My research areas are metric geometry, differential geometry, and geometric group theory. I mainly study the behavior of topological features through Gromov–Hausdorff convergence. This is usually done under additional assumptions such as curvature bounds (Sections 1 and 3) or existence of large isometry groups (Sections 2 and 4). I also work with tools from probability theory, geometric topology, arithmetic combinatorics, and representation theory.

1 Collapse

For $\kappa \in \mathbb{R}$, $n \in \mathbb{N}$, and $D > 0$, one can consider the family $\mathcal{M}(\kappa, n, D)$ of closed $n$-dimensional Riemannian manifolds of diameter $\leq D$, and Ricci curvature $\geq \kappa$. Given a sequence $X_i$ in $\mathcal{M}(\kappa, n, D)$, Gromov’s compactness criterion [11] guarantees the existence (up to subsequence) of a Gromov–Hausdorff limit $X$. The geometric features of both the sequence $X_i$ and the limit space $X$ depend highly on whether $\text{vol}(X_i) \to 0$ as $i \to \infty$, or $\text{vol}(X_i) \geq \nu$ for some $\nu > 0$. In the former case, we say the sequence collapses and in the latter, we say the sequence is non-collapsing.

The behavior of non-collapsing sequences is considerably more stable than the one of collapsing sequences. For example, given a non-collapsing sequence $X_i \in \mathcal{M}(\kappa, n, D)$, Anderson showed that the sequence contains only finitely many fundamental group isomorphism types [1]. Furthermore, Cheeger–Colding showed that if $X$ is a smooth Riemannian manifold, then $X_i$ is diffeomorphic to $X$ for $i$ large enough [7].

Hence, it is important to determine if a sequence $X_i$ collapses or not, and if it does, classify the possible limit spaces (see Figures 1 and 2). Gromov showed that if for a closed smooth manifold $M$ its first Betti number is greater than its dimension, then $M$ cannot collapse to a point [11]. I generalized this result showing that in case of collapse, the first Betti number cannot drop more than the dimension [22].

![Figure 1: The 2-dimensional sphere can collapse to an interval, but Theorem 6 shows that it cannot collapse to a circle.](image)

**Theorem 1.** Let $X_i \in \mathcal{M}(\kappa, n, D)$ be a sequence of spaces whose first Betti number satisfies $\beta_1(X_i) \geq r$. Assume the sequence $X_i$ converges in the Gromov–Hausdorff sense to a space $X$ of dimension $m$, then

$$\beta_1(X) \geq r + m - n.$$
Figure 2: The 2-dimensional torus can collapse to a circle, but by Theorem 1, it cannot collapse to an interval.

**Conjecture 2.** Let $X_i \in \mathcal{M}(\kappa, n, D)$ be a sequence of spaces homeomorphic to the $n$-dimensional torus. If the sequence $X_i$ converges in the Gromov–Hausdorff sense to a $m$-dimensional space $X$, then $X$ is homeomorphic to the $m$-dimensional torus.

Using geometric group theory, Mineyev showed that if $M$ is a closed aspherical $n$-dimensional manifold with non-elementary hyperbolic fundamental group, then any sequence $X_i \in \mathcal{M}(\kappa, n, D)$ of spaces homeomorphic to $M$ cannot collapse [16]. Using Kapovitch–Wilking Theorem on the structure of fundamental groups of manifolds with lower Ricci curvature bounds [12], and systolic inequalities [11], I have studied the fundamental groups of aspherical manifolds that collapse, obtaining a slight generalization of Mineyev’s result [24].

**Theorem 3.** Let $X_i \in \mathcal{M}(\kappa, n, D)$ be a sequence of aspherical manifolds. If the sequence $X_i$ collapses, then for large enough $i$, there are non-trivial finitely generated abelian normal subgroups $1 \neq A_i \triangleleft \pi_1(X_i)$.

**Corollary 4.** Let $X_i \in \mathcal{M}(\kappa, n, D)$ be a sequence of aspherical manifolds with non-elementary hyperbolic fundamental groups $\pi_1(X_i)$. Then the sequence $X_i$ cannot collapse.

Using the Anderson finiteness result [1], and the fact that the Borel conjecture is true for hyperbolic groups [3], one obtains the following corollary:

**Corollary 5.** The class
\[
\{ X \in \mathcal{M}(\kappa, n, D) \mid \text{$X$ is aspherical, $\pi_1(X)$ is non-elementary hyperbolic} \}
\]
contains only finitely many homeomorphism types.

I believe the techniques used to proved Theorem 3 can be pushed to show that certain classes of aspherical manifolds (for example, with relatively hyperbolic fundamental groups) cannot collapse.

### 2 Lower semi-continuity of the fundamental group

It is well known that the fundamental group is lower semi-continuous with respect to Gromov–Hausdorff convergence [20].

