Exam next W: 1/28
1.1-1.10, 2.1-2.3 (today)

- review sheet posted
- practice exams

HW sets for this week will be posted early

Link to old exams: ignore determinants, LU decomposition
will post one with only stuff we did

- M/T go over exam.
How can you tell if a matrix is invertible?

2×2: check if \( ad-bc \neq 0 \) → if yes, not invertible

\[ A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} a & -b \\ -c & a \end{pmatrix} \]

if no, invertible, and

Bigger matrices: it's more complicated.

13 ways to tell. (§2.3)
For a given \( n \times n \) matrix \( A \), all of the following are all true, or all are false:

1) \( A \) is invertible

2) \( A \) is row-equivalent to \( I_n \), the identity 
   (row reduction on \( A \) leads to \( I_n \))

3) \( A \) has \( n \) pivots

4) \( A\hat{x} = \hat{0} \) has only the solution \( \hat{x} = \hat{0} \).

5) columns are linearly independent

6) the transformation determined by \( A \) is one-to-one

7) \( A\hat{x} = \hat{b} \) has at least one solution for every \( \hat{b} \)

8) columns of \( A \) span \( \mathbb{R}^n \)

9) the transformation given by \( A \) is onto
10) there is an \( nxn \) \( C \) so \( CA = In \)

11) there is an \( nxn \) \( D \) so \( AD = In \).

12) \( A^T \) is invertible

\[
13) \det A \neq 0 \quad (\text{we only know this for } 2 \times 2)\]

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- Understanding why these are the same is good
  review: how do these fit together?

- Some tests will be easier for certain matrices.

\[
\text{\textbullet\quad if you know } A \text{ is invertible (check by row reduction)
  you know all these other things too.}\]

\[
\text{\textbullet\quad This is the most important: for example, it tells you that if
  the transformation } A \text{ is one-to-one, it is onto!}
  (\text{THIS IS SPECIFIC TO SQUARE MATRICES})
\]
We won't prove it.

Let's just think through an example and see that everything makes sense.

\[ A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{ad-bc=0} \]

1) Not invertible!

\[ \text{ad-bc} = (1)(4)-(2)(2) = 0 \]

[ didn't go through this in lecture, but maybe it's useful ]

2) \( \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \). Can't get to I!

Remember, if you could,

\[ \begin{pmatrix} 1 & 2 & | & 0 \\ 2 & 4 & | & 0 \end{pmatrix} \]

would you reduce to \[ [ I \mid B ] \].

3) \( \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \) Only one pivot!

4) \( A\hat{x} = 0 \) has only 0 sol.

Nope! \( x_2 \) is free \( x_1 = -2x_2 \) eg. \( (-2, 1) \)

\( (2, 4)(-2) \).
5) Columns of $A$ linearly map.

Nope! We just found a dependence.

\[-2 \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{4} \end{array} \right) + 1 \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{4} \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).\]

So not independent.

6) $x \mapsto Ax$ onto one-to-one.

Nope! This doesn't work.

\[
\left( \begin{array}{cc} 1 & 2 \\ 2 & 4 \end{array} \right) \left( \begin{array}{c} -2 \\ 1 \end{array} \right) \rightarrow 0: \text{we already found a solution.}
\]

7) $Ax = \vec{b}$ at most one sol for every $\vec{b}$.

Nope! Two sols for $\vec{b} = 0$.

\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} -1 \\ 1 \end{array} \right).
\]

(in fact, infinitely many: $\hat{x} = s \left( \begin{array}{c} -1 \\ 1 \end{array} \right)$)
8) columns span \( \mathbb{R}^n \).

nope!

\[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix} \Rightarrow \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix} \quad \text{no pivot in 2nd row}
\]

the span is just a line.

9) \( x \mapsto Ax \) on-to? \((\text{same thing as columns span})\)

no: the image is just multiples of \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

10) there's \( n \times n \ C \) so \( CA = I \). no!

why?

11) there's \( n \times n \ D \) so \( AD = I_n \). no!

why? \( AD = [A\vec{d}_1, A\vec{d}_2] \), each column is a linear combo of the cols of \( A \).

but \( \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \) isn't. so we can't even find \( D \) with \( AD = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \), much less \( I_d \).
in terms of linear transformations

\[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix} =
\begin{pmatrix}
1 \\
2
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}
\begin{pmatrix}
1 \\
3
\end{pmatrix} =
\begin{pmatrix}
7 \\
14
\end{pmatrix}
\]

... all on this line!

No way there can be a

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mirror transformation, because everything
ends up on a single line.
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