3 Voting and agenda manipulation

This section concerns organizations in which decisions are made by a single round or by successive rounds of majority voting. In a case that is of particular interest with respect to organizations as well as to political competition, one member is a “leader,” who has the exclusive right to propose alternatives to the status quo. It turns out that, in the long run, this right can make the leader extremely powerful. This power that agenda manipulation confers on the leader was first recognized by Robert Michel, who described it in Political Parties; a Sociological Study of the Oligarchical Tendencies of Modern Democracy (1911).\(^1\)

3.1 Single-peaked preferences and one-dimensional voting

Electoral competition is often described as a contest between a “right-wing” and a “left-wing” candidate. The idea is that, although there may be a welter of specific issues on which candidates take various positions, their respective positions on those issues can be aggregated to locate each candidate somewhere on a one-dimensional “political spectrum,” and that each voter supports the candidate closest to his or her most preferred point on that spectrum.

A pair of definitions formalize the notion of a political (or, more generally, a voting) spectrum.

**Definition 8.** Relation \(\ll\) is a strict total ordering of the set \(X\) of alternatives if \(\ll\) is asymmetric and transitive and, for every pair of distinct alternatives \(x \neq y\), either \(x \ll y\) or else \(y \ll x\).

**Definition 9.** Strict preference relation \(<\) is single peaked (with respect to strict total ordering \(\ll\)) if there is some alternative \(x\) such that, for all alternatives \(y\) and \(z\), (a) if \(x = y\) \(\ldots\)

\(^1\)Available without charge at http://etext.lib.virginia.edu/toc/modeng/public/MicPoli.html.
or $x \ll y$ and if also $y \ll z$, then $y \prec z$; and (b) if $x = z$ or $z \ll x$ and if also $y \ll z$, then $y \prec z$. The alternative $x$ that satisfies these conditions is the \textit{ideal point} of $\prec$.\footnote{It can easily be seen that the ideal point of a single-peaked preference relation is unique.}

\textbf{Definition 10.} A set consisting of a single alternative, or a set consisting of two \textit{endpoint} alternatives and all the alternatives between them, is an \textit{interval}. The interval with endpoints $x \ll y$ is denoted by $[x,y]$, and the interval consisting of a single alternative $x$ is denoted by $[x]$ or by $[x,x]$. When the notation $[x,y]$ is used, it is implicit that $x = y$ or $x \ll y$. Formally, $x$ and $y$ are the endpoints of the interval $[x,y] = \{x,y\} \cup \{z \mid x \ll z \ll y\}.\footnote{This definition corresponds to the definition of a \textit{nonempty}, \textit{closed}, \textit{bounded interval} in the mathematical theory of orderings.}

\textbf{Definition 11.} The environment specified by $X$, $\ll$, $S$, $I$, and a strict preference relation $\prec$, for each agent $i \in I$ is a \textit{voting spectrum} if $I$ has an odd number higher than 2 (let’s say, specifically, $2n+1$, where $n \geq 1$) of members, $\ll$ is a strict total ordering, every situation is an interval, each $\prec_i$ is single peaked, and no two members have the same ideal point. The agent whose ideal point is larger (with respect to $\ll$) than that of $n$ other agents, and smaller than that of the remaining $n$ agents, is the \textit{median agent}.\footnote{I am coining the term \textit{voting spectrum} here—it is not standard terminology. Voting models in which there are one or more orderings on the space of alternatives are called \textit{spatial models} by political scientists, and if there is a single, strict, total order, then the spatial model is called \textit{one dimensional}. The standard terminology for a median agent is \textit{median voter}.}

\section{The Condorcet winner; Black’s theorem}

\textbf{Definition 12.} For $x$ and $y$ in $X$, let $b(x,y)$ be the set of agents $i$ for whom $y \prec_i x$. (The letter $b$ stands for \textit{ballot}.) Let $\bar{b}(x,y)$ be the number of agents in $b(x,y)$.