**Theorem 6.** (Folklore) If $X_i$ is a sequence of compact geodesic spaces that converges in the Gromov–Hausdorff sense to a semi-locally-simply-connected compact space $X$, then for large enough $i$, there are surjective morphisms
\[
\pi_1(X_i) \rightarrow \pi_1(X).
\]
Theorem 6 does not hold in the non-compact case (see Figure 3), even when the sequence consists of homogeneous spaces ([25], Example 17). On the other hand, using Breuillard–Green–Tao structure of approximate groups [5] and Malcev embedding Theorem [14], I showed that this lower-semi-continuity is recovered when one has discrete groups acting almost transitively [25].

Figure 3: The compactness hypothesis of Theorem 6 cannot be removed. A sequence of ellipsoids can converge to $S^1 \times \mathbb{R}$ in the pointed Gromov–Hausdorff sense.

**Theorem 7.** Let $X_i$ be a sequence of proper geodesic spaces converging in the pointed Gromov–Hausdorff sense to a geodesic space $X$. Assume there is a sequence of discrete groups of isometries $\Gamma_i \leq \text{Iso}(X_i)$ with $\text{diam}(X_i/\Gamma_i) \to 0$. Then $X$ is a nilpotent group equipped with an invariant metric. Furthermore, if either $X$ is semi-locally-simply-connected or has finite topological dimension, then it is a Lie group with an invariant sub-Finsler metric, and for large enough $i$, there are subgroups $\Lambda_i \leq \pi_1(X_i)$ with surjective homomorphisms

$$\Lambda_i \to \pi_1(X).$$

I believe one can drop the hypothesis that the groups $\Gamma_i$ act almost transitively.

**Conjecture 8.** Let $X_i$ be a sequence of proper simply connected geodesic spaces converging in the pointed Gromov–Hausdorff sense to a semi-locally-simply-connected geodesic space $X$. Assume there is a sequence of discrete groups of isometries $\Gamma_i \leq \text{Iso}(X_i)$ satisfying

$$\lim \inf_{i \to \infty} \text{diam}(X_i/\Gamma_i) < \infty.$$

Then $X$ is simply connected.

### 3 $RCD^*(K, N)$ spaces

For $K \in \mathbb{R}$, $N \in [1, \infty)$, the class of $RCD^*(K, N)$ spaces consists of proper metric measure spaces that satisfy a synthetic condition of having Ricci curvature bounded below by $K$ and dimension bounded above by $N$. This class contains the class of complete Riemannian manifolds of Ricci curvature $\geq K$ and dimension $\leq N$, is closed under measured Gromov–Hausdorff convergence [2], and is closed under quotients by compact group actions [9].

$RCD^*(K, N)$ spaces have a well defined notion of dimension called *rectifiable dimension* [6], which is always an integer between 0 and $N$, and is lower semi-continuous with respect to Gromov–Hausdorff convergence [13]. Also, they are semi-locally-simply-connected, so they have simply connected universal covers [21].

In my work with Jaime Santos [19], we were able to recover certain results about the fundamental groups of $RCD^*(K, N)$ spaces, including an analogue of Theorem 1, and some uniform virtual abelianicity results previously known for smooth Riemannian manifolds with lower sectional curvature bounds [15]. We denote by $RCD^*(K, N; D)$ the class of compact $RCD^*(K, N)$ of diameter $\leq D$. 

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Theorem 9. Let \((X_i, d_i, m_i)\) be a sequence of \(RCD^*(0, N; D)\) spaces of rectifiable dimension \(n\) such that the sequence of universal covers \((\tilde{X}_i, \tilde{d}_i, \tilde{m}_i)\) converges in the pointed measured Gromov–Hausdorff sense to some \(RCD^*(0, N)\) space of rectifiable dimension \(n\). Then there is \(C > 0\) such that for each \(i\) there is an abelian subgroup \(A_i \leq \pi_1(X_i)\) with index \([\pi_1(X_i) : A_i] \leq C\).

Theorem 10. Let \((X_i, d_i, m_i)\) be a sequence of \(RCD^*(K, N; D)\) of rectifiable dimension \(n\), and compact universal covers \((\tilde{X}_i, \tilde{d}_i, \tilde{m}_i)\). If the sequence \((\tilde{X}_i, \tilde{d}_i, \tilde{m}_i)\) converges in the pointed measured Gromov–Hausdorff sense to some \(RCD^*(K, N)\) space of rectifiable dimension \(n\), then there is \(C > 0\) such that for each \(i\) there is an abelian subgroup \(A_i \leq \pi_1(X_i)\) with index \([\pi_1(X_i) : A_i] \leq C\).

