Research Statement

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My research areas are metric geometry, differential geometry, and geometric group theory. I mainly study the behavior of topological features through Gromov–Hausdorff convergence. This is usually done under additional assumptions such as curvature bounds (Sections 1 and 3) or existence of large isometry groups (Sections 2 and 4). I also work with tools from probability theory, geometric topology, arithmetic combinatorics, and representation theory.

1 Collapse

For $\kappa \in \mathbb{R}$, $n \in \mathbb{N}$, and $D > 0$, one can consider the family $\mathfrak{M}(\kappa, n, D)$ of closed n-dimensional Riemannian manifolds of diameter $\leq D$, and Ricci curvature $\geq \kappa$. Given a sequence X_i in $\mathfrak{M}(\kappa, n, D)$, Gromov's compactness criterion [11] guarantees the existence (up to subsequence) of a Gromov–Hausdorff limit X. The geometric features of both the sequence X_i and the limit space X depend highly on whether $\text{vol}(X_i) \to 0$ as $i \to \infty$, or $\text{vol}(X_i) \geq \nu$ for some $\nu > 0$. In the former case, we say the sequence **collapses** and in the latter, we say the sequence is **non-collapsing**.

The behavior of non-collapsing sequences is considerably more stable than the one of collapsing sequences. For example, given a non-collapsing sequence $X_i \in \mathfrak{M}(\kappa, n, D)$, Anderson showed that the sequence contains only finitely many fundamental group isomorphism types [1]. Furthermore, Cheeger–Colding showed that if X is a smooth Riemannian manifold, then X_i is diffeomorphic to X for i large enough [7].

Hence, it is important to determine if a sequence X_i collapses or not, and if it does, classify the possible limit spaces (see Figures 1 and 2). Gromov showed that if for a closed smooth manifold M its first Betti number is greater than its dimension, then M cannot collapse to a point [11]. I generalized this result showing that in case of collapse, the first Betti number cannot drop more than the dimension [22].

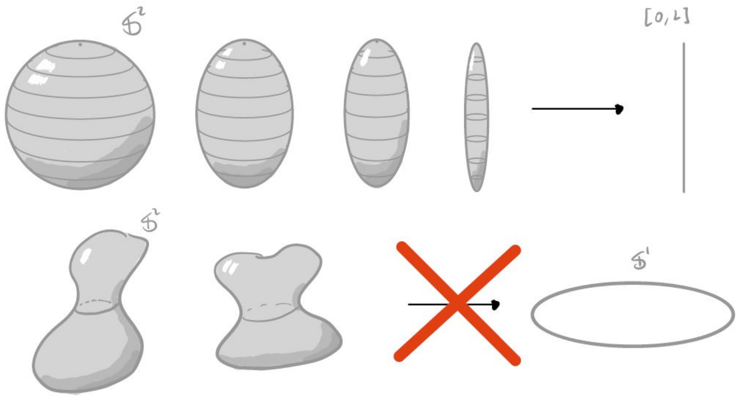

Figure 1: The 2-dimensional sphere can collapse to an interval, but Theorem 6 shows that it cannot collapse to a circle.

Theorem 1. Assume $X_i \in \mathfrak{M}(\kappa, n, D)$ converges in the Gromov–Hausdorff sense to a space X of rectifiable dimension m, then

$$\limsup_{i \to \infty} \beta_1(X_i) - \beta_1(X) \leq n - m.$$

I believe that the ideas behind Theorem 1 can be used to show that tori can only collapse to tori under lower Ricci curvature bounds.

