Astronomy 485 — Problem Set 6 — Weeks 12 and 13

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All problems are worth 10 points.

1. (a) The pseudo-Newtonian potential

$$\Phi = -\frac{GM}{R - R_{\rm S}}, \quad R_{\rm S} = \frac{2GM}{c^2}$$

provides a simple and reasonably accurate potential for particle orbits near a Schwarzschild (non-rotating) black hole of mass M. It only applies for $R > R_{\rm S}$, and it approximates the true effects due to General Relativity. Consider a particle of mass m moving in this potential. Include the usual angular momentum term ($L^2/2mR^2$) to form the 'effective potential' (with units kg m^2 s^{-2}; you can read about this in your physics books if you have not heard of it before — check the section about orbits). Graph this effective potential. Try using the parameters $M = 10$ M$_\odot$, $m = 100$ kg, and $L = 2 \times 10^{15}$ kg m^2 s^{-1}, and consider radii from 30–300 km. Be careful to choose an appropriate ordinate range so that you can properly see the relevant physical behavior of the potential. Compare your graph to a graph of the 'standard' Newtonian effective potential, and do *not* go inside $R_{\rm S}$.

For the pseudo-Newtonian potential, note the 'pit in the potential' for radii close to $R_{\rm S}$. This is an important difference between Newtonian gravity and General Relativity since it allows 'capture orbits' and causes an otherwise Keplerian orbit to precess. If you are interested, you can read more about this matter in chapter 25 of *Gravitation* by C.W. Misner, K.S. Thorne and J.A. Wheeler.

1. (b) Show that the radii of the minimum bound and last stable *circular* orbits for the pseudo-Newtonian potential are $2R_{\rm S}$ and $3R_{\rm S}$, respectively.

1. (c) Show that the binding energy of a *circular* orbit at $3R_{\rm S}$ is $\frac{1}{16}mc^2$ [a longer calculation using General Relativity for a Schwarzschild black hole gives $(1 - (8/9)^{1/2})mc^2$].

If you are interested in further details of this pseudo-Newtonian potential, you can read about it in *Astronomy and Astrophysics* 88, 23 and *Astronomy and Astrophysics* 313, 334. This is entirely optional!

2. A rough lower limit on the number of black hole X-ray novae in the Galaxy. Consider the following facts about black hole X-ray novae:

- We detect about one black hole X-ray nova per year (we have been performing X-ray searches with high effectiveness for about the past 15 years).
- We can detect black hole X-ray novae out to about 3 kpc.
- Black hole X-ray novae lie in the Galactic disk.
- The time between outbursts for the typical black hole X-ray nova is probably at least 40 years (compare with page 265 of Charles & Seward).

Using this information, place a reasonable lower limit on the number of black hole X-ray novae in the Galaxy. You may assume that we are not living in an unusual region of space or epoch of time.

3. Please read the article 'Advection-Dominated Accretion and Black Hole Event Horizons' by Narayan, Garcia & McClintock (*The Astrophysical Journal*, 478, L79); this is available on the course World Wide Web page. Then please write an ≈ 2 page essay describing the main results of this article. You should address issues such as the following:

- What is 'advection' in the context of this article?
- What observations have made people postulate the existence of Advection-Dominated Accretion Flows (ADAFs)? That is, what observational problems can be solved using the ADAF model?
- Why is a 'two-temperature plasma' needed to make the ADAF model work? Given what you have learned in class, why might putting most of the thermal energy into the protons cause the ADAF to be an inefficient radiator?
- Why do the authors think Figure 2 of this paper provides good evidence for existence of black hole event horizons?

If you have any difficulties with this article, feel free to drop by and ask questions. I have put a picture on the course World Wide Web page that will be helpful to look at while you are doing the reading.

4. (a) Using the expression for $v_{\rm app}$ for superluminal motion (see the Fabian notes), make a graph showing how $v_{\rm app}$ varies as a function of θ for $\beta = 0.1$, $\beta = 0.5$ and $\beta = 0.98$. Concentrate on θ values from 0–90°. Explain the observed behavior from a physical point of view. What angle gives the maximum $v_{\rm app}$?

