Convergence notion

\( Q_i^n: \) cell average

\[
Q^\Delta(x, t) = Q_i^n \quad \text{for} \quad (x, t) \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t_n, t_{n+1}]
\]

\( \mathcal{W} = \{ q : q(x, t) \text{ is a weak solution to the conservation law} \} \)

norm for distance

\[
\| v \|_{1,T} \equiv \int_0^T \| v(\cdot, t) \|_1 \, dt = \int_0^T \int_{-\infty}^{\infty} |v(x, t)| \, dx \, dt
\]

Distance/error:

\[
\text{dist}(Q^\Delta, \mathcal{W}) = \inf_{q \in \mathcal{W}} \| Q^\Delta - q \|_{1,T}
\]

Convergence definition:

As \( \Delta \to 0 \) (i.e., \( \Delta x \to 0, \Delta t \to 0 \) with fixed ration \( \Delta t/\Delta x \)), we have \( \text{dist}(Q^\Delta, \mathcal{W}) \to 0 \).
Compactness

A compact set: (Helly’s compactness Theorem)

\[ \{ \nu \in L_1 : TV(\nu) \leq R, \ \text{Supp}(\nu) \subset [-M, M] \} \]

One can extend this to \((x, t)\) space. Function space:

\[ L_{1,T} = \{ \nu : \|\nu\|_{1,T} < \infty \} \]

where

\[ \|\nu\|_{1,T} = \int_{0}^{T} \|\nu(\cdot, t)\|_1 \ dt \]
Total variation in \((x, t)\):

\[
TV_T(q) = \limsup_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{-\infty}^{\infty} |q(x + \epsilon, t) - q(x, t)| \, dx \, dt
\]

\[
+ \limsup_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{-\infty}^{\infty} |q(x, t + \epsilon) - q(x, t)| \, dx \, dt
\]

For discrete data:

\[
TV_T(Q^\Delta) = \frac{T}{\Delta t} \sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \left[ \Delta t \left| Q_{i+1}^n - Q_i^n \right| + \Delta x \left| Q_i^{n+1} - Q_i^n \right| \right]
\]

\[
= \frac{T}{\Delta t} \sum_{n=0}^{\infty} \left[ \Delta t TV(Q^n) + \| Q^{n+1} - Q^n \|_1 \right]
\]

A compact set:

\[
\mathcal{K} = \{ q \in L_{1,T} : TV_T(q) \leq M, \text{ and } \text{Supp}(q(\cdot, t)) \subset [-M, M] \ \forall t \in [0, T] \}\]
A-priori estimates on $Q_i^n$, uniform for all grid size:

1. Maximum principle/uniform bound: $|Q_i^n| \leq M \ \forall n, i$
2. Bound on TV: $\text{TV}(Q^n) \leq M, \ \forall n$
3. $L^1$ continuity in time: $\| Q^{n+1} - Q^n \|_1 \leq \alpha \Delta t, \ \forall n$
4. Discrete version of entropy inequality.

Then:

(1)+(2)+(3): $\Rightarrow$ convergence towards a weak solution.

(1)+(2)+(3)+(4): $\Rightarrow$ convergence towards an entropy weak solution.
Proof for that \((2)+(3) \Rightarrow TV_T(Q^\Delta) \leq M:\)

\[
TV_T(Q^\Delta) = \sum_{n=0}^{T/\Delta t} \left[ \Delta t TV(Q^n) + \| Q^{n+1} - Q^n \|_1 \right]
\]

\[
\leq \sum_{n=0}^{T/\Delta t} [\Delta t M + \alpha \Delta t]
\]

\[
= \Delta t (M + \alpha)(T/\Delta t + 1)
\]

\[
= (T + \Delta t)(M + \alpha) \quad \text{(bounded)}
\]
Remarks:

(1) comes from maximum principle of exact solution for scalar conservation laws.

