The St. Petersburg Paradox and the Quantification of Irrational Exuberance [a]

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Early 1700's: Nikolaus Bernoulli formulated a problem now called the St. Petersburg paradox.

1713: Bernoulli stated the problem in a letter to Réymond de Montmort.

1738: Daniel Bernoulli presented the problem to the Imperial Academy of Sciences in St. Petersburg, Russia.

Fair games of chance

A game of chance is called fair if each player's expected net profit is zero.

In a fair game, neither the player nor the casino has a long-run advantage over the other.

Feller (1945): This definition of "fair game" requires that the player's net profit have finite variance.

This assumption is necessary for technical reasons.

We shall proceed according to this definition even if the variance of the player's profit is not finite.

The St. Petersburg paradox

Peter tosses a fair coin repeatedly until it shows heads and agrees to pay Paul:

\$2 if a head appears on toss 1

\$4 if the first head appears on toss 2

\$8 if the first head appears on toss 3

⋮

$\$2^k$ if the first head appears on toss k

What is Paul's expected revenue? Or, what entrance fee should Peter charge Paul to make the game fair?

Peter $\equiv$ casino,   Paul $\equiv$ gambler

Answer: No finite entrance fee can make the game fair.

Y: Peter's payout to Paul

Y is a discrete random variable with probability distribution

Values of Y	2	4	8	$\cdots$
$P(Y = y)$	1/2	1/4	1/8	$\cdots$

$$E(Y) = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \cdots = \infty$$

Conclusion: Peter should charge Paul an entrance fee of \$$\infty$.

This is paradoxical: No rational Paul is likely to agree to an infinite entrance fee, or to have that much money.

Paul is certain to win only a finite amount, so an infinite entrance fee seems unfair.

Paul is almost certain to receive a small payment:

$$P(\text{Paul receives at most } \$32) = \frac{31}{32} = 97.875\%$$

Even a large, finite entrance fee seems unfair.

How can the paradox be resolved?

Proposed resolutions of the paradox

Nikolaus Bernoulli placed zero probabilities on large values of k. He felt that such values were "morally certain" not to occur so that, from a practical standpoint, little is lost by truncating the series for $E(Y)$.

If $k = 25$ is morally certain not to occur then

$$E(Y) \simeq 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + \cdots + 2^{24} \frac{1}{2^{24}} = 24$$

Bernoulli did not find his argument convincing, so he consulted with Montmort, Buffon, Gabriel Cramer, and others.

Cramer argued that additional amounts of money are possibly meaningless after some point, say, \$$2^{24}$ (\$16,777,216). Then \$$2^{24}$, \$$2^{25}$, ... have the same value to Paul.

The expected payout becomes

$$\sum_{k=1}^{24} 2^k \cdot \frac{1}{2^k} + \sum_{k=25}^{\infty} 2^{24} \cdot \frac{1}{2^k} = 25.$$

Paul should pay an entrance fee of at most \$25.

Nikolaus was dissatisfied with Cramer's solution so he wrote to Daniel, then a member of the Imperial Academy of Sciences in St. Petersburg. Daniel responded in 1731 with a draft of his famous paper, published later in 1738.

D. Bernoulli, “Specimen theoriae novae de mensura sortis,” *Commentarii Academiae Scientiarum Imperialis Petropolitanea*, **V** (1738), 175–192. (Transl. and republ., “Exposition of a new theory on the measurement of risk,” *Econometrica*, **22** (1954), 23–36.)

Bernoulli argued that money should be valued in proportion to its “marginal utility,” the additional use or pleasure one can obtain from extra amounts of money.

“. . . any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed.”

$$\Delta U \propto \frac{\Delta w}{w}$$

Bernoulli used geometrical arguments to obtain the solution, $U \propto \log w$, and concluded that the marginal utility of $\$n$ should be measured by $\log n$.

The expected marginal utility of Peter's payout is

$$\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \cdots = \sum_{k=1}^{\infty} \frac{\log 2^k}{2^k} = \log 4.$$

This corresponds to a payout of $\$4$.

Paul, a rational gambler measuring the use of money through marginal utility, would pay at most \$4 for an entrance fee.

Buffon had a child play the St. Petersburg game 2,048 times:

Tosses (k)	Frequency	Payoff (2^k)
1	1061	2
2	494	4
3	232	8
4	137	16
5	56	32
6	29	64
7	25	128
8	8	256
9	6	512

Buffon concluded that, in practice, the St. Petersburg game becomes fair with an entrance fee of approximately $10.

Whitworth (1901) considered the chance of gambler's ruin and argued that, in repeated plays a rational gambler should risk a percentage, rather than a fixed amount, of his remaining capital.

Whitworth assumed that Paul receives 2^{k-1} "florins" if the first head appears on the kth toss.

If Paul has n florins then Whitworth gave an expected-value argument to show that a fair entrance fee is, approximately

$$\frac{\frac{4}{n}\left(1+n\right)^{1/2}\left(1+\frac{n}{2}\right)^{1/4}\left(1+\frac{n}{4}\right)^{1/8}\left(1+\frac{n}{8}\right)^{1/16}\cdots-2}{\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{4(n+4)}+\frac{1}{8(n+8)}+\cdots}$$

This formula is extremely conservative

n florins	8	32	1024
Entrance fee	3.796	4.025	6.176

Whitworth saw his formula as a satisfactory resolution of the St. Petersburg paradox.

“We have not assigned any new value to the mathematical expectation [in the St. Petersburg paradox]; . . . We have simply determined the terms at which a man may purchase a contingent prospect of advantage, so that by repeating the operation – *each time on a scale proportionate to his funds at that time* – he may be left neither richer nor poorer when each issue of the venture shall have occurred its own average number of times.”

