Change of Basis

Let $V$ be a finite dimensional vector space and suppose that $B$ and $C$ are bases for $V$. For this discussion let $\dim V = 2$.

Say $\vec{x} \in V$ and $[\vec{x}]_B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

I.e. $B = \{ \vec{b}_1, \vec{b}_2 \}$. $\vec{x} = 5\vec{b}_1 + 2\vec{b}_2$.

We want to realize $[\vec{x}]_C$ by multiplying $[\vec{x}]_B$ by some matrix.

How do we find this matrix, $P_{C \leftrightarrow B}$?

$[\vec{x}]_C = [5\vec{b}_1 + 2\vec{b}_2]_C = 5[\vec{b}_1]_C + 2[\vec{b}_2]_C$.

Thus, if put $P_{C \leftrightarrow B} = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix}_C$.

then $P_{C \leftrightarrow B} [\vec{x}]_B = [\vec{x}]_C$

$\begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix}_C \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 5[\vec{b}_1]_C + 2[\vec{b}_2]_C = [\vec{x}]_C$

Let $V$ be the space of linear polynomials, i.e. polynomials of the form $f(x) = ax + b$. 

B consists of \( b_1(x) = 1 + 2x \)
\( b_2(x) = 1 - 2x \)

\( C \) consists of \( c_1(x) = 1 + x \)
\( c_2(x) = 1 - x \)

\[
\begin{bmatrix}
1/2 \\
-1/2 \\
\end{bmatrix}_C = 
\begin{bmatrix}
3/2 \\
-1/2 \\
\end{bmatrix}_B 
\frac{3}{2} (1+x) - \frac{1}{2} (1-x) \\
\frac{3}{2} + \frac{3}{2} x - \frac{1}{2} + \frac{1}{2} x = 1 + 2x \\
= b_1(x) \\
\end{bmatrix}_C
\]

\[
\begin{bmatrix}
1/2 \\
-1/2 \\
\end{bmatrix}_C = 
\begin{bmatrix}
3/2 \\
-1/2 \\
\end{bmatrix}_B .
\]

\[
\begin{bmatrix}
1/2 \\
-1/2 \\
\end{bmatrix}_C = \begin{bmatrix}
3/2 \\
-1/2 \\
\end{bmatrix}_B 
\]

Let's consider \( [f]_B = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \) i.e. \( f(x) = 2(1+2x) + 4(1-2x) = 2 + 4x + 4 - 8x \)

\[
\frac{6 - 4x}{6 - 4x}
\]

What is \( [f]_C \)?

\[
[f]_C = P [f]_B = \begin{bmatrix}
3/2 \\
-1/2 \\
\end{bmatrix}_B \begin{bmatrix}
2 \\
4 \\
\end{bmatrix} = \begin{bmatrix}
1 \\
5 \\
\end{bmatrix}
\]

Thus \( f(x) = 1 \cdot c_1(x) + 5 \cdot c_2(x) \)
\( = (1+x) + 5(1-x) \).
\( = 1 + x + 5 - 5x \)
\( = 6 - 4x \).
Markov Chains:

Example: Each year 5% of the people in the city move to the suburbs. (95% stay)

30% of the people in the suburbs move to city (97% stay put).

\[ \bar{x}_0 = (c \quad s) \quad \text{← # of people in the city} \]

\[ \bar{x}_1 = (\_ \quad \_ ) \quad \text{← # of people in city after one year} \]

\[ \bar{x}_1 = \begin{pmatrix} \bar{x}_0 \end{pmatrix} \]

\[ A \bar{x}_0 = \bar{x}_1 \]

\[ A = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} \]

\[ A \bar{x}_0 = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 0.95c + 0.03s \\ 0.05c + 0.97s \end{pmatrix} \]

In our situation we are going to assume that \( c + s = 1 \). (i.e. 1 represents 100% of the total population and \( c, s \) are the proportions living in the city/suburbs respectively.)
Idea: we can see the long term growth of this system because
$$\overrightarrow{x}_n = A \overrightarrow{x}_{n-1}$$
where $\overrightarrow{x}_n$ is proportions of people living in the city/suburbs after $n$ years.