(2) is usually the hardest estimate. (For systems, (2) might be impossible to show.)

(3) usually is a consequence of (2).
Dafermos, C. M.: 

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x) \]

Assume: \( f : \mathbb{R} \mapsto \mathbb{R} \), locally Lipschitz, and \( \bar{u} \in L^1_{\text{loc}} \).
Kruzhkov’s entropy condition

\[ \int \int \left\{ |u - k| \phi_t + (f(u) - f(k))\text{sgn}(u - k)\phi_x \right\} \, dx \, dt \geq 0 \]

for every constant \( k \in \mathbb{R} \) and non-negative test function \( \phi \in C_c^1(\mathbb{R}^2) \) whose support is contained in the half plane where \( t > 0 \).

Main result: existence of entropy solution via front tracking approximations, which are piecewise constant approximate solutions.
Fix an $\varepsilon > 0$, let $u_j = j\varepsilon$ where $j$ is index (discrete of values).

Let $f^\varepsilon$ be the piecewise affine function which coincides with $f$ at all nodes $u_j$, i.e.,

$$f^\varepsilon(u) = \frac{u - u_j}{\varepsilon} f(u_{j+1}) + \frac{u_{j+1} - u}{\varepsilon} f(u_j) \quad u \in [u_j, u_{j+1}]$$

Approximate $\bar{u}$ with piecewise constant function $\bar{u}^\varepsilon$, taking values in the set

$$\mathcal{Z}^\varepsilon = \{u_j : u_j = i\varepsilon, j \in \mathbb{Z}\}$$
Theorem:

The Cauchy problem

\[ u_t + f^\epsilon(u)_x = 0, \quad u(0, x) = \bar{u}^\epsilon(x) \]

admits piecewise constant solutions \( u(t, x) \), satisfying entropy condition, taking values in the set \( \mathcal{Z}^\epsilon \).
Building block: Riemann problem

\[ u_t + f^\varepsilon(u)_x = 0, \quad u(0, x) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0. \end{cases} \quad u^-, u^+ \in \mathcal{Z}^\varepsilon. \]

Case 1: If \( u^- < u^+ \).

Let \( f_* \) be the biggest convex function such that \( f_*(u) \leq f^\varepsilon(u) \ \forall u \in [u^-, u^+] \).

Then:
\( f_* \) is piecewise affine,
\( (f_*)' \) is piecewise constant, non-decreasing, with jump points

\[ u^- = w_0 < w_1 < w_2 < \cdots < w_q = u^+ \]

where \( w_i \in \mathcal{Z}^\varepsilon \).
Shock speed:

\[ \lambda_\ell = \frac{f^\varepsilon(w_\ell) - f^\varepsilon(w_{\ell-1})}{w_\ell - w_{\ell-1}}, \quad \ell = 1, 2, \ldots, q. \]

Claim A:
The function

\[ w(t, x) = \begin{cases} 
  u^- = w_0, & x < t\lambda_1 \\
  w_\ell, & t\lambda_\ell < x < t\lambda_{\ell+1}, \quad (1 \leq \ell \leq q - 1) \\
  u^+ = w_q, & x > t\lambda_q 
\end{cases} \]

is a weak, entropy-admissible solution for the Riemann problem.

A fan shape, divides the domain into regions of constant state.
Proof for Claim A:

Let $\Omega_\ell$ be the wedge where $w = w_\ell = \text{const}$. Let $k$ be any constant and $\phi$ a test function.

