N)$ spaces of rectifiable dimension n converging in the Gromov–Hausdorff sense to the space (X,p). Then the rectifiable dimension of X is at most n.

The well known Cheeger–Gromoll splitting theorem [6] was extended by Cheeger–Colding for limits of Riemannian manifolds with lower Ricci curvature bounds [5], and later by Gigli to this setting [10].

Theorem 21. (Gigli) Let $(X, d, \mathfrak{m})$ be an $RCD^*(0,N)$ space of rectifiable dimension n. If (X,d) contains an isometric copy of $\mathbb{R}^m$, then there is $c > 0$ and a metric measure space $(Y, d^Y, \mathfrak{n})$ such that $(X, d, c\mathfrak{m})$ is isomorphic to the product $(\mathbb{R}^m \times Y, d^{\mathbb{R}^m} \times d^Y, \mathcal{H}^m \otimes \mathfrak{n})$. Moreover, $(Y, d^Y, \mathfrak{n})$ is an $RCD^*(0, N-m)$ space of rectifiable dimension $n-m$.

2.4 Equivariant Gromov–Hausdorff convergence

Recall from Proposition 12 that if a sequence of pointed proper metric spaces (X_i, p_i) converges in the Gromov–Hausdorff sense to the pointed proper metric space (X,p), one has ε_i-approximations $\phi_i : X_i \to X \cup \{*\}$ and $\psi_i : X \to X_i \cup \{*\}$ with $\varepsilon_i \to 0$ and satisfying Equation 5.

Definition 22. Consider a sequence of pointed proper metric spaces (X_i, p_i) that converges in the Gromov–Hausdorff sense to a pointed proper metric space (X, p) and a sequence of closed groups of isometries $\Gamma_i \leq Iso(X_i)$. We say that the sequence Γ_i *converges equivariantly* to a closed group $\Gamma \leq Iso(X)$ if:

- For each $g \in \Gamma$, there is a sequence $g_i \in \Gamma_i$ with $d_p(\psi_i \circ g_i \circ \phi_i, g) \to 0$ as $i \to \infty$.
- For a sequence $g_i \in \Gamma_i$ and $g \in Iso(X)$, if there is a subsequence g_{i_k} with $d_p(\psi_{i_k} \circ g_{i_k} \circ \phi_{i_k}, g) \to 0$ as $k \to \infty$, then $g \in \Gamma$.

We say that a sequence of isometries $g_i \in Iso(X_i)$ *converges* to an isometry $g \in Iso(X)$ if

$$d_p(\psi_i \circ g_i \circ \phi_i, g) \to 0 \text{ as } i \to \infty.$$

Equivariant convergence allows one to take limits before or after taking quotients [8].

Lemma 23. Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the Gromov–Hausdorff sense to a proper space (Y, q), and $\Gamma_i \leq Iso(Y_i)$ a sequence of closed groups of isometries that converges equivariantly to a closed group $\Gamma \leq Iso(Y)$. Then the sequence $(Y_i/\Gamma_i, [q_i])$ converges in the Gromov–Hausdorff sense to $(Y/\Gamma, [q])$.

Fukaya–Yamaguchi obtained an Arzelá-Ascoli type result for equivariant convergence ([8], Proposition 3.6).

Theorem 24. (Fukaya–Yamaguchi) Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space (Y, q), and take a sequence $\Gamma_i \leq Iso(Y_i)$ of closed groups of isometries. Then there is a subsequence $(Y_{i_k}, q_{i_k}, \Gamma_{i_k})_{k\in\mathbb{N}}$ such that Γ_{i_k} converges equivariantly to a closed group $\Gamma \leq Iso(Y)$.

Proposition 25. Let (X_i, p_i) be a sequence of pointed proper metric spaces that converges in the Gromov–Hausdorff sense to a pointed proper metric space (X, p). Assume a sequence of closed groups $\Gamma_i \leq Iso(X_i)$ converges equivariantly to a closed group $\Gamma \leq Iso(X)$. Then the sequence of pointed metric spaces $(\Gamma_i, d_{p_i}, Id_{X_i})$ converges in the Gromov–Hausdorff sense to (Γ, d_p, Id_X).