I believe the non-collapsing condition on the universal covers in Theorems 9 and 10, can be removed by studying the actions of lattices of nilpotent Lie groups on tori.

Conjecture 11. For each \(N \geq 1\), there is \(C > 0\) such that for any \(RCD^*(0, N; 1)\) space \((X, d, m)\), there is an abelian subgroup \(A \leq \pi_1(X)\) with index \([\pi_1(X) : A] \leq C\).

Conjecture 12. For each \(K \in \mathbb{R}, N \geq 1, D > 0\), there is \(C > 0\) such that for any \(RCD^*(K, N; D)\) space \((X, d, m)\) with compact universal cover, there is an abelian subgroup \(A \leq \pi_1(X)\) with index \([\pi_1(X) : A] \leq C\).

4 Compact universal covers

If \(X\) is a geodesic space, and \(\tilde{X} \to X\) is a compact covering space, it makes sense to control the diameter of \(\tilde{X}\) in terms of the diameter of \(X\) and the number \(k\) of sheets. Specifically, Ivanov [17] has shown that

\[
\text{diam}(\tilde{X}) \leq k \cdot \text{diam}(X).
\]

This bound is sharp, as one can see from the map \(S^1 \to S^1\) given by \(z \mapsto z^k\), where the domain circle is \(k\) times longer than the codomain circle. However, for universal covers, this bound seemed to be too loose, at least as \(k \to \infty\) [18]. In the particular case of universal covers, I found a much better effective bound [23].

Theorem 13. Let \(X\) be a compact semi-locally-simply-connected geodesic space with finite fundamental group, and \(\tilde{X}\) its universal cover. Then

\[
\text{diam}(\tilde{X}) \leq 4\sqrt{\pi_1(X)} \cdot \text{diam}(X).
\]

Stronger but non-effective bounds with arbitrarily small powers of \(|\pi_1(X)|\) have been obtained: Benjamini–Finucane–Tessera [4] showed that if \(\tilde{X}_i\) is a sequence of compact universal covers of geodesic spaces \(X_i\) with \(|\pi_1(X_i)| \to \infty\), then

\[
\frac{\text{diam}(\tilde{X}_i)}{\text{diam}(X_i)} = o(|\pi_1(X_i)|^p).
\]

When the fundamental group is abelian, standard Fourier analysis techniques are available [8]. With them, I obtained an explicit upper bound for the diameter of the universal cover of a Riemannian manifold [23].

Theorem 14. Assume \(M \in \mathfrak{M}(\kappa(n-1), n, D)\) has a point \(p\) whose injectivity radius is \(\geq 2r_0 > 0\). If its universal cover \(\tilde{M}\) is compact and \(\pi_1(M)\) is abelian, then

\[
\frac{\text{diam}(\tilde{M})}{\text{diam}(M)} \leq 3 + \left[\frac{3vn_k^\kappa(2D + r_0)}{2vn_k^\kappa(r_0)} \log |\pi_1(M)|\right],
\]

where \(v_n^\kappa(r)\) denotes the volume of a ball of radius \(r\) in the \(n\)-dimensional simply connected space of constant sectional curvature \(\kappa\).
The above results have room for improvement. We actually believe there is a logarithmic bound.

**Conjecture 15.** There is a universal $c > 0$ such that the following holds. Let $X_i$ be the universal covers of a sequence $X_i$ of compact geodesic spaces with finite fundamental groups $\pi_1(X_i)$. If $|\pi_1(X_i)| \to \infty$ as $i \to \infty$, then

$$\frac{\text{diam}(\tilde{X}_i)}{\text{diam}(X_i)} = o(\log^c|\pi_1(X_i)|).$$

If we have a sequence of compact geodesic spaces $X_i$ with $\text{diam}(X_i) \to 0$ and compact universal covers $\tilde{X}_i$ with $\text{diam}(\tilde{X}_i) = 1$ for all $i$, then the sequence $\tilde{X}_i$ cannot converge with respect to the Gromov–Hausdorff topology [10]. This implies that there is no general “global shape” of compact universal covers. It would be interesting to know if this is caused by some concentration of measure phenomenon.

**Conjecture 16.** Let $X_i$ be a sequence of semi-locally-simply-connected geodesic spaces with $\text{diam}(X_i) \to 0$ and compact universal covers $\tilde{X}_i$ with $\text{diam}(\tilde{X}_i) = 1$ for all $i$. Then the sequence $\tilde{X}_i$ is statistically trivial: that is, for any sequence of $\pi_1(X_i)$-invariant Borel probability measures $\mu_i$ on $\tilde{X}_i$, and 1-Lipschitz functions $f_i : \tilde{X}_i \to \mathbb{R}$, one has

$$\text{Var}((f_i)_{\ast}\mu_i) \to 0 \text{ as } i \to \infty.$$

**References**