4. (b) For some objects showing superluminal expansion it is possible to place a constraint on their distance using the observed superluminal expansion. Consider the black hole candidate GRS 1915+105 which ejected two relativistic 'blobs' of gas in antiparallel directions. We have learned the equations

$$\mu_{\rm a} = \frac{\beta \sin\theta}{1 - \beta\cos\theta}\frac{c}{D} \qquad \mu_{\rm r} = \frac{\beta \sin\theta}{1 + \beta\cos\theta}\frac{c}{D}$$

where $\mu_{\rm a}$ ($\mu_{\rm r}$) is the proper motion in the sky of the blob of gas that is approaching (receding) and D is the distance (the geometry is as shown in the Fabian notes). GRS 1915+105 had $\mu_{\rm a} = 17.6$ mas day^{-1} and $\mu_{\rm r} = 9.0$ mas day^{-1}. Show that the observed motion implies $\theta < 71^\circ$ and hence that the distance to GRS 1915+105 is $D < 13.7$ kpc. This was the first evidence for a source with superluminal motions in our Galaxy.

5. (a) Please roughly calculate the maximum temperature in an accretion disk around a 1.4 $M_\odot$ neutron star (you may take the neutron star to have a very small magnetic field so that the inner accretion disk is not disrupted). Please repeat for a 10^8 $M_\odot$ Schwarzschild black hole. You may assume that both systems are accreting at the Eddington limit.
5. (b) Consider an accretion disk in an active galaxy that has an inner radius of r_I and a very large outer radius. Show that the integrated spectrum of this accretion disk varies with frequency as $\nu^{1/3}$ for low freqencies and $e^{-h\nu/kT_\mathrm{I}}$ for high frequencies. Here T_I is the disk temperature at r_I.
5. (c) Consider spherical accretion onto a highly magnetized neutron star. Please calculate the radius at which magnetic pressure overwhelms the ram pressure of the infalling matter. This radius is known as the Alfvén radius. Your answer should be in terms of the radius of the neutron star, the mass of the neutron star, the surface magnetic field of the neutron star, and the mass accretion rate. For a neutron star with a surface magnetic field of 10^{12} gauss that accretes at the Eddington rate, what is the numerical value of the Alfvén radius (in km)?

6. The importance of the trapping radius for spherical accretion. Consider a spherical flow of ionized hydrogen onto a black hole with mass M that dissipates (and radiates) an energy per gram of $\approx GM/r$ as it falls from r to $r/2$. Even though some of the energy must clearly be escaping as radiation, presume that the matter is still falling inward at roughly the free-fall speed all the way to the black hole.
6. (a) Calculate the optical depth for Thomson scattering from an inner radius r to infinity as a function of the accretion rate (you will need to do an integral here). Simplify your expression into an elegant form using $r_\mathrm{S} = 2GM/c^2$ and $\dot{M}_\mathrm{Edd} = L_\mathrm{Edd}/c^2$. Hint: The answer is $\tau_\mathrm{T} = \frac{\dot{M}}{\dot{M}_\mathrm{Edd}}(\frac{r_\mathrm{S}}{r})^{\frac{1}{2}}$.
6. (b) At what accretion rate (call this $\dot{M}_\mathrm{c}$) does this optical depth become unity for $r = r_\mathrm{S}$? How does $\dot{M}_\mathrm{c}$ relate to $\dot{M}_\mathrm{Edd}$?
6. (c) For $\dot{M} \gg \dot{M}_\mathrm{c}$, the inner parts of the flow become optically thick and the flow can drag back in the photons that are trying to diffuse outward through it. The black hole can then swallow these photons, so this may be a way to escape the Eddington limit on mass accretion.[1] Show that the critical 'trapping radius' (r_t) is equal to $(\dot{M}/\dot{M}_\mathrm{c})r_\mathrm{S}$. Inside this radius, the accretion flow drags the diffusing photons back into the black hole. Hint: The information on pages 35–36 of Rybicki & Lightman may be helpful when thinking about the random walk of a photon that is trying to escape.
6. (d) What is the approximate luminosity, L_esc, that escapes to infinity in the regime $\dot{M} \gg \dot{M}_\mathrm{c}$? How does the efficiency, η, depend on the ratio $\dot{M}/\dot{M}_\mathrm{c}$ in this regime? Here you may assume that all radiation created inside the trapping radius gets dragged back into the black hole, and you do not need to invoke a specific emission mechanism for the radiation.

If you are interested in reading about the 'trapping radius' in more detail, have a look at *Physica Scripta*, 17, 193. You may also be interested in the somewhat more complex papers *Monthly Notices of the Royal Astronomical Society*, 184, 53 and *The Astrophysical Journal*, 326, 223. Perhaps this is a quick way to 'grow' the quasar black holes seen at $z > 6$!

[1] Of course, since the photons are dragged back into the black hole you still cannot easily exceed the Eddington limit on *luminosity*. It is also not known if you can get super-Eddington accretion of this type started in the first place (for example, strong outflows may start to affect the accretion flow significantly). But if you can, then watch out!