Proof. Recall that one has ε_i-approximations $\phi_i : X_i \to X \cup \{*\}$ and $\psi_i : X \to X_i \cup \{*\}$ with $\varepsilon_i \to 0$ and satisfying Equation 5. With these functions, one could define $f_i : \Gamma_i \to \Gamma \cup \{*\}$ in the following way: for each $g \in \Gamma_i$, if there is an element $\gamma \in \Gamma$ with $d_p(\phi_i \circ g \circ \psi_i, \gamma) \leq 1$, choose $f_i(g)$ to be an element of Γ that minimizes $d_p(\phi_i \circ g \circ \psi_i, f_i(g))$. Otherwise, set $f_i(g) = *$. Now we verify that f_i are δ_i-approximations for some $\delta_i \to 0$. The fact that $f_i(Id_{X_i}) \to Id_X$ follows from Equation 5.

The fourth condition, corresponding to Equation 4, follows directly from the first condition in the definition of equivariant convergence and our construction.

To check that f_i satisfy the second condition, corresponding to Equation 2, assume by contradiction that after taking a subsequence we can find $g_i \in \Gamma_i$ such that $d_{p_i}(g_i, Id_{X_i}) \to$

∞ but $d_p(f_i(g_i), Id_X)$ is bounded. This implies that $d(g_i(p_i), p_i) \to \infty$ and since g_i is an isometry, $d(g_i \circ \psi_i(p), p_i) \to \infty$. As ϕ_i are ε_i-approximations for $\varepsilon_i \to 0$, $d(\phi_i \circ g_i \circ \psi_i(p), p) \to \infty$ as well. On the other hand, as $d_p(f_i(g_i), Id_X)$ is bounded, $f_i(g_i) \neq *$ and $d_p(\phi_i \circ g_i \circ \psi_i, f_i(g_i)) \leq 1$ for all i, meaning that $d_p(\phi_i \circ g_i \circ \psi_i, Id_X)$ is bounded, which is a contradiction.

To verify the third condition, corresponding to Equation 3, assume by contradiction that after taking a subsequence we can find two sequences $g_i, h_i \in \Gamma_i$ such that the sequences $d_{p_i}(g_i, Id_{X_i}), d_{p_i}(h_i, Id_{X_i})$ are bounded but $|d_{p_i}(g_i, h_i) - d_p(f_i(g_i), f_i(h_i))| \geq \eta$ for some $\eta > 0$. After again taking a subsequence we can assume g_i converges to $g \in \Gamma$ and h_i converges to $h \in \Gamma$. This means that $d_p(f_i(g_i), \phi_i \circ g_i \circ \psi_i) \to 0$, $d_p(f_i(h_i), \phi_i \circ h_i \circ \psi_i) \to 0$. Hence for i large enough one has $|d_{p_i}(g_i, h_i) - d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i)| \geq \eta/2$.

We first deal with the case when

$$d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i) \geq d_{p_i}(g_i, h_i) + \eta/2 \tag{6}$$

for infinitely many i. By definition of d_{p_i}, there is a sequence $\rho_i > 0$ with

$$d_{p_i}(g_i, h_i) + \frac{\eta}{4} \geq \frac{1}{\rho_i} + \sup_{x \in B(p_i, \rho_i, X_i)} d(g_i x, h_i x).$$

Setting $r_i := \min\{\rho_i, 12/\eta\}$, we obtain a bounded sequence such that

$$d_{p_i}(g_i, h_i) + \frac{\eta}{3} \geq \frac{1}{r_i} + \sup_{x \in B(p_i, r_i, X_i)} d(g_i x, h_i x).$$

For i large enough and $x \in B(p, r_i - 2\varepsilon_i, X)$,

$$\begin{aligned} d(\phi_i \circ g_i \circ \psi_i(x), \phi_i \circ h_i \circ \psi_i(x)) &\leq \varepsilon_i + d(g_i(\psi_i(x)), h_i(\psi_i(x))) \\ &\leq \varepsilon_i - \frac{1}{r_i} + d_{p_i}(g_i, h_i) + \frac{\eta}{3}. \end{aligned}$$

Again from the definition of d_p we deduce

$$\begin{aligned} d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i) &\leq \frac{1}{r_i - 2\varepsilon_i} + \sup_{x \in B(p, r_i - 2\varepsilon_i, X)} d(\phi_i \circ g_i \circ \psi_i(x), \phi_i \circ h_i \circ \psi_i(x)) \\ &\leq \frac{1}{r_i - 2\varepsilon_i} - \frac{1}{r_i} + d_{p_i}(g_i, h_i) + \frac{\eta}{3} + \varepsilon_i. \end{aligned}$$

From the fact that $d_{p_i}(g_i, Id_{X_i}), d_{p_i}(h_i, Id_{X_i})$ are bounded, we get that r_i is bounded away from 0. Then the right hand side is less than $d_{p_i}(g_i, h_i) + \eta/2$ for i large enough, contradicting Equation 6. The other case when

$$d_{p_i}(g_i, h_i) \geq d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i) + \eta/2$$

for infinitely many i is analogous. □

As a consequence of Theorem 21, it is easy to understand the situation when the quotients of a sequence converge to $\mathbb{R}^n$.