\textbf{Definition 13.} An alternative $x$ [that belongs to a situation $S \subseteq X$] is the \textit{Condorcet winner} \textit{from} $S$ if, for every other $y$ [in $S$], $\bar{b}(x,y) > n$ That is, the Condorcet winner is a majority winner against any alternative that might be proposed against it [in the situation].\footnote{When the bracketed phrases are deleted, the notion of a Condorcet winner in the entire set of alternatives is defined. When it is clear that I am referring to the Condorcet winner from a situation $S$, rather than from $I$, I will sometimes simply write “the Condorcet winner” rather than “the Condorcet winner from $S$,” leaving it to the reader to fill in the implicit “from $S$.”}

\textbf{Lemma 7.} In a voting spectrum, the ideal point of the median agent is the unique Condorcet winner from $X$

\textit{Proof.} Let $X = \{1, \ldots, 2n+1\}$, where the agents are ordered by their ideal points. That is, letting $\hat{x}_i$ be the ideal point of agent $i$, $\hat{x}_i \ll \hat{x}_{i+1}$. Thus the median agent’s ideal point is $\hat{x}_{n+1}$. If $y \neq \hat{x}_{n+1}$, then either $y \ll \hat{x}_{n+1}$ or else $\hat{x}_{n+1} \ll y$.

To prove that $\bar{b}(\hat{x}_{n+1},y) > n$, consider the case that $y \ll \hat{x}_{n+1}$. The proof of the opposite case is parallel. Since $y \ll \hat{x}_{n+1} = \hat{x}_{n+1}$, $y \prec_{n+1} \hat{x}_{n+1}$. For the $n$ agents $i \in \{n+2, \ldots, 2n+1\}$, $y \ll \hat{x}_{n+1} \ll \hat{x}_i$, so $y \ll \hat{x}_{n+1}$. Thus $\{n+1, \ldots, 2n+1\} \subseteq b(\hat{x}_{n+1},y)$, so $\bar{b}(\hat{x}_{n+1},y) > n$. \hfill $\square$
Proposition 3. From every situation in a voting spectrum, there is a unique Condorcet winner. In situation \([w, z]\), the Condorcet winner is \(\hat{x}_{n+1}\) if \(\hat{x}_{n+1} \in [w,]\), \(w\) if \(\hat{x}_{n+1} \ll w\), and \(z\) if \(z \ll \hat{x}_{n+1}\). That is, the Condorcet winner from \([w, z]\) is the alternative in the interval that is “closest” to \(\hat{x}_{n+1}\), in the sense that it has the minimal set of alternatives ranked between it and \(\hat{x}_{n+1}\) (in either direction), in \([w, z]\). In all of these cases, the Condorcet winner from \([w, z]\) is the alternative in \([w, z]\) that is ranked highest according to \(\prec_i\).

Proof. Suppose that there were two distinct Condorcet winners, \(x\) and \(y\), from some situation. Then \(\bar{b}(x, y) \geq n+1\) and \(\bar{b}(y, x) \geq n+1\) and, since \(\ll\) is asymmetric, \(b(x, y) \cup b(y, x)\) is empty. These three statements imply that \(I\) must have at least \(2n+2\) members, contradicting the assumption that it has only \(2n+1\) members.

To show that there is at least one Condorcet winner from a situation, consider situation \([w, z]\). Since \(\ll\) is total, either \(\hat{x}_{n+1} \ll w\) or \(\hat{x}_{n+1} \in [w, z]\) or \(z \ll \hat{x}_{n+1}\). If \(\hat{x}_{n+1} \in [w, z]\), then \(\hat{x}_{n+1}\) is the Condorcet winner in \([w, z]\), by lemma 7. (A Condorcet winner from a set of alternatives, that is in a subset, is clearly a Condorcet winner from the subset.) Otherwise, suppose that \(\hat{x}_{n+1} \ll w\). (The case that \(z \ll \hat{x}_{n+1}\) has a parallel proof.) Since \(n+1\) is the median agent, and by transitivity of \(\ll\), \(\hat{x}_i \ll w\) for every \(i \leq n+1\). Let \(y\) be any alternative except \(w\) in \([w, z]\). Then, since \(\prec_i\) is single peaked, \(y \prec_i w\) for every \(i \leq n+1\). Therefore \(\{1, \ldots, n+1\} \subseteq b(w, y)\) and \(n+1 \leq b(w, y)\). That is, \(w\) is a Condorcet winner in \([w, z]\). \(\square\)

Corollary 1. If \(D\) is the decision rule that assigns, to each situation of a voting spectrum, the Condorcet winners from that situation, then \(\prec_n\) rationalizes \(D\), so \(D\) is not certifiably non dictatorial.