Note:
$$\overrightarrow{x}_1 = A \overrightarrow{x}_0$$
$$\overrightarrow{x}_2 = A \overrightarrow{x}_1 = A(A \overrightarrow{x}_0) = A^2 \overrightarrow{x}_0$$
$$\overrightarrow{x}_3 = A \overrightarrow{x}_2 = A(A^2 \overrightarrow{x}_0) = A^3 \overrightarrow{x}_0$$
$$\vdots$$
$$\overrightarrow{x}_n = A^n \overrightarrow{x}_0.$$

Want to know $\lim_{n \to \infty} \overrightarrow{x}_n$.

Definitions: A probability vector is some $\overrightarrow{z} \in \mathbb{R}^n$ where $\overrightarrow{z} = (z_1, z_2, \ldots, z_n)$ with $z_1, z_2, \ldots, z_n \geq 0$.

$$z_1 + z_2 + \ldots + z_n = 1.$$

A stochastic (or Markov) matrix $A$ is a matrix whose columns are probability vectors.

Fact: If $A$ is stochastic and $\overrightarrow{z}$ is a probability vector then $A\overrightarrow{z}$ is a probability vector.

A Markov chain is a sequence $(\overrightarrow{x}_n)$ where
\( \vec{z}_n = A \vec{z}_{n-1} \) and \( \vec{z}_0 \) is probability vector

\( A \) is stochastic.

In our example:

\[ \vec{z}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \] (people in city)

\( \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \) (people in suburbs).

Initial conditions:

\[ A = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} \] stochastic matrix.

\[ \vec{z}_n = A \vec{z}_{n-1} \] so \( (\vec{z}_n) \) is a Markov chain.

Know that \( \vec{z}_n = A^n \vec{z}_0 \).

A priori, requires a lot of computation.

In our experiment, \( \vec{z}_n \) seems to be converging to \( \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix} = \vec{q} \).

One can also see that

\[ A \vec{q} = \vec{q} \]

\( \vec{q} \) is called a steady state. Question:

how do we find steady states without doing lots of computation?

We want to find a steady state probability vector \( \vec{q} \) for \( A \).
We find it by considering the equation
\[
A\vec{q} = \vec{q} \iff A\vec{q} - \vec{q} = 0 \\
\iff A\vec{q} - I\vec{q} = 0 \\
\iff (A - I)\vec{q} = 0.
\]

We see that a steady state vector \( \vec{q} \) is a vector in the nullspace of \( A - I \).

**Thm:** If \( A \) is stochastic, then it admits at least one steady state vector.

Look at \( A - I = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\[
= \begin{pmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{pmatrix}.
\]

\[\begin{pmatrix} 1 \\ -3/5 \end{pmatrix} \]

\[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\( \text{Nul}(A - I) = \{ t\begin{pmatrix} 3/5 \\ 1 \end{pmatrix} : t \in \mathbb{R} \} \)

\[= \text{Span} \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \]

The steady state is a probability vector, so the entries must sum to 1; so look at \( \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \cdot \frac{1}{1 + 3/5} = \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \cdot \frac{5/8}{5/8} = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix} \)

which is the experimentally found steady state!
Question: Can there exist more than one steady state?

Answer: Yes (stupid example).

Let \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \( A \) is stochastic.

but for any probability vector \( \mathbf{p} \).

\( A \mathbf{p} = \mathbf{p} \).

Theo: Def: We call a stochastic matrix \( A \) regular if \( A^k \) has no nonzero entries for some \( k \).

Thm: If \( A \) is a regular stochastic matrix, then \( A \) admits a unique steady state. initial vector \( \mathbf{x}_0 \)

And for any Markov chain \( \mathbf{x}_n = A \mathbf{x}_{n-1} \)

\( \lim_{n \to \infty} \mathbf{x}_n \) converges to this steady state.