2.2. Calculus of Complex Functions.

2.2.1. Differentiation.

We define the derivative $f'(z)$ of a complex valued function $f(z)$ like the derivative of a real function:

$$f'(z) = \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z}$$

where the limit is over all possible ways of approaching z. If the limit exists, the function f is called *differentiable* and $f'(z)$ is the derivative.

Definition. If $f'(z)$ is continuous, then f is called *analytic*.

Continuity is defined like that for real functions of two variables.

Theorem 2.1 (Cauchy-Riemann conditions) The function $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$ is analytic in some region Ω if and only if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist, are continuous, and satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Proof. Let f be continuously differentiable. Then take the special path along x-axis:

$$\begin{aligned} \tfrac{f(z+\Delta x)-f(z)}{\Delta x} &= \tfrac{u(x+\Delta x,y)+iv(x+\Delta x,y)-u(x,y)-iv(x,y)}{\Delta x} = \tfrac{\Delta u}{\Delta x} + i\tfrac{\Delta v}{\Delta x} \\ &\longrightarrow \tfrac{\partial u}{\partial x} + i\tfrac{\partial v}{\partial x}. \end{aligned} \tag{1}$$

Then along the path y-axis:

$$\tfrac{f(x+iy+i\Delta y)-f(z)}{i\Delta y} \longrightarrow \tfrac{1}{i}\tfrac{\partial f}{\partial y} = -i\tfrac{\partial f}{\partial y} = -i\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial y}. \tag{2}$$

The two limits have to be the same by definition, so we have obtained the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Conversely, suppose the Cauchy-Riemann conditions hold; i.e., the existence and continuity of the partial derivatives and the equations of Cauchy-Riemann all hold. Let $z_0 = x_0 + iy_0$. From theory of real variables we have the expansion

$$\begin{aligned} u(x, y) &= u(x_0, y_0) + \tfrac{\partial u}{\partial x}(x_0, y_0)\Delta x + \tfrac{\partial u}{\partial y}(x_0, y_0)\Delta y + R_1(\Delta x, \Delta y), \\ v(x, y) &= v(x_0, y_0) + \tfrac{\partial v}{\partial x}(x_0, y_0)\Delta x + \tfrac{\partial v}{\partial y}(x_0, y_0)\Delta y + R_2(\Delta x, \Delta y), \end{aligned} \tag{3}$$

where $\Delta x = x - x_0, \Delta y = y - y_0$, and

$$\lim_{\Delta x, \Delta y \to 0} \frac{R_i}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

Now we have

$$\begin{aligned} f(z_0+\Delta z)-f(z_0) &= \tfrac{\partial u}{\partial x}(x_0,y_0)\Delta x + \tfrac{\partial u}{\partial y}(x_0,y_0)\Delta y + R_1 \\ &\quad +i[\tfrac{\partial v}{\partial x}(x_0,y_0)\Delta x + \tfrac{\partial v}{\partial y}(x_0,y_0)\Delta y + R_2] \\ &= \tfrac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\tfrac{\partial v}{\partial x}(\Delta x + i\Delta y) + R_1 + R_2 i. \end{aligned} \tag{4}$$

So

$$\begin{aligned} \tfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z} &= \tfrac{\partial u}{\partial x} + i\tfrac{\partial v}{\partial x} + \tfrac{R_1+R_2 i}{\Delta z} \\ &\to \tfrac{\partial u}{\partial x} + i\tfrac{\partial v}{\partial x}. \end{aligned} \tag{5}$$

This completes the proof.

We list some practical rules of differentiation:

$$\begin{array}{ll} f(z)=z^2 \longrightarrow & f'(z)=2z \\ f(z)=z^k \longrightarrow & f'(z)=kz^{k-1}(k \text{ integer }) \\ (e^z)'=e^z & \\ (f(z)g(z))'=f'(z)g(z)+f(z)g'(z) & (f(z)+g(z))'=f'(z)+g'(z) \\ [F(g(z))]'=F'(g(z))g'(z) & (cf(z))'=cf'(z)\ (c: \text{ constant }) \\ (\frac{1}{f})'=-\frac{1}{f^2}f'. & \end{array} \tag{6}$$

2.2.2. Integration.

Integration in the complex plane is defined in terms of real line integrals of the complex function $f=u+iv$. If C is any (geometric) curve in the complex plane we define the line integral

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C u(x,y)dx - v(x,y)dy + i\int_C vdx+udy.$$

Example 1. Find $\int_C zdz$, where $C=\{(x,y)|x=0, 0<y<1\}$ oriented upward.

Solution. Use parametrization $x=0, y=s,\ s\in(0,1)$. Then

$$\int_C zdz = \int_0^1 (0+iy)(dx+idy) = \int_0^1 iyidy = -1/2.$$

Theorem 2.2. If $f(z)$ is analytic in a domain Ω, then

$$\int_C f(z)dz = 0$$

for any closed curve C whose interior lies entirely in Ω.

Note that "a curve C whose interior lies entirely in Ω" is a stronger requirement than "a curve C which lies entirely in Ω". The stronger requirement rules out the situation that the relevant part of Ω is not simply connected.

Proof. Recall Green's Theorem

$$\int_{\partial\Omega} \phi dx + \psi dy = \int_{\Omega} (\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}) dxdy$$

for a simply connected domain Ω. We apply this formula to our complex integral to obtain

$$\begin{aligned} \int_C f(z)dz &= \int_C (u+iv)(dx+idy) \\ &= \int_C u(x,y)dx - v(x,y)dy + i\int_C vdx + udy \\ &= \int_{cC} (\tfrac{\partial}{\partial x}(-v) - \tfrac{\partial}{\partial y}u)dxdy + i\int_{cC} \tfrac{\partial}{\partial x}u - \tfrac{\partial}{\partial y}v)dxdy \\ &= 0 \end{aligned} \tag{7}$$

where we use cC to denote the interior of the contour C. This completes the proof.

Examples. 2. We have

$$\int_C z^n dz = 0$$

for any integer n and any contour C that does not enclose the origin. This follows from Theorem 2.2.

3. We can calculate

$$\int_{|z|=1} z^{-1} dz = \int_0^{2\pi} 1^{-1} e^{-i\theta} \cdot 1 e^{i\theta} i d\theta = 2\pi i.$$

4. We leave as an exercise the claim

$$\int_{|z|=1} z^{-n} dz = 0$$

for all integer $n \neq 1$.

We note that the notation $|z| = 1$ means all points of the unit circle $x^2 + y^2 = 1$. The default direction of the circle is counterclockwise.

==End of Lecture 16====