Proof. Immediate from proposition 3 and the definitions of rationalizability and certifiable non dictatorship.

4 The relevance of market structure

In section 4.1, a simple model of an investment partnership is shown to exhibit rotation of preferences. Thus, rotation of preferences cannot be regarded as an artificial problem. It is an issue for organizations making economic decisions, the members of which have preferences that satisfy the restrictions imposed in microeconomic theory. In view of this analysis (and proposition 2), the standard microeconomic representation of the firm (in this case, the investment partnership) as a single agent whose decisions can be rationalized by a representative preference relation seems questionable.

In section 4.2, this example is re-examined, under the assumption that partners have access to an economy-wide market for temporal reallocation of consumption (i.e., to a bond market) and to an insurance market. The upshot is that, given the partners’ ability to

\[6\text{This result is due to Duncan Black [CITE].}\]
trade investment proceeds for different, individually tailored, patterns of consumption, they become unanimous regarding what investment the partnership should make. Thus, in a complete-markets environment, it is justified to model the firm as a single, rational agent.

4.1 Investment choice by an autarkic partnership

Example 7. Consider 3 partners \(I = \{1, 2, 3\}\) who, in various situations, are presented with three alternative investments \(X = \{x, y, z\}\). Each alternative has a certain immediate payoff but may have a risky future payoff. Assume that each investor receives the payoff. Suppose that investment \(y\) is riskless both now and in the future, but that the future payoffs of investments \(x\) and \(z\) depend on whether a good or a bad event will occur. The bad and good events are equally probable—each will occur with probability \(1/2\).

Consider the specific payoffs in this table:

<table>
<thead>
<tr>
<th></th>
<th>Now</th>
<th>Bad</th>
<th>Good</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>.2</td>
<td>0</td>
<td>.4</td>
</tr>
<tr>
<td>(y)</td>
<td>0</td>
<td>.55</td>
<td>.55</td>
</tr>
<tr>
<td>(z)</td>
<td>.1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Furthermore, let’s suppose that the investors have different utility functions. Consider an investment \(w\) that has current payoff \(n\), future payoff \(b\) in the bad event, and future payment \(g\) in the good event. Let \(u_i(w)\) be the utility level that this investment provides to investor \(i\). The following table specifies the three investor’s utility levels. Its intuitive interpretation is that investor 1 is completely patient and risk neutral, investor 2 is risk neutral but extremely impatient, and investor 3 is patient but risk averse toward future consumption.\(^7\)

<table>
<thead>
<tr>
<th>Investor (i)</th>
<th>Utility (u_i(w))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(n + [b + g]/2)</td>
</tr>
<tr>
<td>2</td>
<td>(n + [(b/10) + (g/10)]/2)</td>
</tr>
<tr>
<td>3</td>
<td>(n + [(b - b^2/2) + (g - g^2/2)]/2)</td>
</tr>
</tbody>
</table>

These assumptions about payoffs and utility functions imply the following utilities of choices to investors.\(^8\)

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\(^7\)Investors are typically assumed to prefer more of a good to less. That is, utility functions are increasing in each good. Risk neutrality is represented by making utility proportional to consumption, and aversion to risk in consumption of a good is represented by strict concavity of the utility of that good. Investor 3’s utility depends on future consumption in each event in a strictly concave way. Moreover, the contribution of consumption of each of those goods to 3’s utility is increasing, until the point where 1 unit of the good has been consumed. Consumption in this example does not exceed that level.