By divergence Theorem:

$$\int_{\Omega_\ell} |w - k| \phi_t + (f^\varepsilon(w) - f^\varepsilon(k)) \text{sign}(w - k) \phi_x \, dx \, dt$$

$$= \int_{\Omega_\ell} [\langle |w - k| \phi \rangle_t + [(f^\varepsilon(w) - f^\varepsilon(k)) \text{sign}(w - k) \phi]_x] \, dx \, dt$$

$$= \int_{\Omega_\ell} \text{div} \left( |w - k| \phi , (f^\varepsilon(w) - f^\varepsilon(k)) \text{sign}(w - k) \phi \right) \, dx \, dt$$

$$= \int_{\partial\Omega_\ell} \left[ |w - k| \phi , (f^\varepsilon(w) - f^\varepsilon(k)) \text{sign}(w - k) \phi \right] \cdot \vec{n} \, ds$$
At $t \to \infty$, we have $\phi = 0$.

Only need line integral along shock lines.

Summing over $\ell$, we only need to integrate the difference from left and right states along rays with slope $\lambda_\ell$. 
\[
\int_0^\infty \int_R |w - k| \phi_t + \left( f^\varepsilon (w) - f^\varepsilon (k) \right) \text{sign}(w - k) \phi_x \, dx \, dt
\]

\[
= \sum_{\ell=1}^q \int_0^\infty \left\{ \left( |w_\ell - k| - |w_{\ell-1} - k| \right) \lambda_\ell - \left( f^\varepsilon (w_\ell) - f^\varepsilon (k) \right) \text{sign}(w_\ell - k) \right. \\
\left. + \left( f^\varepsilon (w_{\ell-1}) - f^\varepsilon (k) \right) \text{sign}(w_{\ell-1} - k) \right\} \phi(t, \lambda_\ell t) \, dt
\]

\[
= \sum_{\ell=1}^q \int_0^\infty (l) \, dt
\]
If $k \leq w_{\ell-1} < w_{\ell}$:

\[ I = (w_{\ell} - w_{\ell-1})\lambda_{\ell} - (f^\varepsilon(w_{\ell})) - f^\varepsilon(w_{\ell-1})) = 0 \]

If $k \geq w_{\ell} > w_{\ell-1}$: similar

\[ I = 0 \]

If $w_{\ell-1} < k < w_{\ell}$:

\[
I = (w_{\ell} + w_{\ell-1} - 2k)\lambda_{\ell} - [f^\varepsilon(w_{\ell}) + f^\varepsilon(w_{\ell-1}) - 2f^\varepsilon(k)] \\
= 2f^\varepsilon(k) - [f^\varepsilon(w_{\ell}) + (k - w_{\ell})\lambda_{\ell}] - [f^\varepsilon(w_{\ell-1}) + (k - w_{\ell-1})\lambda_{\ell}] \\
= 2f^\varepsilon(k) - f_*(k) - f_*(k) \geq 0
\]

Conclusion:

\[
\int_0^\infty \int_R |w - k| \phi_t + (f^\varepsilon(w) - f^\varepsilon(k)) \text{sign}(w - k) \phi_x \, dx \, dt \geq 0
\]
Case 2: \( u^+ < u^- \): treated similarly.

Let \( f^* \) be the smallest concave function such that \( f^*(u) \geq f^c(u) \ \forall u \in [u^+, u^-] \).

\( f^* \) piecewise affine.

\( (f^*)' \) piecewise constant, non-increasing, with jumps at

\[
\begin{align*}
u^+ &= w_0 < w_1 < \cdots < w_q = u^- 
\end{align*}
\]
One can show (leave as an exercise) that

\[ w(t, x) = \begin{cases} 
  u^- = w_q, & x < t\lambda_q \\
  w_\ell, & t\lambda_\ell < x < t\lambda_{\ell+1}, \quad (1 \leq \ell \leq q - 1) \\
  u^+ = w_0, & x > t\lambda_1
\end{cases} \]

is a weak, entropy-admissible solution for the Riemann problem.

As \( t > 0 \) grows, the fronts will interact. New Riemann problems are solved.
Front tracking in a Box

1. Given scalar conservation law $u_t + f(u)_x = 0$, $u(0, x) = \bar{u}$. (1)

2. Fix $\varepsilon > 0$. Approximate $f$ with piecewise affine function $f_\varepsilon$, where $f(u_j) = f_\varepsilon(u_j)$, $\forall u_j \in \mathcal{Z}^\varepsilon = \{\varepsilon j : j \in \mathbb{Z}\}$.