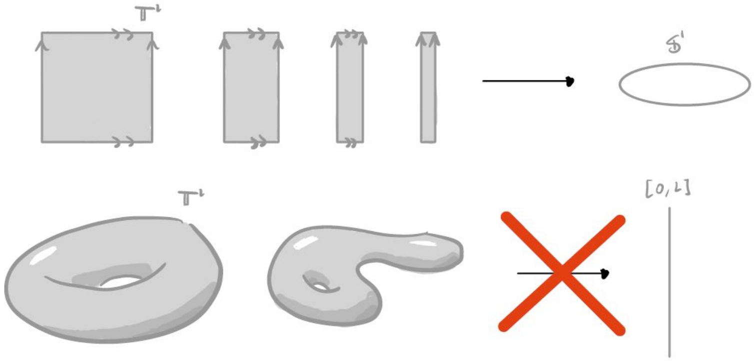

Figure 2: The 2-dimensional torus can collapse to a circle, but by Theorem 1, it cannot collapse to an interval.

Conjecture 2. Let $X_i \in \mathfrak{M}(\kappa, n, D)$ be a sequence of spaces homeomorphic to the n-dimensional torus. If the sequence X_i converges in the Gromov–Hausdorff sense to a m-dimensional space X, then X is homeomorphic to the m-dimensional torus.

Using geometric group theory, Mineyev showed that if M is a closed aspherical n-dimensional manifold with non-elementary hyperbolic fundamental group, then any sequence $X_i \in \mathfrak{M}(\kappa, n, D)$ of spaces homeomorphic to M cannot collapse [16]. Using Kapovitch–Wilking Theorem on the structure of fundamental groups of manifolds with lower Ricci curvature bounds [12], and systolic inequalities [11], I have studied the fundamental groups of aspherical manifolds that collapse, obtaining a slight generalization of Mineyev's result [24].

Theorem 3. Let $X_i \in \mathfrak{M}(\kappa, n, D)$ be a sequence of aspherical manifolds. If the sequence X_i collapses, then for large enough i, there are non-trivial finitely generated abelian normal subgroups

$$1 \neq A_i \triangleleft \pi_1(X_i).$$

Corollary 4. Let $X_i \in \mathfrak{M}(\kappa, n, D)$ be a sequence of aspherical manifolds with non-elementary hyperbolic fundamental groups $\pi_1(X_i)$. Then the sequence X_i cannot collapse.

Using Anderson finiteness [1], and the fact that the Borel conjecture is true for hyperbolic groups [3], one obtains the following corollary:

Corollary 5. The class

$$\{X \in \mathfrak{M}(\kappa, n, D) \mid X \text{ is aspherical, } \pi_1(X) \text{ is non-elementary hyperbolic}\}$$

contains only finitely many homeomorphism types.

I believe the techniques used to proved Theorem 3 can be pushed to show that certain classes of aspherical manifolds (for example, with relatively hyperbolic fundamental groups) cannot collapse.

2 Lower semi-continuity of the fundamental group

It is well knwon that the fundamental group is lower semi-continuous with respect to Gromov–Hausdorff convergence [20].

Theorem 6. (*Folklore*) If X_i is a sequence of compact geodesic spaces that converges in the Gromov–Hausdorff sense to a semi-locally-simply-connected **compact** space X, then for i large enough, $\pi_1(X)$ is isomorphic to a quotient of $\pi_1(X_i)$.

Theorem 6 does not hold in the non-compact case (see Figure 3), even when the sequence consists of homogeneous spaces ([25], Example 17). On the other hand, using Breuillard–Green–Tao structure of approximate groups [5] and Malcev embedding Theorem [14], I showed that this lower-semi-continuity is recovered when one has discrete groups acting almost transitively [25].

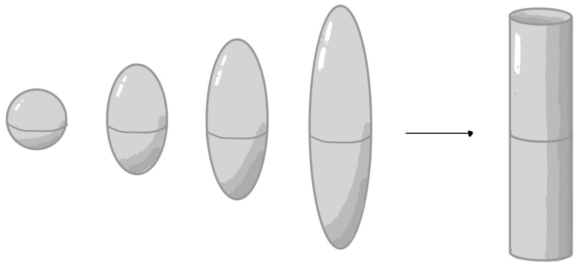

Figure 3: The compactness hypothesis of Theorem 6 cannot be removed. A sequence of ellipsoids can converge to $\mathbb{S}^1 \times \mathbb{R}$ in the pointed Gromov–Hausdorff sense.