\(^8\)\(u_3(y)\) has been rounded.
These utility levels imply preference rankings, as follows

Investor $i$'s preference ranking
1. $x \prec_1 y \prec_1 z$
2. $y \prec_2 z \prec_2 x$
3. $z \prec_3 x \prec_3 y$

This is precisely the rotation-of-preferences pattern that example 5 exhibits. Example 5 shows that, if there are situations of choice between pairs of investments

$\{\{x, y\}, \{y, z\}, \{x, z\}\} \subseteq \mathcal{S}$,

then majority-rule decisions cannot be rationalized. Proposition 1 entails that, if $X \in \mathcal{S}$, then no certifiably non-dictatorial decision rule can be rationalized by a representative preference relation.

4.2 The effect of complete markets.

The preceding example has assumed that partners cannot trade current and future proceeds, or proceeds in different future events, with one another. It has also been assumed, implicitly, that they cannot trade with anyone else. Now, let’s change that second assumption. Let’s suppose that there are a bond market and in insurance market. In the bond market, a person can acquire 1 unit of consumption today in return for paying $R$ units of consumption at the future date, regardless of whether the bad or the good event occurs then. In the insurance market, a person can acquire 1 unit of future consumption in the bad event in return for paying $P$ units of future consumption in the good event.

The individual can trade any quantities of bonds and insurance, and is not restricted to trade 1 unit or whole number of units, and be a buyer rather than a seller. $R$ is the gross interest rate. $P$ is related to an insurance premium.\(^9\)

The 3 investors whom we explicitly model have no market power in these economy-wide financial markets, so each of them behaves as a competitive price taker with respect to $R$ and $P$.

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\(^9\)An insurance premium is paid beforehand. That is, an insured person acquires 1 unit of future consumption in the bad event in return for paying $p$ units of current consumption, where $p$ is the \textit{premium}. (Typically $p < 1/R$ because the premium must be paid with certainty now, while the payment in the future bad event—the \textit{indemnity}—is not always paid (because the bad event does not always occur), and is paid at a future date to which a discount factor is applied.) Suppose that a person buys $q$ units of insurance and also sells $pq$ bonds. Then, consumption now is unaffected because the revenue from the bond sale exactly cancels the insurance premium. Then, in the bad event, the person’s consumption changes by $q - pqR$ as a consequence of the transactions; and in the good event, the person’s consumption changes by $-pqR$. Let $q = 1/(1 - pR)$, so that the person’s consumption increases by 1 unit in the bad event. Then $P = pR/(1 - pR)$. 

5
Consider the investor’s budget constraint, which involves 3 goods. Denote the goods by \( N \) (consumption now), \( B \) (future consumption in the bad event), and \( G \) (future consumption in the good event). A unit \( B \) costs \( P \) units of \( G \). A unit of \( N \) costs \( R \) units of \( B \) and \( R \) units of \( G \). If someone endowed only with \( G \) wanted to purchase a unit of \( N \), he or she would first have to purchase \( R \) units of \( B \) with \( RP \) units of \( G \), and then would purchase the unit of \( N \) with the \( R \) units of \( B \) thus acquired, together with a further \( R \) units of \( G \). Thus, the price of \( N \) in terms of \( G \) is \( R(1 + P) \). Taking \( G \) to be the numéraire good, and taking \( R(1 + P) \) and \( P \) to be the prices of \( N \) and \( B \), the value of a bundle \((n, b, g)\) of the goods is \( Rn + R(1 + P)b + g \). An investor can trade \((n, b, g)\) for another bundle, \((n', b', g')\), if \( Rn' + R(1 + P)b' + g' \leq Rn + R(1 + P)b + g \).

Thus, in the presence of financial markets, an investor cares about which alternative is chosen because the choice determines his budget constraint. All investors have wider scope for their respective consumption choices, the higher is \( Rn + R(1 + P)b + g \). That is, the investors unanimously prefer the project with highest value at market prices. Since there is unanimity in the complete-markets regime, rotation of preferences does not occur in this regime.