3. Approximate $\bar{u}$ by piecewise constant function $\bar{u}_\varepsilon$, taking values in $\mathcal{Z}^\varepsilon$.

4. Solve $u_t + f_\varepsilon(u)_x = 0$, $u(0, x) = \bar{u}_\varepsilon$ exactly. Denote the solution by $u_\varepsilon(t, x)$.

5. As $\varepsilon \to 0$ ($\bar{u}_\varepsilon \to \bar{u}$ and $f_\varepsilon \to f$), we have $u_\varepsilon \to u$ where $u$ is an entropy weak solution of (1).
Claim:
For a fixed $\varepsilon$, the total number of fronts remain finite, and the total variation $TV\{u(t, \cdot)\}$ is non-increasing in time.

Indeed, if $\bar{u}(\cdot)$ has bounded variation, then the initial number of fronts is finite.

It suffices to show that the total variation is non-increasing at every wave interaction.
Let $u_1, u_2, u_3$ be the states of the two interacting waves.

Total variation before interaction: $= |u_1 - u_2| + |u_2 - u_3|$

At the interaction, one solves a Riemann problem with $u_1, u_3$ as the left and right states.

Any Riemann problem is solved by waves taking intermediate values between $u^-$ and $u^+$. (i.e., Maximum principle)

Total variation after interaction $= |u_1 - u_3| \leq |u_1 - u_2| + |u_2 - u_3|$

Thus: $TV(u^\varepsilon(\cdot, t_2)) \leq TV(u^\varepsilon(\cdot, t_1)), \ \forall t_2 \geq t_1,$

and number of fronts at $t \leq \frac{TV(\bar{u}^\varepsilon)}{\varepsilon}$.
Front tracking solution for $u_t + (u^2(1 - u^2))_x = 0, u(0, x) = \sin(\pi x)$
Dimension splitting and front tracking

\[ u_t + f(u)_x + g(u)_y = 0 \]

Given \( \delta > 0 \). Let \( f_\delta, g_\delta \) be the piecewise linear continuous interpolations.

Let \( S_{f_\delta}^x u \) be the one-dimensional solution to

\[ u_t + f(u)_x = 0 \]

and \( S_{g_\delta}^y u \) be the one-dimensional solution to

\[ u_t + g(u)_y = 0 \]

Grid cell

\[ l_{ij} = \{(x, y)| i\Delta x \leq x \leq (i + 1)\Delta x, j\Delta y \leq y \leq (j + 1)\Delta y\}. \]

Projection operator \( \pi \):

\[ \pi u(x, y) = \frac{1}{\Delta x \Delta y} \int \int_{l_{ij}} u(x, y) \, dx \, dy \quad \text{for} \ (x, y) \in l_{ij}. \]
Set $u^0 = \pi u_0$.

Given $u^n$, one computes

$$u^{n+1/2} = \pi \circ S_{\Delta t}^{f_\delta, x} u^n$$

and then

$$u^{n+1} = \pi \circ S_{\Delta t}^{g_\delta, y} u^{n+1/2}$$

Collect the discretization parameters in $\eta = (\delta, \Delta x, \Delta y, \Delta t)$, and we define the approximate solution $u_\eta$ as

$$u_\eta(t) = \begin{cases} 
S_{2(t-t_n)}^{f_\delta, x} u^n & (t_n \leq t < t_{n+1/2}) \\
 u^{n+1/2} & t = t_{n+1/2} \\
S_{2(t-t_{n+1/2})}^{g_\delta, y} u^{n+1/2} & t_{n+1/2} \leq t < t_{n+1} \\
 u^{n+1} & t = t_{n+1}
\end{cases}$$
Dimension splitting

\[ u^n(0) \xrightarrow{\pi} n \rightarrow n + 1 \xrightarrow{\pi} u^n(\Delta t) \]

\[ u^{n+1/2}(\Delta t) \xrightarrow{S_{\Delta t}^{y,z,u}} u^{n+1/2}(0) \]
Nonlinear systems of conservation laws

The shallow water:

Water shapes its course according to the nature of the ground over which it flows.