Kelly (1956) developed another formula for determining how much gamblers should risk on each venture.

Whitworth (1901) also gave a formula for the fair entrance fee to a generalized reverse St. Petersburg game.

Paul has n florins and opts to play a generalized St. Petersburg game having outcomes $O_1, O_2, \ldots$ with probabilities $p_1, p_2, \ldots$, respectively, where $p_1 + p_2 + \ldots = 1$. In the event of outcome O_k, Paul must pay P_k florins. Then a fair entrance fee is

$$\frac{1 - \left(1 - \frac{P_1}{n}\right)^{p_1} \left(1 - \frac{P_2}{n}\right)^{p_2} \left(1 - \frac{P_3}{n}\right)^{p_3} \cdots}{\frac{p_1}{n-P_1} + \frac{p_2}{n-P_2} + \frac{p_3}{n-P_3} + \cdots}$$

Steinhaus (1944), Cšorgö and Simons (1993)

2 4 2 8 2 4 2 16 2 4 2 8 2 4 2 32 2 4 ...

Denote this sequence of entrance fees by $a_1, a_2, \ldots$

Steinhaus proved that, with probability 1, the sequence of Paul's gains is the same as the distribution function of the sequence a_n

More recent games

1970: Cleveland Indians, major league baseball team

Ken Harrelson played with the Indians from 1969-1971

1970: Harrelson offered a deal to Gabe Paul, the Indians' GM.

Harrelson offered to play the entire season without salary "except for 50 cents doubled for every HR he hit." Harrelson would get 50 cents if he hit only 1 HR, \$1 if he hit 2 HR, etc.

Harrelson hit 30 HRs that season (note: \$$2^{28}=$268,435,456)

A recent TV show, "Who Wants to be a Millionaire?" is a modified St. Petersburg game.

I was amazed to see players push their luck beyond \$10,000.

The valuation of growth stocks

Growth company: One whose revenues and earnings are growing significantly faster than the overall economy.

Fall 1999: Almost everyone seemed to buying growth stocks.

12/15/96: Alan Greenspan asked,

> “But how do we know when irrational exuberance has unduly escalated asset values, which then become subject to unexpected and prolonged contractions as they have in Japan over the past decade?”

We need a quantitative measure of irrational exuberance.

11/19/99: The Wall Street Journal reported that 59 mutual funds had increased by more than 100% since Jan. 1.

Nicholas-Applegate Global Technology Fund: Up by 325%

The Wall Street Journal was unsure if Nicholas-Applegate investors would think that 1% per day was sufficient.

Hot tech stocks: Akamai, Altera, Broadvision, Ciena, Citrix, CMGI, Daleen, DoubleClick, Excite@Home, Go2Net, I2, Inktomi, JDS Uniphase, Liberte, Lucent, Network Solutions, Portal Software, Triquint, Verisign, Veritas, Xilinx.

David Durand, "Growth stocks and the Petersburg paradox," The Journal of Finance, 12 (1957), 348–363.

Classical St. Petersburg game:

Y-values	2	4	8	$\cdots$
$P(Y=y)$	1/2	1/4	1/8	$\cdots$

Modified game: $P(\text{Heads}) = 1/(1+i), \quad P(\text{Tails}) = i/(1+i)$

Peter: A growth company

Paul: A buyer of stock in Peter

g: The compound annual growth rate of Peter's revenues

Peter pays dividends annually to Paul, as revenues and dividends increase each year at rate g

Outcomes	Prob.	Dividends	Cumulative Dividends
H	$i/(1+i)$	0	0
TH	$i/(1+i)^2$	c	c
TTH	$i/(1+i)^2$	$c(1+g)$	$c+c(1+g)$
TTTH	$i/(1+i)^4$	$c(1+g)^2$	$c+c(1+g)+c(1+g)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Paul buys shares and holds them forever; his expected cumulative dividends is

$$0\cdot\frac{i}{1+i}+c\cdot\frac{i}{(1+i)^2}+[c+c(1+g)]\cdot\frac{i}{(1+i)^3}+\cdots$$

By rearranging this series, we get

$$\frac{c}{1+i} + \frac{c(1+g)}{(1+i)^2} + \frac{c(1+g)^2}{(1+i)^3} + \cdots = \begin{cases} c/(i-g), & g < i \\ \infty, & g \geq i \end{cases}$$

Stocks are valued intrinsically as the perpetual series of dividends they pay, starting at c, growing at rate g, and discounted at rate i.

i: A discount rate

If the company pays no dividends and retains all earnings, a similar argument based on "book value" leads to

$$\text{Value of stock} = \begin{cases} \text{const.}/(i-g), & g < i \\ \infty, & g \geq i \end{cases}$$

Conclusion: If $g \geq i$ then Paul should be willing to pay $\$\infty$ for shares of Peter.

A minor caveat: Astronomers report that the Sun will consume the Earth in less than 5 billion years. So it seems unlikely that Google's dividends or retained earnings will grow in perpetuity.

Fall, 1999

The Federal Reserve Bank had lowered discount rates, so i was small.

Wall Street analysts estimated high growth everywhere, so many tech stocks sported high g's.

The outcome: $i < g$.

Many tech stocks even had $i/g \simeq 0$.

Analysts estimated nearly infinite intrinsic values for many tech stocks by using standard discount valuation formulas.

No price was too high for Yahoo!, JDS Uniphase, etc.

Conclusion

Walter Bagehot: “Life is a school of probability.”

Mark Twain noted, for the benefit of stock market speculators, that

> “There are two times in a man’s life when he should not speculate: when he can’t afford it, and when he can.”

Twain’s “Pudd’nhead Wilson’s Calendar” commented,

> “October. This is one of the peculiarly dangerous months to speculate in stocks in. The others are July, January, September, April, November, May, March, June, December, August and February.”