Theorem 7. Let X_i be a sequence of proper geodesic spaces converging in the pointed Gromov–Hausdorff sense to a semi-locally-simply-connected space X. Assume there is a sequence of discrete groups of isometries $\Gamma_i \leq Iso(X_i)$ with $\mathrm{diam}(X_i/\Gamma_i) \to 0$. Then for i large enough, $\pi_1(X)$ can be embedded in a quotient of $\pi_1(X_i)$.

I believe one can drop the hypothesis that the groups Γ_i act almost transitively.

Conjecture 8. Let X_i be a sequence of proper simply connected geodesic spaces converging in the pointed Gromov–Hausdorff sense to a semi-locally-simply-connected geodesic space X. Assume there is a sequence of discrete groups of isometries $\Gamma_i \leq Iso(X_i)$ satisfying

$$\liminf_{i \to \infty} \mathrm{diam}(X_i/\Gamma_i) < \infty.$$

Then X is simply connected.

3 $RCD^*(K,N)$ spaces

For $K \in \mathbb{R}$, $N \in [1,\infty)$, the class of $RCD^*(K,N)$ spaces consists of proper metric measure spaces that satisfy a synthetic condition of having Ricci curvature bounded below by K and dimension bounded above by N. This class contains the class of complete Riemannian manifolds of Ricci curvature $\geq K$ and dimension $\leq N$, is closed under measured Gromov–Hausdorff convergence [2], and is closed under quotients by discrete group actions [9].

$RCD^*(K,N)$ spaces have a well defined notion of dimension called *rectifiable dimension* [6], which is always an integer between 0 and N, and is lower semi-continuous with respect to Gromov–Hausdorff convergence [13]. Also, they are semi-locally-simply-connected, so they have simply connected universal covers [21].

In my work with Jaime Santos [19], we were able to recover certain results about the fundamental groups of $RCD^*(K,N)$ spaces, including an analogue of Theorem 1, and some uniform virtual abelianicity results previously known for smooth Riemannian manifolds with lower sectional curvature bounds [15]. We denote by $RCD^*(K,N;D)$ the class of compact $RCD^*(K,N)$ of diameter $\leq D$.

Theorem 9. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(0,N;D)$ spaces of rectifiable dimension n such that the sequence of universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converges in the pointed measured Gromov–Hausdorff sense to some $RCD^*(0,N)$ space of rectifiable dimension n. Then there is $C > 0$ such that for each i there is an abelian subgroup $A_i \leq \pi_1(X_i)$ with index $[\pi_1(X_i) : A_i] \leq C$.

Theorem 10. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of $RCD^*(K,N;D)$ of rectifiable dimension n, and compact universal covers $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$. If the sequence $(\tilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i)$ converges in the pointed measured Gromov–Hausdorff sense to some $RCD^*(K,N)$ space of rectifiable dimension n, then there is $C > 0$ such that for each i there is an abelian subgroup $A_i \leq \pi_1(X_i)$ with index $[\pi_1(X_i) : A_i] \leq C$.

I believe the non-collapsing condition on the universal covers in Theorems 9 and 10, can be removed by studying the actions of lattices of nilpotent Lie groups on tori.

Conjecture 11. For each $N \geq 1$, there is $C > 0$ such that for any $RCD^*(0,N;1)$ space $(X, d, \mathfrak{m})$, there is an abelian subgroup $A \leq \pi_1(X)$ with index $[\pi_1(X) : A] \leq C$.

Conjecture 12. For each $K \in \mathbb{R}$, $N \geq 1$, $D > 0$, there is $C > 0$ such that for any $RCD^*(K,N;D)$ space $(X, d, \mathfrak{m})$ with compact universal cover, there is an abelian subgroup $A \leq \pi_1(X)$ with index $[\pi_1(X) : A] \leq C$.