*Sun Tzu, The Art of War (6th-5th century B.C.)*

\[ h(x, t): \text{height of water, } h \ll 1 \]
\[ \bar{\rho}: \text{density of water, constant (incompressible)} \]
\[ u(x, t): \text{velocity of water} \]
Conservation of mass

\[
(\bar{\rho}h)_t + (\bar{\rho}hu)_x = 0, \quad \rightarrow \quad h_t + (hu)_x = 0
\]

Conservation of momentum :

\[
(\bar{\rho}hu)_t + (\bar{\rho}hu^2 + P)_x = 0
\]

\(P\): pressure, determined by hydrostatic law

\[
P_y = \bar{\rho}g(h - y), \quad \rightarrow \quad P = \int_0^h \bar{\rho}g(h - y) \, dy = \frac{1}{2} \bar{\rho}gh^2
\]

so

\[
(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0
\]
One dimensional shallow water equations:

\[
\begin{bmatrix}
  h \\
  hu
\end{bmatrix}_t + \begin{bmatrix}
  hu \\
  hu^2 + \frac{1}{2}gh^2
\end{bmatrix}_x = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

notation:

\[
q = \begin{bmatrix}
  q_1 \\
  q_2
\end{bmatrix} = \begin{bmatrix}
  h \\
  hu
\end{bmatrix},
\]

\[
f(q) = \begin{bmatrix}
  f_1(q) \\
  f_2(q)
\end{bmatrix} = \begin{bmatrix}
  hu \\
  hu^2 + \frac{1}{2}gh^2
\end{bmatrix} = \begin{bmatrix}
  \frac{q_2^2}{q_1} + \frac{1}{2}gq_1^2
\end{bmatrix}
\]

so

\[
q_t + f(q)_x = 0
\]
Quasi-linear form $q_t + f'(q)q_x = 0$ where

$$f'(q) = \begin{bmatrix} 0 & 1 \\ -(q_2/q_1)^2 + gq_1 & 2q_2/q_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix}$$

Eigenvalues of $f'(q)$:

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh}, \quad \lambda_1 < \lambda_2$$

Right eigenvectors

$$r_1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}$$

Let $c_0 = \sqrt{gh}$. Then $\lambda_1, \lambda_2$ differ from water speed $u$ by $c_0$.

$c_0$ is larger for larger $h$. 
Shallow Water Equations: solution examples

$h$ at $t = 0$

$h$ at $t = 0.5$

$h$ at $t = 1$

$hu$ at $t = 0$

$hu$ at $t = 0.5$

$hu$ at $t = 1$
$h$ at $t = 2$

$hu$ at $t = 2$

$h$ at $t = 3$

$hu$ at $t = 3$
Shallow Water Equations: Radial solution in 2D
Riemann Problem example 1: Dam break

\[ h(x,0) = \begin{cases} 
  h_l, & x < 0, \\
  h_r, & x > 0.
\end{cases} \quad u(x,0) = 0, \quad h_l > h_r > 0. \]
Characteristics:

1-characteristics in the $x$-$t$ plane

2-characteristics in the $x$-$t$ plane
Riemann problem example 2: a Two Shock Riemann Problem

\[ h(x, 0) \equiv h_0 > 0, \quad u(x, 0) = \begin{cases} u_0, & x < 0, \\ -u_0, & x > 0. \end{cases} \]
Characteristics:

1–characteristics in the $x$–$t$ plane

2–characteristics in the $x$–$t$ plane