Announcements

- next quiz will be a take-home; stay tuned.

- we'll have a full lecture on W to finish 6.5 & 6.6, then do some general review if there's time.

- please do your course evals! you should have gotten an email already.
A set of vectors \( \vec{v}_1, \ldots, \vec{v}_n \) is called **orthonormal** if the vectors are all orthogonal, and length 1.

→ it's easy to turn an orthogonal set into an orthonormal set: just divide each one by its length.

\[
\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

orthogonal. but \( \|\vec{x}_1\| = \sqrt{1^2+0^2+1^2} = \sqrt{2} \)

\( \|\vec{x}_2\| = 1 \).

\[
\vec{v}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \frac{\vec{x}_2}{\|\vec{x}_2\|} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ is orthonormal.}
\]
If $U$ is a matrix $(mxn)$, with orthonormal cols, then $U^TU = nnx$ identity matrix.

\[
U = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 \\
0 & 1 \\
\frac{1}{\sqrt{2}} & 0 \\
\end{pmatrix}
\]

\[
U^TU = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 \\
0 & 1 \\
\frac{1}{\sqrt{2}} & 0 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]

**Fact:** If $\hat{u}_1, \ldots, \hat{u}_n$ is an orthonormal basis for a subspace $W \subset \mathbb{R}^m$ (e.g. $\hat{u}_1, \hat{u}_2$, basis for plane in $\mathbb{R}^3$) another formula for projection:

\[
proj_w \hat{x} = (U(U^T))^T \hat{x}
\]

where $U$ is a matrix with basis vector as columns.

**Note:** If $U$ is $m \times n$, then $U^TU$ is $nnx$ identity but $UU^T$ is $m \times n$, not identity! Don't mix them up! If $U$ is square $nxn$, $UU^T$ is identity too.
Gram-Schmidt orthonormalization.

input: basis for a subspace \( \mathbb{R}^n \)
output: orthonormal basis for same subspace
which you can use in our formulas.

\[
\begin{align*}
\text{2D example.} \\
\vec{x}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\vec{x}_2 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{align*}
\]

to start, take original vector and don't change it.

\[
\vec{v}_1 = \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
write \( \vec{x}_2 \) as \( \vec{x}_2^\parallel + \vec{x}_2^\perp \)
__parallel to \( \vec{x}_1 \) __perp to \( \vec{x}_1 \); use as 2nd basis vector.

\[
\begin{aligned}
\vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1 \\
&= \vec{x}_2 - \frac{1 \cdot 1}{1 \cdot 1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\end{aligned}
\]

\[
\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[
\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\]
to get an orthonormal basis: \( \vec{u}_1, \vec{u}_2 \).
In general, say you're handed $\hat{x}_1, \ldots, \hat{x}_n$ basis for $U \subset \mathbb{R}^m$.

(eg $m=3$, $n=2$: a plane in $\mathbb{R}^3$)

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take

**Step 1:** get orthogonal basis.

take $\vec{V}_1 = \hat{x}_1$

$$\vec{V}_2 = \hat{x}_2 - \frac{\hat{x}_2 \cdot \hat{V}_1}{\hat{V}_1 \cdot \hat{V}_1} \hat{V}_1$$

($\hat{V}_1$ & $\hat{V}_2$ are orthogonal)

$$\vec{V}_3 = \hat{x}_3 - \frac{\hat{x}_3 \cdot \hat{V}_1}{\hat{V}_1 \cdot \hat{V}_1} \hat{V}_1 - \frac{\hat{x}_3 \cdot \hat{V}_2}{\hat{V}_2 \cdot \hat{V}_2} \hat{V}_2$$

... this is the component of $\hat{x}_3$ parallel to $\hat{V}_1$ & $\hat{V}_2$; after subtracting, what's left is orthogonal to both.

(ground picture)

(on page 357)

**Step 2:**
the $\hat{v}_i$'s are an orthogonal basis, but not orthonormal.

an orthonormal basis is given by

$$\hat{u}_1 = \frac{\hat{v}_1}{||\hat{v}_1||}$$

... 

$$\hat{u}_2 = \frac{\hat{v}_2}{||\hat{v}_2||}$$
Example:

\[\begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix}, \begin{pmatrix}
3 \\
1 \\
5
\end{pmatrix}, \begin{pmatrix}
1 \\
-2 \\
3
\end{pmatrix}\]

basis for a 3-dim subspace of \(\mathbb{R}^4\)

\[\hat{v}_1 = \hat{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\]

\[\hat{v}_2 = \hat{x}_2 - \frac{\hat{x}_2 \cdot \hat{v}_1}{\hat{v}_1 \cdot \hat{v}_1} \hat{v}_1 = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} - \frac{18}{6} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}\]

\[\hat{v}_3 = \hat{x}_3 - \frac{\hat{x}_3 \cdot \hat{v}_1}{\hat{v}_1 \cdot \hat{v}_1} \hat{v}_1 - \frac{\hat{x}_3 \cdot \hat{v}_2}{\hat{v}_2 \cdot \hat{v}_2} \hat{v}_2 = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} - \frac{12}{6} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\]

\[\hat{u}_1 = \frac{\hat{v}_1}{||\hat{v}_1||} = \begin{pmatrix} 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{pmatrix}, \hat{u}_2 = \frac{\hat{v}_2}{||\hat{v}_2||} = \begin{pmatrix} -1/\sqrt{6} \\ 0 \\ \sqrt{6} \end{pmatrix}, \hat{u}_3 = \frac{\hat{v}_3}{||\hat{v}_3||} = \begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}\]

to get an orthonormal basis:
QR factorization

if $A$ is $m \times n$, lin indep cols. then can write

$A = QR$ where:

- $Q$ $m \times n$, orthonormal cols.
- $R$ $n \times n$, upper triangular.

cols of $Q$ = result of Gram-Schmidt on cols of $A$.

$R = Q^T A$. (Just multiply it out.)

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this is sort of similar to LU decomposition and simplifies some calculations. Whereas LU decomposition "remembers" how do to do row reduction, QR factorization "remembers" how to do Gram-Schmidt.