Lemma 26. For each $i \in \mathbb{N}$, let (X_i, p_i) be a pointed $RCD^*(-\varepsilon_i, N)$ space of rectifiable dimension n with $\varepsilon_i \to 0$. Assume (X_i, p_i) converges in the Gromov–Hausdorff sense to a pointed $RCD^*(0, N)$ space (X, p), there is a sequence of closed groups of isometries $\Gamma_i \leq Iso(X_i)$ that converges equivariantly to $\Gamma \leq Iso(X)$, and the sequence of pointed proper metric spaces $(X_i/\Gamma_i, [p_i])$ converges in the Gromov–Hausdorff sense to $(\mathbb{R}^n, 0)$. Then Γ is trivial.

Proof. One can use the submetry $X \to X/\Gamma = \mathbb{R}^n$ to lift the lines of $\mathbb{R}^n$ to lines in X passing through p. By iterated applications of Theorem 21, we get that $X = \mathbb{R}^n \times Y$ for some $RCD^*(0, N - n)$ space Y. But from Theorem 20, the rectifiable dimension of X is at most n, so Y is a point. Since $\Gamma \leq Iso(\mathbb{R}^n)$ satisfies $\mathbb{R}^n/\Gamma = \mathbb{R}^n$, it must be trivial. □

2.5 Group norms

Let (X, p) be a pointed proper geodesic space and $\Gamma \leq Iso(X)$ a group of isometries. The *norm* $\|\cdot\|_p : \Gamma \to \mathbb{R}$ associated to p is defined as $\|g\|_p := d(gp, p)$. The *spectrum* $\sigma(\Gamma, X, p)$ is defined as the set of $r \geq 0$ such that

$$\langle \{g \in \Gamma | \|g\|_p \leq r\} \rangle \neq \langle \{g \in \Gamma | \|g\|_p \leq r - \varepsilon\} \rangle \text{ for all } \varepsilon > 0.$$

This spectrum is closely related to the covering spectrum introduced by Sormani–Wei in [15], and it also satisfies a continuity property.

Proposition 27. Let (X_i, p_i) be a sequence of pointed proper metric spaces that converges in the Gromov–Hausdorff sense to (X, p) and consider a sequence of closed isometry groups $\Gamma_i \leq Iso(X_i)$ that converges equivariantly to a closed group $\Gamma \leq Iso(X)$. Then for any convergent sequence $r_i \in \sigma(\Gamma_i, X_i, p_i)$, we have $\lim_{i\to\infty} r_i \in \sigma(\Gamma, X, p)$.

Proof. Let $r = \lim_{i\to\infty} r_i$ By definition, there is a sequence $g_i \in \Gamma_i$ with $\|g_i\|_p = r_i$, and $g_i \notin \langle \{\gamma \in \Gamma_i | \|\gamma\|_{p_i} \leq r_i - \varepsilon\} \rangle$ for all $\varepsilon > 0$. Up to subsequence, we can assume that g_i converges to some $g \in Iso(X)$ with $\|g\|_p = r$.

If $r \notin \sigma(\Gamma, X, p)$, it would mean there are $h_1, \ldots, h_k \in \Gamma$ with $\|h_j\|_p < r$ for each $j \in \{1, \ldots, k\}$, and $h_1 \cdots h_k = g$. For each j, choose sequences $h_j^i \in \Gamma_i$ that converge to h_j. As the norm is continuous with respect to convergence of isometries, for i large enough one has $\|h_j^i\|_p < r_i$ for each j.

The sequence $g_i(h_1^i \cdots h_k^i)^{-1} \in \Gamma_i$ converges to $g(h_1 \cdots h_k)^{-1} = e \in \Gamma$, so its norm is less than r_i for i large enough, allowing us to write g_i as a product of $k+1$ elements with norm $< r_i$, thus a contradiction. □

Definition 28. Let G be a group and $S \subset G$ a generating subset containing the identity. We say that S is a *determining set* if G has a presentation $G = \langle S | R \rangle$ with R consisting of words of length 3 using as letters the elements of $S \cup S^{-1}$.