4 Compact universal covers

If X is a geodesic space, and $\tilde{X} \to X$ is a compact covering space, it makes sense to control the diameter of $\tilde{X}$ in terms of the diameter of X and the number k of sheets. Specifically, Ivanov [17] has shown that

$$\mathrm{diam}(\tilde{X}) \leq k \cdot \mathrm{diam}(X).$$

This bound is sharp, as one can see from the map $\mathbb{S}^1 \to \mathbb{S}^1$ given by $z \to z^k$, where the domain circle is k times longer than the codomain circle. However, for universal covers, this bound seemed to be too loose, at least as $k \to \infty$ [18]. In the particular case of universal covers, I found a much better effective bound [23].

Theorem 13. Let X be a compact semi-locally-simply-connected geodesic space with finite fundamental group, and $\tilde{X}$ its universal cover. Then

$$\mathrm{diam}(\tilde{X}) \leq 4\sqrt{|\pi_1(X)|} \cdot \mathrm{diam}(X).$$

Stronger but non-effective bounds with arbitrarily small powers of $|\pi_1(X)|$ have been obtained: Benjamini–Finucane–Tessera [4] showed that if $\tilde{X}_i$ is a sequence of compact universal covers of geodesic spaces X_i with $|\pi_1(X_i)| \to \infty$, then

$$\frac{\mathrm{diam}(\tilde{X}_i)}{\mathrm{diam}(X_i)} = o\left(|\pi_1(X_i)|^p\right).$$

When the fundamental group is abelian, standard Fourier analysis techniques are available [8]. With them, I obtained an explicit upper bound for the diameter of the universal cover of a Riemannian manifold [23].

Theorem 14. Assume $M \in \mathfrak{M}(\kappa(n-1), n, D)$ has a point p whose injectivity radius is $\geq 2r_0 > 0$. If its universal cover $\tilde{M}$ is compact and $\pi_1(M)$ is abelian, then

$$\frac{\mathrm{diam}(\tilde{M})}{\mathrm{diam}(M)} \leq 3 + \left\lfloor \frac{3v_n^\kappa(2D + r_0)}{2v_n^\kappa(r_0)} \log|\pi_1(M)| \right\rfloor,$$

where $v_n^\kappa(r)$ denotes the volume of a ball of radius r in the n-dimensional simply connected space of constant sectional curvature κ.

The above results have room for improvement. We actually believe there is a logarithmic bound.

Conjecture 15. There is a universal $c > 0$ such that the following holds. Let $\tilde{X}_i$ be the universal covers of a sequence X_i of compact geodesic spaces with finite fundamental groups $\pi_1(X_i)$. If $|\pi_1(X_i)| \to \infty$ as $i \to \infty$, then

$$\frac{\operatorname{diam}(\tilde{X}_i)}{\operatorname{diam}(X_i)} = o\left(\log^c |\pi_1(X_i)|\right).$$

If we have a sequence of compact geodesic spaces X_i with $\operatorname{diam}(X_i) \to 0$ and compact universal covers $\tilde{X}_i$ with $\operatorname{diam}(\tilde{X}_i) = 1$ for all i, then the sequence $\tilde{X}_i$ cannot converge with respect to the Gromov–Hausdorff topology [10]. This implies that there is no general "global shape" of compact universal covers. It would be interesting to know if this is caused by some concentration of measure phenomenon.

Conjecture 16. Let X_i be a sequence of semi-locally-simply-connected geodesic spaces with $\operatorname{diam}(X_i) \to 0$ and compact universal covers $\tilde{X}_i$ with $\operatorname{diam}(\tilde{X}_i) = 1$ for all i. Then the sequence $\tilde{X}_i$ is *statistically trivial*: that is, for any sequence of $\pi_1(X_i)$-invariant Borel probability measures μ_i on $\tilde{X}_i$, and 1-Lipschitz functions $f_i : \tilde{X}_i \to \mathbb{R}$, one has

$$\operatorname{Var}((f_i)_* \mu_i) \to 0 \text{ as } i \to \infty.$$

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