- Office hrs this week: M 10-11  
  W 10-11  
  (or email me; some other time W)

- HW 4.4.14 fixed in pdf

- Quiz W: 4.3, 4.4, 4.5

- Attendance sign-in today!
Using coordinates

You have a vector space $V$ and a basis $B = \{ \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \}$.

Want to know if $\mathbf{x}$ (in $V$) is a combo of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

What do: write down $[\mathbf{x}]_B$, $[\mathbf{v}_1]_B$, $[\mathbf{v}_2]_B$, $[\mathbf{v}_3]_B$.

Actual vectors in $\mathbb{R}^3$.

Check $[\mathbf{x}]_B$ is combo of $[\mathbf{v}_1]_B$, $[\mathbf{v}_2]_B$, $[\mathbf{v}_3]_B$.

If it is, $\mathbf{x}$ is combo of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, with same weights!

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Laguerre polynomials.

- $\mathbf{v}_1(t) = 1$
- $\mathbf{v}_2(t) = 1 - t$
- $\mathbf{v}_3(t) = 2 - 4t + t^2$

Given the polynomial $\mathbf{x}(t) = t^2$.

Write as combo of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$.

These are vectors in $\mathbb{P}_2$.

A basis for $\mathbb{P}_2$ is $1, t, t^2$.

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In basis, want $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathbf{v}_1|_B$, $\mathbf{v}_2|_B$, $\mathbf{v}_3|_B$, $[\mathbf{x}]_B$.
this is just a linear system!

\[
\begin{bmatrix}
1 & 1 & 2 & 0 \\
0 & -1 & -4 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

\begin{align*}
c_1 &= 2 \\
c_2 &= -4 \\
c_3 &= 1.
\end{align*}

Check:

\begin{align*}
2(1) + (-4)(1-t) + (1)(2 - 4 + t^2) \\
= 2 - 4 + t + 2 - 4 + t^2 \\
= t^2
\end{align*}
The row space of a matrix & rank

If $A$ is an $m \times n$ matrix, then the row space of $A$ is the subspace of $\mathbb{R}^m$ spanned by the rows of $A$.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, Row $A = \text{span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \right\}$

(in $\mathbb{R}^3$).

How to find it

Do row reduction on $A$ until it's echelon form (don't need to call echelon form $B$).

A basis for the row space of $A$ is given by nonzero rows of $B$. (not $A$)

Another way:

note: Row $A$ is the same thing as $\text{Col}(A^T)$!

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \]

\[ \text{Row} \ A = \text{span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\} = \text{Col} \ A^T. \]
Example:

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \text{ find Row } A, \text{ Col } A, \text{ Nul } A \]

(give a basis)

Once we get \( A \) into rref form, can find all 3!

\[ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \]

\( B = \text{rref}(A) \).

Row \( A = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} \) (rows of \( B \))

Col \( A = \text{span} \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\} \) (pNof cols)

\( \text{Nul } A = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} \)

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \]

\[ A\vec{x} = 0 \rightarrow \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \]

\[ x_1 = 5 \\
\begin{cases} x_2 = -25 \\ x_3 = 5 \end{cases} \text{ is true} \]
Notes

1. putting A in echelon form is enough to find Col A, Row A

2. for Nul A, need to use rref.

3. things in the basis of Col A are a bunch of cols of A.
   The other ones are less similar.
Definition

- The rank of $A$ is the dimension of the column space of $A$.

Rank-nullity theorem (fundamental thm of linear algebra)

- $\text{dim Row } A = \text{dim Col } A$ (always!)  
  \[ \text{rk } A \leq \text{rank} \]

- $\text{rk } A + \dim \text{Nul } A = n$ (where $A$ is an $m \times n$ matrix)

Why?
- row $A$ has a basis vector for every pivot in echelon form
  (since every nonzero row has a pivot)
- col $A$ has a basis vector for every pivot: basis is the pivot cols.
  $\rightarrow$ so dimension of each one is equal to number of pivots.
- nul $A$ has a basis vector for each free variable.
  
  the number of free variables = (number of cols) - (pivot columns)

  \[ n - \text{rk } A \]

so \[ \dim \text{Nul } A + \text{rk } A = n. \]
How to use it.

\[ A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix} \]

What's \( \dim \text{Nul} \ A \)? (What's \( \dim \) of set of sols to \( A \mathbf{x} = \mathbf{0} \)?)

\( \text{col} \ A = \text{span} \left\{ \frac{1}{2} \right\} \; \text{and all multiples of} \; \left( \frac{1}{2} \right) \).

\[ \text{so} \; \dim \; \text{col} \ A = 1 \]

\[ \dim \; \text{Nul} \; A = 3 \; \text{c} \; \# \text{cols} - \dim \; \text{col} \ A. \]

\[ \text{can a} \; 6 \times 9 \; \text{matrix have 2-dimensional nullspace?} \]

\[ \dim \; \text{Nul} \; A + \dim \; \text{col} \; A = 9 \]

\[ \text{so if Nul} \; A \; \text{is 2-dim}, \; \text{Col} \; A \; \text{is 7-dim!} \]

but Col A is a subspace of \( \mathbb{R}^6 \); (each col has six entries)

\[ \text{so dimension is at most 6, can't be 7.} \]

so Nul A can't be 2-dim (must be at least 3-dim)

in other words... a system of 6 equations in 9 variables.

must have at least 3-dim set of solutions.
"Application"

A scientist has 40 eqns in 42 variables.

found two linearly indep solutions to $A\vec{x} = \vec{0}$, and there are no others.

Does $A\vec{x} = \vec{b}$ have a solution for any $\vec{b}$?

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dim Nul $A = 2$, since there are two independent sols.

Theorem says: dim Col $A = 40$. (42 - 2)

so the columns span $\mathbb{R}^{40}$!

$\Rightarrow$ column space is $\mathbb{R}^{40}$.

$\Rightarrow A\vec{x} = \vec{b}$ always has a sol.