Proposition 29. For each $M \in \mathbb{N}$, there are only finitely many isomorphism types of groups admitting a determining set of size $\leq M$.

The following lemma can be found in ([17], Section 2.12).

Lemma 30. Let $D > 0$, (Y, q) a pointed proper geodesic space and $G \leq Iso(Y)$ a closed group of isometries with $\operatorname{diam}(Y/G) \leq D$. If $\{g \in G | \|g\|_q \leq 10D\}$ is not a determining set, then there is a non-trivial covering map $\tilde{Y} \to Y$.

3 Proof of main theorems

Proof of Theorem 5. By contradiction, assume there is a sequence (X_i, p_i) of pointed $RCD^*(K, N)$ spaces, discrete groups $\Gamma_i \leq Iso(X_i)$ of measure preserving isometries such that $\mathrm{vol}^*_{\mathrm{K,N}}\ (X_i/\Gamma_i, [p_i]) \geq \nu$, and elements $g_i \in \Gamma_i \backslash \{Id_{X_i}\}$ with $d_{p_i}(g_i, Id_{X_i}) \to 0$. After taking a subsequence, we can assume the spaces X_i have dimension n for some $n \in \mathbb{N} \cap [0, N]$, the sequence $(X_i/\Gamma_i, [p_i])$ converges to a pointed $RCD^*(K, N)$ space (Y, q) of rectifiable dimension n. By Corollary 3, we can also assume that q is n-regular.

Choose $\eta_i \to \infty$ diverging so slowly that $(\eta_i X_i/\Gamma_i, [p_i])$ converges to $(\mathbb{R}^n, 0)$ and $\eta_i d_{p_i}(g_i, Id_{X_i}) \to 0$, and set $Y_i := \eta_i X_i$. By Theorem 18, we can find a Reifenberg sequence $y_i \in Y_i/\Gamma_i$ and $\tilde{y}_i \in B(p_i, 1, Y_i)$ with $[\tilde{y}_i] = y_i$, which by Lemma 9 can be taken so that $\|g_i\|_{\tilde{y}_i} \neq 0$.

Notice that by Corollary 3, we still have $d_{\tilde{y}_i}(g_i, Id_{Y_i}) \to 0$. Hence we can find $1/\lambda_i \in \sigma(\Gamma_i, Y_i, \tilde{y}_i)$ with $\lambda_i \to \infty$. As y_i is a Reifenberg sequence, $(\lambda_i Y_i/\Gamma_i, y_i)$ converges to $(\mathbb{R}^n, 0)$, so by Lemma 26, the actions of Γ_i on $\lambda_i Y_i$ converge equivariantly to the trivial group. This contradicts Proposition 27, as by construction we have $1 \in \sigma(\Gamma_i, \lambda_i Y_i, \tilde{y}_i)$ for all i. □

Proof of Theorem 4: Assuming the contrary, we could find a sequence (X_i, p_i) of pointed $RCD^*(K, N)$ spaces of diameter $\leq D$ and collapsing volume $\geq \nu$ whose fundamental groups are pairwise non-isomorphic. After taking a subsequence, we may assume their universal covers $(\tilde{X}_i, \tilde{p}_i)$ converge to an $RCD^*(K, N)$ space $(\tilde{X}, \tilde{p})$, and the actions of $\pi_1(X_i)$ converge to the action of a group Γ in $\tilde{X}$.

By Theorem 5, there is $\varepsilon > 0$ such that the elements of Γ_i are at pairwise $d_{\tilde{p}_i}$-distance $\geq \varepsilon$. By Lemma 30, for each i the set $S_i := \{g \in \Gamma_i | \|g\|_{\tilde{p}_i} \leq 10D\}$ is determining in Γ_i, and by plugging $r = 1$ in Equation 1, we get $S_i \subset \{g \in \Gamma_i | d_{\tilde{p}_i}(g, Id_{\tilde{X}_i}) \leq 10D + 3\}$. As $(\Gamma_i, d_{\tilde{p}_i}, Id_{\tilde{X}_i})$ converges in the Gromov–Hausdorff sense to $(\Gamma, d_{\tilde{p}}, Id_{\tilde{X}})$, Corollary 13 implies that $|S_i| \leq M$ for some $M \in \mathbb{N}$, so by Proposition 29 there are only finitely many isomorphism types in the sequence $\{\pi_1(X_i)\}_{i \in \mathbb{N}}$, contradicting our initial assumption. □

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